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A new global divergence free and pressure-robust HDG method for tangential boundary control of Stokes equations

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Abstract

In Gong et al. (2020), we proposed an HDG method to approximate the solution of a tangential boundary control problem for the Stokes equations and obtained an optimal convergence rate for the optimal control that reflects its global regularity. However, the error estimates depend on the pressure, and the velocity is not divergence free. The importance of pressure-robust numerical methods for fluids was addressed by John et al. (2017). In this work, we devise a new HDG method to approximate the solution of the Stokes tangential boundary control problem; the HDG method is also of independent interest for solving the Stokes equations. This scheme yields a $\boldsymbol{H}(\mathrm{div})$ conforming, globally divergence free, and pressure-robust solution. To the best of our knowledge, this is the first time such a numerical scheme has been obtained for an optimal boundary control problem for the Stokes equations. We also provide numerical experiments to show the performance of the new HDG method and the advantage over the non pressure-robust scheme.

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Keywords: Dirichlet optimal control; Stokes system; Hybridizable discontinuous Galerkin method; Pressure-robust method

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1. Introduction

Control of fluid flows modeled by Stokes or Navier–Stokes equations is an important area of research that has undergone major developments in the recent past. The model poses many theoretical and computational challenges and there is an extensive body of literature devoted to this subject; see, e.g., [1–9]. In [10] we investigated an HDG discretization for the tangential boundary control of a fluid governed by the Stokes system and proved optimal error estimates with respect to the global regularity of the optimal control; however, the numerical method is not pressure-robust, i.e., the discretization errors depend on the norm of the pressure.

As pointed out by John et al. in the 2017 review article [11], many mixed finite element methods, such as Taylor–Hood finite element, Crouzeix–Raviart and MINI elements are not pressure-robust. The key for a numerical scheme to be pressure-robust is the way the null divergence condition is discretized. In the above mentioned review, at least three ways to obtain pressure-robust mixed methods are described: building H^1-conforming divergence-free schemes, using discontinuous Galerkin methods, or committing some variational crime. In 2014, Linke [12] slightly modified the classical lowest order Crouzeix–Raviart element with a variational crime by noticing that the Raviart–Thomas interpolation – see (3.5) below – maps divergence-free vector fields onto divergence-free discrete vector fields. In this way, the discrete velocity of the numerical solution is not affected when the external force is modified with a gradient field, which is a property that is satisfied by the continuous solution: if $-\Delta \boldsymbol{y} + \nabla p = \boldsymbol{f}$ and $\nabla \cdot \boldsymbol{y} = 0$, then for any scalar field ϕ, $-\Delta \boldsymbol{y} + \nabla(p+\phi) = \boldsymbol{f} + \nabla\phi$, $\nabla \cdot \boldsymbol{y} = 0$ and only the pressure is modified. In 2007, Cockburn et al. [13] had already studied a DG method for the Navier–Stokes equations which yields divergence-free solutions.

Hybridizable discontinuous Galerkin (HDG) methods were proposed by Cockburn et al. in [14] as an improvement of traditional DG methods; for a recent didactic exposition, see, e.g., [15]. The HDG algorithm proposed and analyzed in our work [10] is not pressure-robust: although the convergence rate is optimal, the magnitude of the error strongly depends on the pressures; see Example 4.1 below.

In 2016, Lehrenfeld and Schöberl [16] first proposed a pressure-robust HDG method for the Navier–Stokes equations and used a divergence-conforming velocity space; see also Lederer, Lehrenfeld, and Schöberl [17] for an improvement of this method. Recently, Rhebergen and Wells, in [18], used standard cell and facet discontinuous Galerkin spaces that do not involve a divergence-conforming finite element space for the velocity. They obtained pressure-robust scheme for the Navier–Stokes equation; see also Kirk and Rhebergen in [19] for a detailed analysis of this method. For other pressure-robust HDG methods, see [20–23]. In this paper, we propose a new HDG scheme with less degrees of freedom than that of [16], apply it to a tangential boundary control problem governed by the Stokes equation, and prove that the method is pressure-robust.

Despite the large amount of existing work on numerical methods for fluid flow control problems, the authors are only aware of one work dealing with pressure-robustness in the context of optimal control problems, the very recent preprint [24], where a distributed control problem governed by the Stokes equation is discretized by means of a pressure-robust variant of a classical finite element discretization. We, on the other hand, propose a pressure-robust HDG scheme for solving the following tangential boundary control problem:

$$\min_{\boldsymbol{u}\in\boldsymbol{U}} J(\boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{y}_{\boldsymbol{u}} - \boldsymbol{y}_d\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\gamma}{2}\|\boldsymbol{u}\|^2_{\boldsymbol{U}}, \tag{1.1}$$

where $\boldsymbol{y_d}$ is the desired state, $\boldsymbol{y_u}$ is the unique solution in the transposition sense (see, e.g., [10, Definition 2.3]) of

$$-\Delta \boldsymbol{y} + \nabla p = \boldsymbol{f} \text{ in } \Omega, \quad \nabla\cdot\boldsymbol{y} = 0 \text{ in } \Omega, \quad \boldsymbol{y} = \boldsymbol{u} \text{ on } \Gamma, \quad \int_\Omega p = 0, \tag{1.2}$$

γ is a positive constant, and we take the control space

$$\boldsymbol{U} = \{\boldsymbol{u} = u\boldsymbol{\tau} : u \in L^2(\Gamma)\}$$

with norm $\|\boldsymbol{u}\|_{\boldsymbol{U}} = \|u\|_{L^2(\Gamma)}$ and $\boldsymbol{\tau}$ the unit tangential vector.

Formally, the optimal control $u \in L^2(\Gamma)$ and the optimal state $\boldsymbol{y} \in \boldsymbol{L}^2(\Omega)$ satisfy the first order optimality system

$$-\Delta \boldsymbol{y} + \nabla p = \boldsymbol{f} \text{ in } \Omega, \quad \nabla\cdot\boldsymbol{y} = 0 \text{ in } \Omega, \quad \boldsymbol{y} = u\boldsymbol{\tau} \text{ on } \Gamma, \tag{1.3a}$$

$$-\Delta \boldsymbol{z} - \nabla q = \boldsymbol{y} - \boldsymbol{y}_d \text{ in } \Omega, \quad \nabla\cdot\boldsymbol{z} = 0 \text{ in } \Omega, \quad \boldsymbol{z} = 0 \text{ on } \Gamma, \tag{1.3b}$$

$$\partial_{\boldsymbol{n}}\boldsymbol{z}\cdot\boldsymbol{\tau} = \gamma u \text{ on } \Gamma. \tag{1.3c}$$

In [10], we proved that the optimal control is indeed determined by a very weak formulation of the above optimality system and we proved a regularity result for the solution in 2D polygonal domains. The optimal control satisfies (see [10, Theorem 2.4]) $u \in H^s(\Gamma)$ with $s \in (0, 3/2)$. We utilized an existing HDG method to discretize the optimality system and obtained the following a priori error estimate (see [10, Theorem 4.1]):

$$\|u - u_h\|_{L^2(\Gamma)} \le Ch^s(\|\boldsymbol{y}\|_{\boldsymbol{H}^{s+1/2}(\Omega)} + \|\boldsymbol{z}\|_{\boldsymbol{H}^{s+3/2}(\Omega)} + \|p\|_{H^{s-1/2}(\Omega)} + \|q\|_{H^{s+1/2}(\Omega)} + \|u\|_{H^s(\Gamma)}). \tag{1.4}$$

The error estimate (1.4) implies that the error is dependent on the pressure p and dual pressure q.

In this paper, we propose a new HDG method to revisit the problem (1.1)–(1.2). Our new HDG method is pressure-robust; i.e., we obtain the a priori error estimate (see Theorem 3.1):

$$\|u - u_h\|_{L^2(\Gamma)} \le Ch^s(\|\boldsymbol{y}\|_{\boldsymbol{H}^{s+1/2}(\Omega)} + \|\boldsymbol{z}\|_{\boldsymbol{H}^{s+3/2}(\Omega)}). \tag{1.5}$$

The error estimate (1.5) shows the same convergence rate as obtained in [10], but the errors no longer depend on the pressures.

As in [18], our method introduces a numerical trace to approximate the pressure on the boundary edge, but in that reference, the authors use polynomials of degree $k+1$ to approximate the trace of the velocity and we use polynomials of degree k. Hence, the degrees of freedoms of our scheme are less than that in [18]. The price, of course, is that we obtain lower orders of convergence than those obtained in [19] for the method proposed in [18], but on the other hand, our error estimates are valid for problems with very low regularity solutions, as the ones we find when solving Dirichlet control problems.

We find that a pressure-robust method is specially appropriate for the tangential control problem that we address. Notice that if we perturb $\boldsymbol{y}_d$ with a conservative field $\nabla\phi$ for some scalar function ϕ, the optimal solution would not change at all. We should just replace q by $q + \phi$ to obtain the solution of the optimality system.

The plan of this paper is as follows. In Section 2 we present the functional framework, the optimality system for the control problem, and the new HDG formulation; we prove that, for any given control, both the discrete velocity and adjoint velocity are divergence free. Section 3 is devoted to the error analysis; we present and prove our main result. The scheme of our proof largely follows the structure in our previous work [10], but here we needed to use new techniques to show in every auxiliary lemma that the obtained estimates are independent of the pressure. Finally, in Section 4 we provide the results of two numerical experiments to compare the performance of the present pressure-robust method with the method in [10].

2. Background: Regularity and HDG formulation

In this section, we briefly review the regularity results for the tangential boundary control problem and give the HDG formulation.

First, we define some notation. Let Ω be a bounded polygonal domain. We use the standard notation $H^m(\Omega)$ to denote the Sobolev space with norm $\|\cdot\|_{m,\Omega}$. In many places, we use $\|\cdot\|_m$ to replace $\|\cdot\|_{m,\Omega}$ if the context makes the norm clear. Let $\mathbb{H}^m(\Omega) = [H^m(\Omega)]^{2\times 2}$, $\boldsymbol{H}^m(\Omega) = [H^m(\Omega)]^2$ and $\boldsymbol{H}_0^1(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega); \boldsymbol{v} = 0 \text{ on } \Gamma\}$. Let $\langle\cdot,\cdot\rangle_\Gamma$ denote the inner product in $L^2(\Gamma)$ and let $[\cdot,\cdot]_\Gamma$ denote the duality product between $H^{-s}(\Gamma)$ and $H^s(\Gamma)$. We introduce the spaces

$$\begin{aligned}
&\boldsymbol{V}^s(\Omega) = \{\boldsymbol{y} \in \boldsymbol{H}^s(\Omega) : \nabla\cdot\boldsymbol{y} = 0,\ [\boldsymbol{y}\cdot\boldsymbol{n}, 1]_\Gamma = 0\}, \text{ for } s \ge 0,\\
&\boldsymbol{V}_0^s(\Omega) = \{\boldsymbol{y} \in \boldsymbol{H}^s(\Omega) : \nabla\cdot\boldsymbol{y} = 0,\ \boldsymbol{y} = 0 \text{ on } \Gamma\}, \text{ for } s > 1/2,\\
&\boldsymbol{V}^s(\Gamma) = \{\boldsymbol{u} \in \boldsymbol{H}^s(\Gamma) : \langle\boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_\Gamma = 0\}, \text{ for } 0 \le s < 3/2.
\end{aligned}$$

We denote the L^2-inner products on $\mathbb{L}^2(\Omega)$, $\boldsymbol{L}^2(\Omega)$, $L^2(\Omega)$ and $\boldsymbol{L}^2(\Gamma)$ by

$$(\mathbb{L}, \mathbb{G})_\Omega = \sum_{i,j=1}^{2} \int_\Omega L_{ij}G_{ij}, \quad (\boldsymbol{y}, \boldsymbol{z})_\Omega = \sum_{j=1}^{2} \int_\Omega y_j z_j, \quad (p, q)_\Omega = \int_\Omega pq, \quad \langle\boldsymbol{y}, \boldsymbol{z}\rangle_\Gamma = \sum_{j=1}^{2} \int_\Gamma y_j z_j.$$

Define the spaces $\mathbb{H}(\text{div}; \Omega)$ and $L_0^2(\Omega)$ as

$$\mathbb{H}(\text{div}, \Omega) = \{\mathbb{K} \in \mathbb{L}^2(\Omega), \nabla\cdot\mathbb{K} \in \boldsymbol{L}^2(\Omega)\}, \quad L_0^2(\Omega) = \left\{p \in L^2(\Omega), (p, 1)_\Omega = 0\right\}.$$

2.1. Regularity

In [10, Theorem 2.8 and Corollary 2.9], we proved the following well-posedness and regularity result for the tangential Dirichlet boundary control problem (1.1)–(1.2). Set $\mathbb{L} = \nabla \boldsymbol{y}$ and $\mathbb{G} = \nabla \boldsymbol{z}$, let ω be the largest interior angle of Γ, and let $\xi \in (0.5, 4]$ be the real part of the smallest root different from zero of the equation

$$\sin^2(\lambda\omega) - \lambda^2 \sin^2 \omega = 0. \tag{2.1}$$

It is known that $\xi > \pi/\omega$ if $\omega < \pi$ and $0.5 < \xi < \pi/\omega$ if $\omega > \pi$.

Theorem 2.1. *If Ω is a convex polygonal domain, $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{y}_d \in \boldsymbol{H}^{\min\{2,\xi\}}(\Omega)$, then there is a unique solution $u \in L^2(\Gamma)$ of problem (1.1)–(1.2). The solution u satisfies $u \in H^s(\Gamma)$ for all $1/2 < s < \min\{3/2, \xi - 1/2\}$ and there exists*

$$\boldsymbol{y} \in \boldsymbol{V}^{s+1/2}(\Omega), \qquad \mathbb{L} \in \mathbb{H}^{s-1/2}(\Omega), \qquad p \in H^{s-1/2}(\Omega) \cap L_0^2(\Omega),$$
$$\boldsymbol{z} \in \boldsymbol{V}_0^{r+1}(\Omega), \qquad \mathbb{G} \in \mathbb{H}^{r}(\Omega), \qquad q \in H^{r}(\Omega) \cap L_0^2(\Omega)$$

for all $1 < r < \min\{3, \xi\}$, and $\mathbb{L} - p\mathbb{I} \in \mathbb{H}(\mathrm{div}, \Omega)$ such that

$$(\mathbb{L}, \mathbb{T})_\Omega + (\boldsymbol{y}, \nabla \cdot \mathbb{T})_\Omega = \langle u\boldsymbol{\tau}, \mathbb{T}\boldsymbol{n}\rangle_\Gamma, \tag{2.2a}$$
$$-(\nabla \cdot (\mathbb{L} - p\mathbb{I}), \boldsymbol{v})_\Omega = (\boldsymbol{f}, \boldsymbol{v})_\Omega, \tag{2.2b}$$
$$(\nabla \cdot \boldsymbol{y}, w)_\Omega = 0, \tag{2.2c}$$
$$(\mathbb{G}, \mathbb{T})_\Omega + (\boldsymbol{z}, \nabla \cdot \mathbb{T})_\Omega = 0, \tag{2.2d}$$
$$-(\nabla \cdot (\mathbb{G} + q\mathbb{I}), \boldsymbol{v})_\Omega = (\boldsymbol{y} - \boldsymbol{y}_d, \boldsymbol{v})_\Omega, \tag{2.2e}$$
$$(\nabla \cdot \boldsymbol{z}, w)_\Omega = 0, \tag{2.2f}$$
$$\langle \gamma u\boldsymbol{\tau} - \mathbb{G}\boldsymbol{n}, \mu\boldsymbol{\tau}\rangle_\Gamma = 0 \tag{2.2g}$$

for all $(\mathbb{T}, \boldsymbol{v}, w, \mu) \in \mathbb{H}(\mathrm{div}, \Omega) \times \boldsymbol{L}^2(\Omega) \times L_0^2(\Omega) \times L^2(\Gamma)$. Moreover,

$$u \in \prod_{i=1}^{m} H^{r-1/2}(\Gamma_i) \text{ for all } r < \min\{3, \xi\}, \tag{2.3}$$

where Γ_i denotes the smooth segment of Γ such that $\Gamma = \bigcup_{i=1}^m \Gamma_i$.

2.2. The HDG formulation

We use the same notation as in [10] to describe the HDG method. Let $\{\mathcal{T}_h\}$ be a family of conforming and quasi-uniform triangular meshes of Ω. This assumption on the meshes is stronger than in [10]; there we assumed $\{\mathcal{T}_h\}$ is a family of conforming and quasi-uniform polygonal meshes. Let $\partial\mathcal{T}_h$ denote the set $\{\partial K : K \in \mathcal{T}_h\}$. For an element K of the collection $\mathcal{T}_h$, $e = \partial K \cap \Gamma$ is the boundary edge if the length of e is non-zero. For two elements K^+ and K^- of the collection $\mathcal{T}_h$, $e = \partial K^+ \cap \partial K^-$ is the interior edge between K^+ and K^- if the length of e is non-zero. Let $\mathcal{E}_h^o$ and $\mathcal{E}_h^\partial$ denote the set of interior and boundary edges, respectively. We denote by $\mathcal{E}_h$ the union of $\mathcal{E}_h^o$ and $\mathcal{E}_h^\partial$. We introduce various inner products:

$$(\eta, \zeta)_{\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h} (\eta, \zeta)_K, \quad (\boldsymbol{\eta}, \boldsymbol{\zeta})_{\mathcal{T}_h} = \sum_{i=1}^{2} (\eta_i, \zeta_i)_{\mathcal{T}_h}, \quad (\mathbb{L}, \mathbb{G})_{\mathcal{T}_h} = \sum_{i,j=1}^{2} (L_{ij}, G_{ij})_{\mathcal{T}_h},$$

$$\langle \eta, \zeta\rangle_{\partial\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h} \langle \eta, \zeta\rangle_{\partial K}, \quad \langle \boldsymbol{\eta}, \boldsymbol{\zeta}\rangle_{\partial\mathcal{T}_h} = \sum_{i=1}^{2} \langle \eta_i, \zeta_i\rangle_{\partial\mathcal{T}_h}.$$

The norms induced by the above inner products are defined accordingly.

Let $\mathcal{P}^k(D)$ denote the set of polynomials of degree at most k on a domain D. We introduce the following discontinuous finite element spaces:

$$\begin{aligned}
\mathbb{K}_h &:= \{\mathbb{L} \in \mathbb{L}^2(\Omega) : \mathbb{L}|_K \in [\mathcal{P}^k(K)]^{2\times 2},\ \forall K \in \mathcal{T}_h\},\\
\boldsymbol{V}_h &:= \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{v}|_K \in [\mathcal{P}^{k+1}(K)]^2,\ \forall K \in \mathcal{T}_h\},\\
W_h &:= \{w \in L^2(\Omega) : w|_K \in \mathcal{P}^k(K),\ \forall K \in \mathcal{T}_h\},\\
\boldsymbol{M}_h &:= \{\mu \in \boldsymbol{L}^2(\mathcal{E}_h) : \boldsymbol{\mu}|_e \in [\mathcal{P}^k(e)]^2,\ \forall e \in \mathcal{E}_h\},\\
M_h &:= \{\mu \in L^2(\mathcal{E}_h^\partial) : \mu|_e \in \mathcal{P}^k(e),\ \forall e \in \mathcal{E}_h^\partial\},\\
Q_h &:= \{\mu \in L^2(\mathcal{E}_h) : \mu|_e \in \mathcal{P}^{k+1}(e),\ \forall e \in \mathcal{E}_h\}.
\end{aligned}$$

Let $\boldsymbol{M}_h(o)$ denote the space defined in the same way as $\boldsymbol{M}_h$, but with $\mathcal{E}_h$ replaced by $\mathcal{E}_h^o$. We use $\nabla\boldsymbol{v}$ and $\nabla\cdot\mathbb{L}$ to denote the gradient of $\boldsymbol{v}$ and the divergence of $\mathbb{L}$ taken piecewise on each element $K \in \mathcal{T}_h$. Finally, we define

$$W_h^0 = \left\{w \in L^2(\Omega) : w|_K \in \mathcal{P}^k(K),\ \forall K \in \mathcal{T}_h \text{ and } (w,1)_\Omega = 0\right\}.$$

The HDG method seeks approximate fluxes $\mathbb{L}_h, \mathbb{G}_h \in \mathbb{K}_h$, states $\boldsymbol{y}_h, \boldsymbol{z}_h \in \boldsymbol{V}_h$, pressures $p_h, q_h \in W_h^0$, interior element boundary traces $\widehat{\boldsymbol{y}}_h^o, \widehat{\boldsymbol{z}}_h^o \in \boldsymbol{M}_h(o)$ and $\widehat{p}_h, \widehat{q}_h \in Q_h$, and boundary control $u_h \in M_h$ satisfying

$$(\mathbb{L}_h, \mathbb{T}_1)_{\mathcal{T}_h} + (\boldsymbol{y}_h, \nabla\cdot\mathbb{T}_1)_{\mathcal{T}_h} - \left\langle\widehat{\boldsymbol{y}}_h^o, \mathbb{T}_1\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} = \langle u_h\boldsymbol{\tau}, \mathbb{T}_1\boldsymbol{n}\rangle_{\mathcal{E}_h^\partial}, \tag{2.4a}$$

$$\begin{aligned}
&-(\nabla\cdot\mathbb{L}_h, \boldsymbol{v}_1)_{\mathcal{T}_h} - (p_h, \nabla\cdot\boldsymbol{v}_1)_{\mathcal{T}_h} + \langle\widehat{p}_h, \boldsymbol{v}_1\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}\\
&\qquad+\langle h^{-1}\boldsymbol{P_M}\boldsymbol{y}_h, \boldsymbol{v}_1\rangle_{\partial\mathcal{T}_h} - \langle h^{-1}\widehat{\boldsymbol{y}}_h^o, \boldsymbol{v}_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} = (\boldsymbol{f}, \boldsymbol{v}_1)_{\mathcal{T}_h} + \langle h^{-1}u_h\boldsymbol{\tau}, \boldsymbol{v}_1\rangle_{\mathcal{E}_h^\partial},
\end{aligned}\tag{2.4b}$$

$$(\nabla\cdot\boldsymbol{y}_h, w_1)_{\mathcal{T}_h} = 0, \tag{2.4c}$$

$$\langle\boldsymbol{y}_h\cdot\boldsymbol{n}, \widehat{w}_1\rangle_{\partial\mathcal{T}_h} = 0 \tag{2.4d}$$

for all $(\mathbb{T}_1, \boldsymbol{v}_1, w_1, \widehat{w}_1) \in \mathbb{K}_h \times \boldsymbol{V}_h \times W_h^0 \times Q_h$,

$$(\mathbb{G}_h, \mathbb{T}_2)_{\mathcal{T}_h} + (\boldsymbol{z}_h, \nabla\cdot\mathbb{T}_2)_{\mathcal{T}_h} - \left\langle\widehat{\boldsymbol{z}}_h^o, \mathbb{T}_2\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} = 0, \tag{2.4e}$$

$$\begin{aligned}
&-(\nabla\cdot\mathbb{G}_h, \boldsymbol{v}_2)_{\mathcal{T}_h} + (q_h, \nabla\cdot\boldsymbol{v}_2)_{\mathcal{T}_h} - \langle\widehat{q}_h, \boldsymbol{v}_2\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}\\
&\qquad+\langle h^{-1}\boldsymbol{P_M}\boldsymbol{z}_h, \boldsymbol{v}_2\rangle_{\partial\mathcal{T}_h} - \langle h^{-1}\widehat{\boldsymbol{z}}_h^o, \boldsymbol{v}_2\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} = (\boldsymbol{y}_h - \boldsymbol{y}_d, \boldsymbol{v}_2)_{\mathcal{T}_h},
\end{aligned}\tag{2.4f}$$

$$(\nabla\cdot\boldsymbol{z}_h, w_2)_{\mathcal{T}_h} = 0, \tag{2.4g}$$

$$\langle\boldsymbol{z}_h\cdot\boldsymbol{n}, \widehat{w}_2\rangle_{\partial\mathcal{T}_h} = 0 \tag{2.4h}$$

for all $(\mathbb{T}_2, \boldsymbol{v}_2, w_2, \widehat{w}_2) \in \mathbb{K}_h \times \boldsymbol{V}_h \times W_h^0 \times Q_h$,

$$\langle\mathbb{L}_h\boldsymbol{n} - h^{-1}(\boldsymbol{P}_M\boldsymbol{y}_h - \widehat{\boldsymbol{y}}_h^o), \boldsymbol{\mu}_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} = 0 \tag{2.4i}$$

for all $\boldsymbol{\mu}_1 \in \boldsymbol{M}_h(o)$,

$$\langle\mathbb{G}_h\boldsymbol{n} - h^{-1}(\boldsymbol{P}_M\boldsymbol{z}_h - \widehat{\boldsymbol{z}}_h^o), \boldsymbol{\mu}_2\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} = 0 \tag{2.4j}$$

for all $\boldsymbol{\mu}_2 \in \boldsymbol{M}_h(o)$,

$$\langle\mathbb{G}_h\boldsymbol{n} - h^{-1}\boldsymbol{P}_M\boldsymbol{z}_h - \gamma u_h\boldsymbol{\tau}, \mu_3\boldsymbol{\tau}\rangle_{\mathcal{E}_h^\partial} = 0 \tag{2.4k}$$

for all $\mu_3 \in M_h$. Here $\boldsymbol{P}_M$ denotes the standard L^2-orthogonal projection from $\boldsymbol{L}^2(\mathcal{E}_h)$ onto $\boldsymbol{M}_h$; see (3.3c) below. This completes the formulation of the HDG method.

Remark 2.2. Our method resembles the one introduced in [18] and analyzed in [19] in the sense that the numerical trace of the pressure plays the role of Lagrange multipliers enforcing continuity of the normal component of the velocity across element boundaries. Nevertheless, to approximate the trace of the velocity, we use polynomials of degree k instead of $k+1$. In this way, our method has fewer degrees of freedom, but at the price of a lower order of convergence. This feature can be seen as a drawback when solving an uncontrolled Stokes problem or even a distributed control problem governed by the Stokes equation. But for the problem at hand the regularity of the solution is usually very low, see Theorem 2.1, and the order of convergence will be mainly limited by this fact, so it makes sense to use a method with suboptimal rates of convergence.

Notice also that the HDG method developed in this paper has more degrees of freedom than the scheme in [10], since we introduced two more numerical traces $\widehat{p}_h$ and $\widehat{q}_h$ to approximate the traces of the pressures p_h and q_h, respectively in order to obtain a pressure-robust method.

Next, we show that the discrete system (2.4) yields a *globally* divergence free state $\boldsymbol{y}_h$ and dual state $\boldsymbol{z}_h$.

Proposition 2.3. *Let $\boldsymbol{y}_h$ and $\boldsymbol{z}_h$ be the solutions of (2.4), then we have $\boldsymbol{y}_h, \boldsymbol{z}_h \in \boldsymbol{H}(\mathrm{div};\Omega)$ and $\nabla\cdot\boldsymbol{y}_h = \nabla\cdot\boldsymbol{z}_h = 0$.*

Proof. We only prove the result for $\boldsymbol{y}_h$ since the proof for $\boldsymbol{z}_h$ is similar. Let K_1, $K_2 \in \mathcal{T}_h$ be any two adjacent elements sharing a common edge e. Define $\widehat{r} \in Q_h$ as follows:

$$\begin{aligned} \widehat{r}|_e &= -(\boldsymbol{y}_h\cdot\boldsymbol{n}_e)|_{K_1\cap e} - (\boldsymbol{y}_h\cdot\boldsymbol{n}_e)|_{K_2\cap e} && \forall e \in \mathcal{E}_h^o,\\ \widehat{r}|_e &= 0 && \forall e \in \mathcal{E}_h^\partial. \end{aligned}$$

Let $c_0 = \frac{1}{|\Omega|}\sum_{K\in\mathcal{T}_h}\int_K \nabla\cdot\boldsymbol{y}_h$ and take $(w_1,\widehat{w}_1) = (\nabla\cdot\boldsymbol{y}_h - c_0, \widehat{r}-c_0)$ in (2.4c)–(2.4d) to get

$$\begin{aligned} 0 &= -(\nabla\cdot\boldsymbol{y}_h, \nabla\cdot\boldsymbol{y}_h - c_0)_{\mathcal{T}_h} + \langle \boldsymbol{y}_h\cdot\boldsymbol{n}, \widehat{r}-c_0\rangle_{\partial\mathcal{T}_h}\\ &= -(\nabla\cdot\boldsymbol{y}_h, \nabla\cdot\boldsymbol{y}_h)_{\mathcal{T}_h} + \langle \boldsymbol{y}_h\cdot\boldsymbol{n}, \widehat{r}\rangle_{\partial\mathcal{T}_h}\\ &= -(\nabla\cdot\boldsymbol{y}_h, \nabla\cdot\boldsymbol{y}_h)_{\mathcal{T}_h} - \sum_{e\in\mathcal{E}_h^o} \|(\boldsymbol{y}_h\cdot\boldsymbol{n}_e)|_{K_1} + (\boldsymbol{y}_h\cdot\boldsymbol{n}_e)|_{K_2}\|_{0,e}^2. \end{aligned}$$

This implies $\boldsymbol{y}_h \in \boldsymbol{H}(\mathrm{div};\Omega)$ and $\nabla\cdot\boldsymbol{y}_h = 0$.

3. Error analysis

We assume that the solution of (2.2a)–(2.2g) satisfies

$$\mathbb{L}\in\mathbb{H}^{r_{\mathbb{L}}}(\Omega),\quad \boldsymbol{y}\in\boldsymbol{H}^{r_y}(\Omega),\quad \mathbb{G}\in\mathbb{H}^{r_{\mathbb{G}}}(\Omega),\quad \boldsymbol{z}\in\boldsymbol{H}^{r_z}(\Omega),$$

where

$$r_y > 1,\quad r_z > 2,\quad r_{\mathbb{L}} > 1/2,\quad r_{\mathbb{G}} > 1. \tag{3.1}$$

We now state our main result.

Theorem 3.1. *For*

$$s_{\mathbb{L}} = \min\{r_{\mathbb{L}}, k+1\},\quad s_y = \min\{r_y, k+2\},\quad s_{\mathbb{G}} = \min\{r_{\mathbb{G}}, k+1\},\quad s_z = \min\{r_z, k+2\},$$

if the regularity assumption (3.1) holds we have

$$\begin{aligned} \|u-u_h\|_{\mathcal{E}_h^\partial} &\lesssim h^{s_{\mathbb{L}}+\frac12}\|\mathbb{L}\|_{s_{\mathbb{L}},\Omega} + h^{s_y-\frac12}\|\boldsymbol{y}\|_{s_y,\Omega} + h^{s_{\mathbb{G}}-\frac12}\|\mathbb{G}\|_{s_{\mathbb{G}},\Omega} + h^{s_z-\frac32}\|\boldsymbol{z}\|_{s_z,\Omega},\\ \|\boldsymbol{y}-\boldsymbol{y}_h\|_{\mathcal{T}_h} &\lesssim h^{s_{\mathbb{L}}+\frac12}\|\mathbb{L}\|_{s_{\mathbb{L}},\Omega} + h^{s_y-\frac12}\|\boldsymbol{y}\|_{s_y,\Omega} + h^{s_{\mathbb{G}}-\frac12}\|\mathbb{G}\|_{s_{\mathbb{G}},\Omega} + h^{s_z-\frac32}\|\boldsymbol{z}\|_{s_z,\Omega},\\ \|\mathbb{G}-\mathbb{G}_h\|_{\mathcal{T}_h} &\lesssim h^{s_{\mathbb{L}}+\frac12}\|\mathbb{L}\|_{s_{\mathbb{L}},\Omega} + h^{s_y-\frac12}\|\boldsymbol{y}\|_{s_y,\Omega} + h^{s_{\mathbb{G}}-\frac12}\|\mathbb{G}\|_{s_{\mathbb{G}},\Omega} + h^{s_z-\frac32}\|\boldsymbol{z}\|_{s_z,\Omega},\\ \|\boldsymbol{z}-\boldsymbol{z}_h\|_{\mathcal{T}_h} &\lesssim h^{s_{\mathbb{L}}+\frac12}\|\mathbb{L}\|_{s_{\mathbb{L}},\Omega} + h^{s_y-\frac12}\|\boldsymbol{y}\|_{s_y,\Omega} + h^{s_{\mathbb{G}}-\frac12}\|\mathbb{G}\|_{s_{\mathbb{G}},\Omega} + h^{s_z-\frac32}\|\boldsymbol{z}\|_{s_z,\Omega}. \end{aligned}$$

If $k\geq 1$, then

$$\|\mathbb{L}-\mathbb{L}_h\|_{\mathcal{T}_h} \lesssim h^{s_{\mathbb{L}}}\|\mathbb{L}\|_{s_{\mathbb{L}},\Omega} + h^{s_y-1}\|\boldsymbol{y}\|_{s_y,\Omega} + h^{s_{\mathbb{G}}-1}\|\mathbb{G}\|_{s_{\mathbb{G}},\Omega} + h^{s_z-2}\|\boldsymbol{z}\|_{s_z,\Omega}.$$

Remark 3.2. The error estimates in Theorem 3.1 are independent of the pressures p and q, which are different from the error estimates in [10, Theorem 4.1]. Therefore, our HDG method is pressure-robust. We note that the HDG method considered here has more degrees of freedom than that in [10], since we have introduced numerical traces for the pressures. We also note that the technique used in [10] cannot be applied here to treat the case when $r_{\mathbb{L}} \leq 1/2$. This low regularity for $\mathbb{L} = \nabla\boldsymbol{y}$ may appear when $\xi \leq 3/2$, which corresponds to a value of ω greater than $\omega_{3/2} \approx 0.839138753489667\pi$; see more details in Remark 3.11. Moreover, the meshes here are restricted to be triangular, while in [10] we can use general polygonal meshes.

Noticing that for $\omega \in [\pi/3, \omega_{3/2})$ we have that $\xi \in (3/2, 4]$, the application of Theorems 3.1 and 2.1 gives the following result.

Corollary 3.3. *Suppose $\boldsymbol{y}_d \in \boldsymbol{H}^{\xi}(\Omega)$. Let $\omega \in [\pi/3, \omega_{3/2})$ be the largest interior angle of Γ, and define r_Ω by*

$$r_\Omega = \min\left\{\frac{3}{2}, \xi - \frac{1}{2}\right\} \in (1, \frac{3}{2}].$$

Then the regularity condition (3.1) is satisfied. Also, if $k \geq 1$, then for any $r < r_\Omega$ we have

$$h^{\frac{1}{2}}\|\mathbb{L} - \mathbb{L}_h\|_{\mathcal{T}_h} + \|\boldsymbol{y} - \boldsymbol{y}_h\|_{\mathcal{T}_h} + \|\mathbb{G} - \mathbb{G}_h\|_{\mathcal{T}_h} + \|\boldsymbol{z} - \boldsymbol{z}_h\|_{\mathcal{T}_h} + \|u - u_h\|_{\mathcal{E}_h^\partial} \lesssim h^r.$$

Moreover, if $k = 0$, we have

$$\|u - u_h\|_{\mathcal{E}_h^\partial} + \|\boldsymbol{y} - \boldsymbol{y}_h\|_{\mathcal{T}_h} + \|\boldsymbol{z} - \boldsymbol{z}_h\|_{\mathcal{T}_h} + \|\mathbb{G} - \mathbb{G}_h\|_{\mathcal{T}_h} \lesssim h^{1/2}.$$

3.1. Preliminary material

We use the standard L^2 projections $\boldsymbol{\Pi}_{\mathbb{K}} : \mathbb{L}^2(\Omega) \to \mathbb{K}_h$, $\boldsymbol{\Pi}_V : \boldsymbol{L}^2(\Omega) \to \boldsymbol{V}_h$, and $\Pi_W : L^2(\Omega) \to W_h$ satisfying

$$(\boldsymbol{\Pi}_{\mathbb{K}}\mathbb{L}, \mathbb{T})_K = (\mathbb{L}, \mathbb{T})_K \qquad \forall\, \mathbb{T} \in [\mathcal{P}^k(K)]^{2\times 2}, \tag{3.2a}$$

$$(\boldsymbol{\Pi}_V \boldsymbol{y}, \boldsymbol{v})_K = (\boldsymbol{y}, \boldsymbol{v})_K \qquad \forall\, \boldsymbol{v} \in [\mathcal{P}^{k+1}(K)]^2, \tag{3.2b}$$

$$(\Pi_W p, w)_K = (p, w)_K \qquad \forall\, w \in \mathcal{P}^k(K). \tag{3.2c}$$

For all edges e of the triangle K, we also need the L^2-orthogonal projections P_M onto M_h, P_Q onto Q_h, and $\boldsymbol{P}_M$ onto $\boldsymbol{M}_h$ satisfying

$$\langle P_M u - u, \mu\rangle_e = 0 \quad \forall \mu \in M_h, \tag{3.3a}$$

$$\langle P_Q p - p, \mu\rangle_e = 0 \quad \forall \mu \in Q_h, \tag{3.3b}$$

$$\langle \boldsymbol{P}_M \boldsymbol{y} - \boldsymbol{y}, \boldsymbol{\mu}\rangle_e = 0 \quad \forall \boldsymbol{\mu} \in \boldsymbol{M}_h. \tag{3.3c}$$

In the analysis, we use the following classical results [25, Section 4.2]:

$$\|\boldsymbol{\Pi}_{\mathbb{K}}\mathbb{L} - \mathbb{L}\|_{\mathcal{T}_h} \lesssim h^{s_{\mathbb{L}}}\|\mathbb{L}\|_{s_{\mathbb{L}},\Omega}, \quad \|\boldsymbol{\Pi}_V \boldsymbol{y} - \boldsymbol{y}\|_{\mathcal{T}_h} \lesssim h^{s_y}\|\boldsymbol{y}\|_{s_y,\Omega}, \tag{3.4a}$$

$$\|\boldsymbol{\Pi}_{\mathbb{K}}\mathbb{L} - \mathbb{L}\|_{\partial\mathcal{T}_h} \lesssim h^{s_{\mathbb{L}}-\frac{1}{2}}\|\mathbb{L}\|_{s_{\mathbb{L}},\Omega}, \quad \|\boldsymbol{\Pi}_V \boldsymbol{y} - \boldsymbol{y}\|_{\partial\mathcal{T}_h} \lesssim h^{s_y-\frac{1}{2}}\|\boldsymbol{y}\|_{s_y,\Omega}, \tag{3.4b}$$

$$\|\Pi_W p - p\|_{\mathcal{T}_h} \lesssim h^{s_p}\|p\|_{s_p,\Omega}, \quad \|\boldsymbol{P}_M \boldsymbol{y} - \boldsymbol{y}\|_{\partial\mathcal{T}_h} \lesssim h^{s_y-\frac{1}{2}}\|\boldsymbol{y}\|_{s_y,\Omega}, \tag{3.4c}$$

$$\|P_M u - u\|_{\partial\mathcal{T}_h} \lesssim h^{s_y-\frac{1}{2}}\|\boldsymbol{y}\|_{s_y,\Omega}, \quad \|P_Q p - p\|_{\partial\mathcal{T}_h} \lesssim h^{s_p-\frac{1}{2}}\|p\|_{s_p,\Omega}. \tag{3.4d}$$

We have the same projection error bounds for $\mathbb{G}$, $\boldsymbol{z}$ and q.

For the error analysis in this section, we need to introduce the classical Raviart–Thomas (RT) space:

$$\mathcal{R}^k(K) = [\mathcal{P}^k(K)]^2 + \boldsymbol{x}\mathcal{P}^k(K),$$

and define the RT projection $\Pi^{\mathrm{RT}} : \boldsymbol{H}^1(K) \to \mathcal{R}^{k+1}(K)$

$$\langle \Pi^{\mathrm{RT}}\boldsymbol{v}\cdot\boldsymbol{n}, w\rangle_e = \langle \boldsymbol{v}\cdot\boldsymbol{n}, w\rangle_e \qquad \forall w \in \mathcal{P}^{k+1}(e), e \subset \partial K, \tag{3.5a}$$

$$(\Pi^{\mathrm{RT}}\boldsymbol{v}, \boldsymbol{w})_K = (\boldsymbol{v}, \boldsymbol{w})_K \qquad \forall \boldsymbol{w} \in [\mathcal{P}^k(K)]^2. \tag{3.5b}$$

We also need the following classical results [26, Theorem 3.1]:

$$\|\Pi^{\mathrm{RT}}\boldsymbol{y} - \boldsymbol{y}\|_{\mathcal{T}_h} \lesssim h^{s_y}\|\boldsymbol{y}\|_{s_y,\Omega}, \qquad \|\Pi^{\mathrm{RT}}\boldsymbol{y} - \boldsymbol{y}\|_{\partial\mathcal{T}_h} \lesssim h^{s_y-1/2}\|\boldsymbol{y}\|_{s_y,\Omega}.$$

By the well-known commutative diagram [26, Equation (38)] we have

$$\nabla\cdot(\Pi^{\mathrm{RT}}\boldsymbol{v}) = \Pi(\nabla\cdot\boldsymbol{v}),$$

where Π is the standard L^2 projection from $L^2(K)$ onto $\mathcal{P}^{k+1}(K)$. If $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega)$ and $\nabla\cdot\boldsymbol{v} = 0$, then

$$\nabla\cdot(\Pi^{\mathrm{RT}}\boldsymbol{v}) = 0.$$

Applying [26, Lemma 3.1] we have the following lemma.

Lemma 3.4. *For any $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega)$ and $\nabla\cdot\boldsymbol{v} = 0$, we have $\Pi^{\mathrm{RT}}\boldsymbol{v} \in \boldsymbol{V}_h$.*

To simplify notation, we define an HDG operator $\mathscr{B}$. For all $(\mathbb{L}_h, \boldsymbol{y}_h, p_h, \widehat{p}_h, \widehat{\boldsymbol{y}}_h^o) \in \mathbb{K}_h \times \boldsymbol{V}_h \times W_h^0 \times Q_h \times \boldsymbol{M}_h(o)$, we define

$$
\begin{aligned}
&\mathscr{B}(\mathbb{L}_h, \boldsymbol{y}_h, p_h, \widehat{p}_h, \widehat{\boldsymbol{y}}_h^o; \mathbb{T}_1, \boldsymbol{v}_1, w_1, \widehat{w}_1, \boldsymbol{\mu}_1)\\
&= (\mathbb{L}_h, \mathbb{T}_1)_{\mathcal{T}_h} + (\boldsymbol{y}_h, \nabla\cdot\mathbb{T}_1)_{\mathcal{T}_h} - \left\langle \widehat{\boldsymbol{y}}_h^o, \mathbb{T}_1\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} - (\nabla\cdot\mathbb{L}_h, \boldsymbol{v}_1)_{\mathcal{T}_h}\\
&\quad - (p_h, \nabla\cdot\boldsymbol{v}_1)_{\mathcal{T}_h} + \langle \widehat{p}_h, \boldsymbol{v}_1\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + \langle h^{-1}P_{\boldsymbol{M}}\boldsymbol{y}_h, \boldsymbol{v}_1\rangle_{\partial\mathcal{T}_h} - \langle h^{-1}\widehat{\boldsymbol{y}}_h^o, \boldsymbol{v}_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}\\
&\quad + (\nabla\cdot\boldsymbol{y}_h, w_1)_{\mathcal{T}_h} - \langle \boldsymbol{y}_h\cdot\boldsymbol{n}, \widehat{w}_1\rangle_{\partial\mathcal{T}_h} + \langle \mathbb{L}_h\boldsymbol{n} - h^{-1}(\boldsymbol{P}_M\boldsymbol{y}_h - \widehat{\boldsymbol{y}}_h^o), \boldsymbol{\mu}_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}
\end{aligned}
\tag{3.6}
$$

for all $(\mathbb{T}_1, \boldsymbol{v}_1, w_1, \widehat{w}_1, \boldsymbol{\mu}_1) \in \mathbb{K}_h \times \boldsymbol{V}_h \times W_h^0 \times Q_h \times \boldsymbol{M}_h(o)$.

By the definition of $\mathscr{B}$, we can rewrite the HDG formulation (2.4) as follows: find $(\mathbb{L}_h, \boldsymbol{y}_h, p_h, \widehat{p}_h, \widehat{\boldsymbol{y}}_h^o; \mathbb{G}_h, \boldsymbol{z}_h, q_h, \widehat{q}_h, \widehat{\boldsymbol{z}}_h^o) \in [\mathbb{K}_h \times \boldsymbol{V}_h \times W_h^0 \times Q_h \times \boldsymbol{M}_h(o)]^2$ and $u_h \in M_h$ such that

$$
\mathscr{B}(\mathbb{L}_h, \boldsymbol{y}_h, p_h, \widehat{p}_h, \widehat{\boldsymbol{y}}_h^o; \mathbb{T}_1, \boldsymbol{v}_1, w_1, \widehat{w}_1, \boldsymbol{\mu}_1) = \langle u_h\boldsymbol{\tau}, \mathbb{T}_1\boldsymbol{n} + h^{-1}\boldsymbol{v}_1\rangle_{\mathcal{E}_h^\partial} + (\boldsymbol{f}, \boldsymbol{v}_1)_{\mathcal{T}_h}, \tag{3.7a}
$$

$$
\mathscr{B}(\mathbb{G}_h, \boldsymbol{z}_h, -q_h, -\widehat{q}_h, \widehat{\boldsymbol{z}}_h^o; \mathbb{T}_2, \boldsymbol{v}_2, w_2, \widehat{w}_2, \boldsymbol{\mu}_2) = (\boldsymbol{y}_h - \boldsymbol{y}_d, \boldsymbol{v}_2)_{\mathcal{T}_h}, \tag{3.7b}
$$

$$
\langle \mathbb{G}_h\boldsymbol{n} - h^{-1}\boldsymbol{P}_M\boldsymbol{z}_h, \mu_3\boldsymbol{\tau}\rangle_{\mathcal{E}_h^\partial} = \gamma\langle u_h, \mu_3\rangle_{\mathcal{E}_h^\partial} \tag{3.7c}
$$

for all $(\mathbb{T}_1, \boldsymbol{v}_1, w_1, \widehat{w}_1, \boldsymbol{\mu}_1; \mathbb{T}_2, \boldsymbol{v}_2, w_2, \widehat{w}_2, \boldsymbol{\mu}_2) \in [\mathbb{K}_h \times \boldsymbol{V}_h \times W_h^0 \times Q_h \times \boldsymbol{M}_h(o)]^2$ and $\mu_3 \in M_h$.

Lemma 3.5. *For any* $(\mathbb{L}_h, \boldsymbol{y}_h, p_h, \widehat{p}_h, \widehat{\boldsymbol{y}}_h^o) \in \mathbb{K}_h \times \boldsymbol{V}_h \times W_h \times Q_h \times \boldsymbol{M}_h(o)$,

$$
\begin{aligned}
&\mathscr{B}(\mathbb{L}_h, \boldsymbol{y}_h, p_h, \widehat{p}_h, \widehat{\boldsymbol{y}}_h^o; \mathbb{L}_h, \boldsymbol{y}_h, p_h, \widehat{p}_h, \widehat{\boldsymbol{y}}_h^o)\\
&= \|\mathbb{L}_h\|_{\mathcal{T}_h}^2 + h^{-1}\|\boldsymbol{P}_M\boldsymbol{y}_h - \widehat{\boldsymbol{y}}_h^o\|_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}^2 + h^{-1}\|\boldsymbol{P}_M\boldsymbol{y}_h\|_{\mathcal{E}_h^\partial}^2.
\end{aligned}
\tag{3.8}
$$

Proof. According to the definition of $\mathscr{B}$ in (3.6) and integration by parts, we get

$$
\begin{aligned}
&\mathscr{B}(\mathbb{L}_h, \boldsymbol{y}_h, p_h, \widehat{p}_h, \widehat{\boldsymbol{y}}_h^o; \mathbb{L}_h, \boldsymbol{y}_h, p_h, \widehat{p}_h, \widehat{\boldsymbol{y}}_h^o)\\
&= (\mathbb{L}_h, \mathbb{L}_h)_{\mathcal{T}_h} + (\boldsymbol{y}_h, \nabla\cdot\mathbb{L}_h)_{\mathcal{T}_h} - \left\langle \widehat{\boldsymbol{y}}_h^o, \mathbb{L}_h\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} - (\nabla\cdot\mathbb{L}_h, \boldsymbol{y}_h)_{\mathcal{T}_h}\\
&\quad - (p_h, \nabla\cdot\boldsymbol{y}_h)_{\mathcal{T}_h} + \langle \widehat{p}_h, \boldsymbol{y}_h\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + \left\langle h^{-1}(\boldsymbol{P}_M\boldsymbol{y}_h - \widehat{\boldsymbol{y}}_h^o), \boldsymbol{y}_h\right\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}\\
&\quad + \langle h^{-1}\boldsymbol{P}_M\boldsymbol{y}_h, \boldsymbol{y}_h\rangle_{\mathcal{E}_h^\partial} + (\nabla\cdot\boldsymbol{y}_h, p_h)_{\mathcal{T}_h} - \langle \boldsymbol{y}_h\cdot\boldsymbol{n}, \widehat{p}_h\rangle_{\partial\mathcal{T}_h}\\
&\quad + \langle \mathbb{L}_h\boldsymbol{n} - h^{-1}(\boldsymbol{P}_M\boldsymbol{y}_h - \widehat{\boldsymbol{y}}_h^o), \widehat{\boldsymbol{y}}_h^o\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}\\
&= \|\mathbb{L}_h\|_{\mathcal{T}_h}^2 + h^{-1}\|\boldsymbol{P}_M\boldsymbol{y}_h - \widehat{\boldsymbol{y}}_h^o\|_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}^2 + h^{-1}\|\boldsymbol{P}_M\boldsymbol{y}_h\|_{\mathcal{E}_h^\partial}^2.
\end{aligned}
$$

Similarly, for any $(\mathbb{G}_h, \boldsymbol{z}_h, q_h, \widehat{q}_h, \widehat{\boldsymbol{z}}_h^o) \in \mathbb{K}_h \times \boldsymbol{V}_h \times W_h \times Q_h \times \boldsymbol{M}_h(o)$, we have

$$
\begin{aligned}
&\mathscr{B}(\mathbb{G}_h, \boldsymbol{z}_h, -q_h, -\widehat{q}_h, \widehat{\boldsymbol{z}}_h^o; \mathbb{G}_h, \boldsymbol{z}_h, -q_h, -\widehat{q}_h, \widehat{\boldsymbol{z}}_h^o)\\
&= \|\mathbb{G}_h\|_{\mathcal{T}_h}^2 + h^{-1}\|\boldsymbol{P}_M\boldsymbol{z}_h - \widehat{\boldsymbol{z}}_h^o\|_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}^2 + h^{-1}\|\boldsymbol{P}_M\boldsymbol{z}_h\|_{\mathcal{E}_h^\partial}^2.
\end{aligned}
\tag{3.9}
$$

Next we give a property of $\mathscr{B}$ that is critically important to our error analysis of this method.

Lemma 3.6. *For any* $(\mathbb{L}_h, \boldsymbol{y}_h, p_h, \widehat{p}_h, \widehat{\boldsymbol{y}}_h^o; \mathbb{G}_h, \boldsymbol{z}_h, q_h, \widehat{q}_h, \widehat{\boldsymbol{z}}_h^o) \in [\mathbb{K}_h \times \boldsymbol{V}_h \times W_h \times Q_h \times \boldsymbol{M}_h(o)]^2$,

$$
\mathscr{B}(\mathbb{L}_h, \boldsymbol{y}_h, p_h, \widehat{p}_h, \widehat{\boldsymbol{y}}_h^o; -\mathbb{G}_h, \boldsymbol{z}_h, q_h, \widehat{q}_h, \widehat{\boldsymbol{z}}_h^o) = \mathscr{B}(\mathbb{G}_h, \boldsymbol{z}_h, -q_h, -\widehat{q}_h, \widehat{\boldsymbol{z}}_h^o; -\mathbb{L}_h, \boldsymbol{y}_h, p_h, \widehat{p}_h, \widehat{\boldsymbol{y}}_h^o).
$$

Proof. By the definition of $\mathscr{B}$ in (3.6) we have

$$
\begin{aligned}
&\mathscr{B}(\mathbb{L}_h, \boldsymbol{y}_h, p_h, \widehat{p}_h, \widehat{\boldsymbol{y}}_h^o; -\mathbb{G}_h, \boldsymbol{z}_h, q_h, \widehat{q}_h, \widehat{\boldsymbol{z}}_h^o)\\
&= -(\mathbb{L}_h, \mathbb{G}_h)_{\mathcal{T}_h} - (\boldsymbol{y}_h, \nabla\cdot\mathbb{G}_h)_{\mathcal{T}_h} + \left\langle \widehat{\boldsymbol{y}}_h^o, \mathbb{G}_h\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} - (\nabla\cdot\mathbb{L}_h, \boldsymbol{z}_h)_{\mathcal{T}_h}\\
&\quad - (p_h, \nabla\cdot\boldsymbol{z}_h)_{\mathcal{T}_h} + \langle \widehat{p}_h, \boldsymbol{z}_h\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + \langle h^{-1}P_{\boldsymbol{M}}\boldsymbol{y}_h, \boldsymbol{z}_h\rangle_{\partial\mathcal{T}_h} - \langle h^{-1}\widehat{\boldsymbol{y}}_h^o, \boldsymbol{z}_h\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}\\
&\quad + (\nabla\cdot\boldsymbol{y}_h, q_h)_{\mathcal{T}_h} - \langle \boldsymbol{y}_h\cdot\boldsymbol{n}, \widehat{q}_h\rangle_{\partial\mathcal{T}_h} + \langle \mathbb{L}_h\boldsymbol{n} - h^{-1}(\boldsymbol{P}_M\boldsymbol{y}_h - \widehat{\boldsymbol{y}}_h^o), \widehat{\boldsymbol{z}}_h^o\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}.
\end{aligned}
$$

Rearrange the terms above to get

$$
\begin{aligned}
&\mathscr{B}(\mathbb{L}_h, \boldsymbol{y}_h, p_h, \widehat{p}_h, \widehat{\boldsymbol{y}}_h^o; -\mathbb{G}_h, \boldsymbol{z}_h, q_h, \widehat{q}_h, \widehat{\boldsymbol{z}}_h^o)\\
&= -(\mathbb{G}_h, \mathbb{L}_h)_{\mathcal{T}_h} - (\boldsymbol{z}_h, \nabla\cdot\mathbb{L}_h)_{\mathcal{T}_h} + \left\langle \widehat{\boldsymbol{z}}_h^o, \mathbb{L}_h\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h\setminus\mathcal{E}_h^\partial} - (\nabla\cdot\mathbb{G}_h, \boldsymbol{y}_h)_{\mathcal{T}_h}\\
&\quad + (q_h, \nabla\cdot\boldsymbol{y}_h)_{\mathcal{T}_h} - \langle \widehat{q}_h, \boldsymbol{y}_h\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + \langle h^{-1}\boldsymbol{P}_{\boldsymbol{M}}\boldsymbol{z}_h, \boldsymbol{y}_h\rangle_{\partial\mathcal{T}_h} - \langle h^{-1}\widehat{\boldsymbol{z}}_h^o, \boldsymbol{y}_h\rangle_{\partial\mathcal{T}_h\setminus\mathcal{E}_h^\partial}\\
&\quad - (\nabla\cdot\boldsymbol{z}_h, p_h)_{\mathcal{T}_h} + \langle \boldsymbol{z}_h\cdot\boldsymbol{n}, \widehat{p}_h\rangle_{\partial\mathcal{T}_h} + \langle \mathbb{G}_h\boldsymbol{n} - h^{-1}(\boldsymbol{P}_{\boldsymbol{M}}\boldsymbol{z}_h - \widehat{\boldsymbol{z}}_h^o), \widehat{\boldsymbol{y}}_h^o\rangle_{\partial\mathcal{T}_h\setminus\mathcal{E}_h^\partial}\\
&= \mathscr{B}(\mathbb{G}_h, \boldsymbol{z}_h, -q_h, -\widehat{q}_h, \widehat{\boldsymbol{z}}_h^o; -\mathbb{L}_h, \boldsymbol{y}_h, p_h, \widehat{p}_h, \widehat{\boldsymbol{y}}_h^o),
\end{aligned}
$$

where we used the fact that $\boldsymbol{z}_h \in \boldsymbol{H}(\mathrm{div};\Omega)$ and $\nabla\cdot\boldsymbol{z}_h = 0$ in Proposition 2.3.

To prove the uniqueness of solution of the HDG formulation, we need to recall the following BDM projection.

Lemma 3.7 *([27, Equation (2.3)]). For any $K \in \mathcal{T}_h$ and $\boldsymbol{v} \in \left[H^1(K)\right]^2$, there exists a unique $\Pi^{\mathrm{BDM}}\boldsymbol{v} \in [\mathcal{P}^{k+1}(K)]^2$ such that*

$$
\begin{aligned}
\left\langle \Pi^{\mathrm{BDM}}\boldsymbol{v}\cdot\boldsymbol{n}_e, w_{k+1}\right\rangle_e &= \langle \boldsymbol{v}\cdot\boldsymbol{n}_e, w_{k+1}\rangle_e && \forall w_{k+1}\in\mathcal{P}^{k+1}(e), e\in\partial K, && (3.10a)\\
\left(\Pi^{\mathrm{BDM}}\boldsymbol{v}, \nabla p_k\right)_K &= (\boldsymbol{v}, \nabla p_k)_K && \forall p_k\in\mathcal{P}^k(K), && (3.10b)\\
\left(\Pi^{\mathrm{BDM}}\boldsymbol{v}, \mathrm{curl}\,(b_K p_{k-1})\right)_K &= (\boldsymbol{v}, \mathrm{curl}\,(b_K p_{k-1}))_K\,, && \forall p_{k-1}\in\mathcal{P}^{k-1}(K), && (3.10c)
\end{aligned}
$$

where $b_K = \lambda_1\lambda_2\lambda_3$ is a "bubble" function and $\mathrm{curl}\phi = [\partial_y\phi, -\partial_x\phi]^\top$. If $k = 0$, then (3.10c) is vacuous and Π^{BDM} is defined by (3.10a) and *(3.10b).*

Remark 3.8. In [27, Lemma 2.1], Brezzi, Douglas and Marini proved that the system (3.10) determines Π^{BDM} uniquely. In other words, the matrix formed from the left hand side of (3.10) is non-singular. Hence, for any $z_1 \in H^1(e)$, $\boldsymbol{z}_2, \boldsymbol{z}_3 \in \left[H^1(K)\right]^2$, we can uniquely determine $\boldsymbol{v}_h \in [\mathcal{P}^{k+1}(K)]^2$ such that

$$
\begin{aligned}
\langle \boldsymbol{v}_h\cdot\boldsymbol{n}_e, w_{k+1}\rangle_e &= \langle z_1, w_{k+1}\rangle_e && \forall w_{k+1}\in\mathcal{P}^{k+1}(e), e\in\partial K, && (3.11a)\\
(\boldsymbol{v}_h, \nabla p_k)_K &= (\boldsymbol{z}_2, \nabla p_k)_K && \forall p_k\in\mathcal{P}^k(K), && (3.11b)\\
(\boldsymbol{v}_h, \mathrm{curl}\,(b_K p_{k-1}))_K &= (\boldsymbol{z}_3, \mathrm{curl}\,(b_K p_{k-1}))_K\,, && \forall p_{k-1}\in\mathcal{P}^{k-1}(K). && (3.11c)
\end{aligned}
$$

Theorem 3.9. *There exists a unique solution of the HDG discrete optimality system (3.7).*

Proof. Since the system (3.7) is finite dimensional, we only need to prove the uniqueness. Therefore, we assume $\boldsymbol{y}_d = \boldsymbol{f} = 0$ and we show the system (3.7) only has the trivial solution.

First, take $(\mathbb{T}_1, \boldsymbol{v}_1, w_1, \widehat{w}_1, \boldsymbol{\mu}_1) = (-\mathbb{G}_h, \boldsymbol{z}_h, -q_h, -\widehat{q}_h, \widehat{\boldsymbol{z}}_h^o)$, $(\mathbb{T}_2, \boldsymbol{v}_2, w_2, \widehat{w}_2, \boldsymbol{\mu}_2) = (-\mathbb{L}_h, \boldsymbol{y}_h, p_h, \widehat{p}_h, \widehat{\boldsymbol{y}}_h^o)$, and $\mu_3 = -u_h$ in (3.7), respectively. By Lemma 3.6 we have

$$
\begin{aligned}
&\mathscr{B}(\mathbb{L}_h, \boldsymbol{y}_h, p_h, \widehat{p}_h, \widehat{\boldsymbol{y}}_h^o; -\mathbb{G}_h, \boldsymbol{z}_h, -q_h, -\widehat{q}_h, \widehat{\boldsymbol{z}}_h^o) - \mathscr{B}(\mathbb{G}_h, \boldsymbol{z}_h, q_h, \widehat{q}_h, \widehat{\boldsymbol{z}}_h^o; -\mathbb{L}_h, \boldsymbol{y}_h, p_h, \widehat{p}_h, \widehat{\boldsymbol{y}}_h^o)\\
&= -(\boldsymbol{y}_h, \boldsymbol{y}_h)_{\mathcal{T}_h} - \gamma\langle u_h, u_h\rangle_{\mathcal{E}_h^\partial}\\
&= 0.
\end{aligned}
$$

This implies $\boldsymbol{y}_h = u_h = 0$ since $\gamma > 0$.

Next, taking $(\mathbb{T}_1, \boldsymbol{v}_1, w_1, \widehat{w}_1, \boldsymbol{\mu}_1) = (\mathbb{L}_h, \boldsymbol{y}_h, p_h, \widehat{p}_h, \widehat{\boldsymbol{y}}_h^o)$ in (3.7a) and $(\mathbb{T}_2, \boldsymbol{v}_2, w_2, \widehat{w}_2, \boldsymbol{\mu}_2) = (\mathbb{G}_h, \boldsymbol{z}_h, q_h, \widehat{q}_h, \widehat{\boldsymbol{z}}_h^o)$ in (3.7b) and using Lemma 3.5, we obtain $\mathbb{L}_h = \mathbb{G}_h = 0$, $\widehat{\boldsymbol{y}}_h^o = \widehat{\boldsymbol{z}}_h^o = 0$.

Next, taking $(\mathbb{T}_1, w_1, \widehat{w}_1, \boldsymbol{\mu}_1) = (0, 0, 0, 0)$ and $(\mathbb{T}_2, \boldsymbol{v}_2, w_2, \widehat{w}_2, \boldsymbol{\mu}_2) = (0, 0, 0, 0, 0)$ and applying integration by parts gives

$$
(\nabla p_h, \boldsymbol{v}_1)_{\mathcal{T}_h} + \langle \widehat{p}_h - p_h, \boldsymbol{v}_1\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0. \tag{3.12}
$$

Next, set $z_1 = \widehat{p}_h - p_h$ in (3.11a), $\boldsymbol{z}_2 = \boldsymbol{0}$ in (3.11b), and $\boldsymbol{z}_3 = \boldsymbol{0}$ in (3.11c). Then there exists a unique $\boldsymbol{v}_1 \in [\mathcal{P}^{k+1}(K)]^2$ such that on each element K we have

$$
\begin{aligned}
\langle \boldsymbol{v}_1\cdot\boldsymbol{n}_e, w_{k+1}\rangle_e &= \langle \widehat{p}_h - p_h, w_{k+1}\rangle_e && \forall w_{k+1}\in\mathcal{P}^{k+1}(e), e\in\partial K,\\
(\boldsymbol{v}_1, \nabla p_k)_K &= 0 && \forall p_k\in\mathcal{P}^k(K).
\end{aligned}
$$

This implies that $(\boldsymbol{v}_1, \nabla p_h)_K = 0$ and $\boldsymbol{v}_1\cdot\boldsymbol{n} = \widehat{p}_h - p_h$ on ∂K. This gives $\widehat{p}_h = p_h$.

Finally, taking $\boldsymbol{v}_1 = \nabla p_h$ in (3.12) we have $\nabla p_h = 0$, which together with the fact that $\widehat{p}_h$ is single-valued on each edge implies p_h is a constant on the whole domain. Moreover, $p_h \in L_0^2(\Omega)$ gives $\widehat{p}_h = p_h = 0$. Following the same idea gives $\widehat{q}_h = q_h = 0$.

3.2. Proof of Theorem 3.1

We follow the strategy of our earlier work [10] and split the proof into eight steps. Consider the following auxiliary problem: find $(\mathbb{L}_h(u), \boldsymbol{y}_h(u), p_h(u), \widehat{p}_h(u), \widehat{\boldsymbol{y}}_h^o(u);$
$\mathbb{G}_h(u), z_h(u), q_h(u), \widehat{q}_h(u), \widehat{z}_h^o(u)) \in [\mathbb{K}_h \times \boldsymbol{V}_h \times W_h^0 \times Q_h \times \boldsymbol{M}_h(o)]^2$ such that

$$\mathscr{B}(\mathbb{L}_h(u), \boldsymbol{y}_h(u), p_h(u), \widehat{p}_h(u), \widehat{\boldsymbol{y}}_h^o(u); \mathbb{T}_1, \boldsymbol{v}_1, w_1, \widehat{w}_1, \boldsymbol{\mu}_1) = \langle (P_M u)\boldsymbol{\tau}, h^{-1}\boldsymbol{v}_1 + \mathbb{T}_1\boldsymbol{n}\rangle_{\mathcal{E}_h^\partial} + (\boldsymbol{f}, \boldsymbol{v}_1)_{\mathcal{T}_h}, \tag{3.13a}$$

$$\mathscr{B}(\mathbb{G}_h(u), z_h(u), -q_h(u), -\widehat{q}_h(u), \widehat{z}_h^o(u); \mathbb{T}_2, \boldsymbol{v}_2, w_2, \widehat{w}_2, \boldsymbol{\mu}_2) = (\boldsymbol{y}_h(u) - \boldsymbol{y}_d, \boldsymbol{v}_2)_{\mathcal{T}_h} \tag{3.13b}$$

for all $(\mathbb{T}_1, \boldsymbol{v}_1, w_1, \widehat{w}_1, \boldsymbol{\mu}_1; \mathbb{T}_2, \boldsymbol{v}_2, w_2, \widehat{w}_2, \boldsymbol{\mu}_2) \in [\mathbb{K}_h \times \boldsymbol{V}_h \times W_h^0 \times Q_h \times \boldsymbol{M}_h(o)]^2$.

We also note that although the proof strategy is very similar to [10], a simple rewriting of the proofs for the settings of this paper is not enough. For each of the following lemmas, we must take care of the spaces of velocity and pressure so that estimates are independent of the pressure.

We begin by bounding the error between the solutions of the auxiliary problem and the mixed form (2.2a)–(2.2g) of the optimality system. Define

$$\begin{aligned}
\delta^{\mathbb{L}} &= \mathbb{L} - \boldsymbol{\Pi}_{\mathbb{K}}\mathbb{L}, & \varepsilon_h^{\mathbb{L}} &= \boldsymbol{\Pi}_{\mathbb{K}}\mathbb{L} - \mathbb{L}_h(u),\\
\delta^{\boldsymbol{y}} &= \boldsymbol{y} - \Pi^{\mathrm{RT}}\boldsymbol{y}, & \varepsilon_h^{\boldsymbol{y}} &= \Pi^{\mathrm{RT}}\boldsymbol{y} - \boldsymbol{y}_h(u),\\
\delta^{p} &= p - \Pi_W p, & \varepsilon_h^{p} &= \Pi_W p - p_h(u),\\
\delta^{\widehat{p}} &= p - P_Q p, & \varepsilon_h^{\widehat{p}} &= P_Q p - \widehat{p}_h(u),\\
\delta^{\widehat{\boldsymbol{y}}} &= \boldsymbol{y} - \boldsymbol{P}_M\boldsymbol{y}, & \varepsilon_h^{\widehat{\boldsymbol{y}}} &= \boldsymbol{P}_M\boldsymbol{y} - \widehat{\boldsymbol{y}}_h(u),
\end{aligned} \tag{3.14}$$

where $\widehat{\boldsymbol{y}}_h(u) = \widehat{\boldsymbol{y}}_h^o(u)$ on $\mathcal{E}_h^o$ and $\widehat{\boldsymbol{y}}_h(u) = (P_M u)\boldsymbol{\tau}$ on $\mathcal{E}_h^\partial$, then $\varepsilon_h^{\widehat{\boldsymbol{y}}} = \boldsymbol{0}$ on $\mathcal{E}_h^\partial$.

Step 1: The error equation for part 1 of the auxiliary problem (3.13a)

Lemma 3.10. *Let $(\mathbb{L}, \boldsymbol{y}, p)$ be the solution of the optimality system (1.3). Then we have for all $(\mathbb{T}_1, \boldsymbol{v}_1, w_1, \widehat{w}_1, \boldsymbol{\mu}_1) \in \mathbb{K}_h \times \boldsymbol{V}_h \times W_h^0 \times Q_h \times \boldsymbol{M}_h(o)$ that*

$$\begin{aligned}
&\mathscr{B}(\boldsymbol{\Pi}_{\mathbb{K}}\mathbb{L}, \Pi^{\mathrm{RT}}\boldsymbol{y}, \Pi_W p, P_Q p, \boldsymbol{P}_M\boldsymbol{y}; \mathbb{T}_1, \boldsymbol{v}_1, w_1, \widehat{w}_1, \boldsymbol{\mu}_1)\\
&= (\boldsymbol{f}, \boldsymbol{v}_1)_{\mathcal{T}_h} + \langle (P_M u)\boldsymbol{\tau}, \mathbb{T}_1\boldsymbol{n} + h^{-1}\boldsymbol{v}_1\rangle_{\mathcal{E}_h^\partial} - \langle h^{-1}\boldsymbol{P}_M\delta^{\boldsymbol{y}}, \boldsymbol{v}_1\rangle_{\partial\mathcal{T}_h}\\
&\quad + \langle \delta^{\mathbb{L}}\boldsymbol{n}, \boldsymbol{v}_1\rangle_{\partial\mathcal{T}_h} - \langle \delta^{\mathbb{L}}\boldsymbol{n}, \boldsymbol{\mu}_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} + \langle h^{-1}\boldsymbol{P}_M\delta^{\boldsymbol{y}}, \boldsymbol{\mu}_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}.
\end{aligned}$$

Proof. Since $\nabla\cdot\boldsymbol{y} = 0$, by Lemma 3.4 we have $\Pi^{\mathrm{RT}}\boldsymbol{y} \in \boldsymbol{V}_h$. By the definition of the operator $\mathscr{B}$ in (3.6) we obtain

$$\begin{aligned}
&\mathscr{B}(\boldsymbol{\Pi}_{\mathbb{K}}\mathbb{L}, \Pi^{\mathrm{RT}}\boldsymbol{y}, \Pi_W p, P_Q p, \boldsymbol{P}_M\boldsymbol{y}; \mathbb{T}_1, \boldsymbol{v}_1, w_1, \widehat{w}_1, \boldsymbol{\mu}_1)\\
&= (\boldsymbol{\Pi}_{\mathbb{K}}\mathbb{L}, \mathbb{T}_1)_{\mathcal{T}_h} + (\Pi^{\mathrm{RT}}\boldsymbol{y}, \nabla\cdot\mathbb{T}_1)_{\mathcal{T}_h} - \langle \boldsymbol{P}_M\boldsymbol{y}, \mathbb{T}_1\boldsymbol{n}\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} - (\nabla\cdot\Pi_{\mathbb{K}}\mathbb{L}, \boldsymbol{v}_1)_{\mathcal{T}_h}\\
&\quad - (\Pi_W p, \nabla\cdot\boldsymbol{v}_1)_{\mathcal{T}_h} + \langle P_Q p, \boldsymbol{v}_1\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + \langle h^{-1}\boldsymbol{P}_M\Pi^{\mathrm{RT}}\boldsymbol{y}, \boldsymbol{v}_1\rangle_{\partial\mathcal{T}_h}\\
&\quad - \langle h^{-1}\boldsymbol{P}_M\boldsymbol{y}, \boldsymbol{v}_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} + (\nabla\cdot\Pi^{\mathrm{RT}}\boldsymbol{y}, w_1)_{\mathcal{T}_h} - \langle \Pi^{\mathrm{RT}}\boldsymbol{y}\cdot\boldsymbol{n}, \widehat{w}_1\rangle_{\partial\mathcal{T}_h}\\
&\quad + \langle \Pi_{\mathbb{K}}\mathbb{L}\boldsymbol{n} - h^{-1}(\boldsymbol{P}_M\Pi^{\mathrm{RT}}\boldsymbol{y} - \boldsymbol{P}_M\boldsymbol{y}), \boldsymbol{\mu}_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}.
\end{aligned}$$

By definition of the L^2 projections and the RT projection, we have

$$\begin{aligned}
&\mathscr{B}(\boldsymbol{\Pi}_{\mathbb{K}}\mathbb{L}, \Pi^{\mathrm{RT}}\boldsymbol{y}, \Pi_W p, P_M p, \boldsymbol{P}_M\boldsymbol{y}; \mathbb{T}_1, \boldsymbol{v}_1, w_1, \widehat{w}_1, \boldsymbol{\mu}_1)\\
&= (\mathbb{L}, \mathbb{T}_1)_{\mathcal{T}_h} + (\boldsymbol{y}, \nabla\cdot\mathbb{T}_1)_{\mathcal{T}_h} - \langle \boldsymbol{y}, \mathbb{T}_1\boldsymbol{n}\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} + (\nabla\cdot\delta^{\mathbb{L}}, \boldsymbol{v}_1)_{\mathcal{T}_h}
\end{aligned}$$

$$
\begin{aligned}
&- (\nabla\cdot\mathbb{L}, \boldsymbol{v}_1)_{\mathcal{T}_h} - (p, \nabla\cdot\boldsymbol{v}_1)_{\mathcal{T}_h} + \langle p, \boldsymbol{v}_1\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + \langle h^{-1}\boldsymbol{P}_M\boldsymbol{y}, \boldsymbol{v}_1\rangle_{\partial\mathcal{T}_h}\\
&- \langle h^{-1}\boldsymbol{P}_M\delta^{\boldsymbol{y}}, \boldsymbol{v}_1\rangle_{\partial\mathcal{T}_h} - \langle h^{-1}\boldsymbol{P}_M\boldsymbol{y}, \boldsymbol{v}_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} + (\nabla\cdot\Pi^{\mathrm{RT}}\boldsymbol{y}, w_1)_{\mathcal{T}_h}\\
&- \langle \boldsymbol{y}\cdot\boldsymbol{n}, \widehat{w}_1\rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{\Pi}_{\mathbb{K}}\mathbb{L}\boldsymbol{n}, \boldsymbol{\mu}_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} + \langle h^{-1}\boldsymbol{P}_M\delta^{\boldsymbol{y}}, \boldsymbol{\mu}_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}.
\end{aligned}
$$

Moreover, integration by parts gives

$$
\begin{aligned}
(\nabla\cdot\Pi^{\mathrm{RT}}\boldsymbol{y}, w_1)_{\mathcal{T}_h} &= \langle \Pi^{\mathrm{RT}}\boldsymbol{y}\cdot\boldsymbol{n}, w_1\rangle_{\partial\mathcal{T}_h} - (\Pi^{\mathrm{RT}}\boldsymbol{y}, \nabla w_1)_{\mathcal{T}_h}\\
&= \langle \boldsymbol{y}\cdot\boldsymbol{n}, w_1\rangle_{\partial\mathcal{T}_h} - (\boldsymbol{y}, \nabla w_1)_{\mathcal{T}_h}\\
&= (\nabla\cdot\boldsymbol{y}, w_1)_{\mathcal{T}_h}\\
&= 0.
\end{aligned}
$$

Note that the exact solutions $\mathbb{L}$, $\boldsymbol{y}$ and p satisfy

$$
\begin{aligned}
(\mathbb{L}, \mathbb{T}_1)_{\mathcal{T}_h} + (\boldsymbol{y}, \nabla\cdot\mathbb{T}_1)_{\mathcal{T}_h} - \langle \boldsymbol{y}, \mathbb{T}_1\boldsymbol{n}\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} &= \langle u\boldsymbol{\tau}, \mathbb{T}_1\boldsymbol{n}\rangle_{\mathcal{E}_h^\partial},\\
-(\nabla\cdot(\mathbb{L}-p\mathbb{I}), \boldsymbol{v}_1)_{\mathcal{T}_h} &= (\boldsymbol{f}, \boldsymbol{v}_1)_{\mathcal{T}_h},\\
(\nabla\cdot\boldsymbol{y}, w_1)_{\mathcal{T}_h} &= 0,\\
\langle \boldsymbol{y}\cdot\boldsymbol{n}, \widehat{w}_1\rangle_{\partial\mathcal{T}_h} &= 0
\end{aligned}
$$

for all $(\mathbb{T}_1, \boldsymbol{v}_1, w_1, \widehat{w}_1) \in \mathbb{K}_h \times \boldsymbol{V}_h \times W_h^0 \times Q_h$ and $\boldsymbol{y} = u\boldsymbol{\tau}$ on $\mathcal{E}_h^\partial$. Then we have

$$
\begin{aligned}
&\mathscr{B}(\boldsymbol{\Pi}_{\mathbb{K}}\mathbb{L}, \Pi^{\mathrm{RT}}\boldsymbol{y}, \Pi_W p, P_M p, \boldsymbol{P}_M\boldsymbol{y}; \mathbb{T}_1, \boldsymbol{v}_1, w_1, \widehat{w}_1, \boldsymbol{\mu}_1)\\
&= (\boldsymbol{f}, \boldsymbol{v}_1)_{\mathcal{T}_h} + \langle (P_M u)\boldsymbol{\tau}, \mathbb{T}_1\boldsymbol{n} + h^{-1}\boldsymbol{v}_1\rangle_{\mathcal{E}_h^\partial} - \langle h^{-1}\boldsymbol{P}_M\delta^{\boldsymbol{y}}, \boldsymbol{v}_1\rangle_{\partial\mathcal{T}_h}\\
&\quad + (\nabla\cdot\delta^{\mathbb{L}}, \boldsymbol{v}_1)_{\mathcal{T}_h} + \langle \boldsymbol{\Pi}_{\mathbb{K}}\mathbb{L}\boldsymbol{n}, \boldsymbol{\mu}_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} + \langle h^{-1}\boldsymbol{P}_M\delta^{\boldsymbol{y}}, \boldsymbol{\mu}_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}.
\end{aligned}
$$

Since $\mathbb{L} \in \mathbb{H}^{r_{\mathbb{L}}}(\Omega)$ with $r_{\mathbb{L}} > 1/2$, then $\langle \mathbb{L}\boldsymbol{n}, \boldsymbol{\mu}_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} = 0$. This implies

$$
\begin{aligned}
&\mathscr{B}(\boldsymbol{\Pi}_{\mathbb{K}}\mathbb{L}, \Pi^{\mathrm{RT}}\boldsymbol{y}, \Pi_W p, P_M p, \boldsymbol{P}_M\boldsymbol{y}; \mathbb{T}_1, \boldsymbol{v}_1, w_1, \widehat{w}_1, \boldsymbol{\mu}_1)\\
&= (\boldsymbol{f}, \boldsymbol{v}_1)_{\mathcal{T}_h} + \langle (P_M u)\boldsymbol{\tau}, \mathbb{T}_1\boldsymbol{n} + h^{-1}\boldsymbol{v}_1\rangle_{\mathcal{E}_h^\partial} - \langle h^{-1}\boldsymbol{P}_M\delta^{\boldsymbol{y}}, \boldsymbol{v}_1\rangle_{\partial\mathcal{T}_h}\\
&\quad + \langle \delta^{\mathbb{L}}\boldsymbol{n}, \boldsymbol{v}_1\rangle_{\partial\mathcal{T}_h} - \langle \delta^{\mathbb{L}}\boldsymbol{n}, \boldsymbol{\mu}_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial} + \langle h^{-1}\boldsymbol{P}_M\delta^{\boldsymbol{y}}, \boldsymbol{\mu}_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial},
\end{aligned}
$$

where we used the fact that $(\mathbb{L} - \boldsymbol{\Pi}_{\mathbb{K}}\mathbb{L}, \nabla\boldsymbol{v}_1)_{\mathcal{T}_h} = 0$.

Remark 3.11. In [10], we used $\mathbb{L} - p\mathbb{I} \in \mathbb{H}(\mathrm{div}, \Omega)$ when $s_{\mathbb{L}} \leq 1/2$. However, $\mathbb{L} \in \mathbb{H}(\mathrm{div}, \Omega)$ does not hold here. Hence, we assume $r_{\mathbb{L}} > 1/2$ so that $\mathbb{L}$ has a well-defined trace. Improving the analysis to handle the case $s_{\mathbb{L}} \leq 1/2$ is left to be considered elsewhere.

Subtract part 1 of (3.13a) from Lemma 3.10 to obtain the following lemma.

Lemma 3.12. *For all* $(\mathbb{T}_1, \boldsymbol{v}_1, w_1, \widehat{w}_1, \boldsymbol{\mu}_1) \in \mathbb{K}_h \times \boldsymbol{V}_h \times W_h^0 \times Q_h \times \boldsymbol{M}_h(o)$, *we have*

$$
\begin{aligned}
\mathscr{B}(\varepsilon_h^{\mathbb{L}}, \varepsilon_h^{\boldsymbol{y}}, \varepsilon_h^{p}, \varepsilon_h^{\widehat{p}}, \varepsilon_h^{\widehat{\boldsymbol{y}}}; \mathbb{T}_1, \boldsymbol{v}_1, w_1, \widehat{w}_1, \boldsymbol{\mu}_1) = &-\langle h^{-1}\boldsymbol{P}_M\delta^{\boldsymbol{y}}, \boldsymbol{v}_1\rangle_{\partial\mathcal{T}_h} + \langle h^{-1}\boldsymbol{P}_M\delta^{\boldsymbol{y}}, \boldsymbol{\mu}_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}\\
&+ \langle \delta^{\mathbb{L}}\boldsymbol{n}, \boldsymbol{v}_1\rangle_{\partial\mathcal{T}_h} - \langle \delta^{\mathbb{L}}\boldsymbol{n}, \boldsymbol{\mu}_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}.
\end{aligned}
\tag{3.15}
$$

Step 2: Estimate for $\varepsilon_h^{\mathbb{L}}$

We first provide a key inequality which was proven in [10, Lemma 4.7].

Lemma 3.13. *We have*

$$
\|\nabla\varepsilon_h^{\boldsymbol{y}}\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|\varepsilon_h^{\boldsymbol{y}} - \varepsilon_h^{\widehat{\boldsymbol{y}}}\|_{\partial\mathcal{T}_h} \lesssim \|\varepsilon_h^{\mathbb{L}}\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|\boldsymbol{P}_M\varepsilon_h^{\boldsymbol{y}} - \varepsilon_h^{\widehat{\boldsymbol{y}}}\|_{\partial\mathcal{T}_h}. \tag{3.16}
$$

Lemma 3.14. *We have*

$$
\|\varepsilon_h^{\mathbb{L}}\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|\boldsymbol{P}_M\varepsilon_h^{\boldsymbol{y}} - \varepsilon_h^{\widehat{\boldsymbol{y}}}\|_{\partial\mathcal{T}_h} \lesssim h^{s_{\mathbb{L}}}\|\mathbb{L}\|_{s_{\mathbb{L}},\Omega} + h^{s_{\boldsymbol{y}}-1}\|\boldsymbol{y}\|_{s_{\boldsymbol{y}},\Omega}.
$$

Proof. First, since $\varepsilon_h^{\widehat{y}} = 0$ on $\mathcal{E}_h^\partial$, the basic property of $\mathscr{B}$ in Lemma 3.5 gives

$$\mathscr{B}(\varepsilon_h^{\mathbb{L}}, \varepsilon_h^{\boldsymbol{y}}, \varepsilon_h^p, \varepsilon_h^{\widehat{p}}, \varepsilon_h^{\widehat{\boldsymbol{y}}}; \varepsilon_h^{\mathbb{L}}, \varepsilon_h^{\boldsymbol{y}}, \varepsilon_h^p, \varepsilon_h^{\widehat{p}}, \varepsilon_h^{\widehat{\boldsymbol{y}}}) = \|\varepsilon_h^{\mathbb{L}}\|_{\mathcal{T}_h}^2 + h^{-1}\|\boldsymbol{P}_M \varepsilon_h^{\boldsymbol{y}} - \varepsilon_h^{\widehat{\boldsymbol{y}}}\|_{\partial\mathcal{T}_h}^2.$$

On the other hand, taking $(\mathbb{T}_1, \boldsymbol{v}_1, p_1, \widehat{p}_1, \boldsymbol{\mu}_1) = (\varepsilon_h^{\mathbb{L}}, \varepsilon_h^{\boldsymbol{y}}, \varepsilon_h^p, \varepsilon_h^{\widehat{p}}, \varepsilon_h^{\widehat{\boldsymbol{y}}})$ in (3.15) gives

$$\|\varepsilon_h^{\mathbb{L}}\|_{\mathcal{T}_h}^2 + h^{-1}\|\boldsymbol{P}_M \varepsilon_h^{\boldsymbol{y}} - \varepsilon_h^{\widehat{\boldsymbol{y}}}\|_{\partial\mathcal{T}_h}^2 = \langle \delta^{\mathbb{L}}\boldsymbol{n}, \varepsilon_h^{\boldsymbol{y}} - \varepsilon_h^{\widehat{\boldsymbol{y}}}\rangle_{\partial\mathcal{T}_h} - \langle h^{-1}\delta^{\boldsymbol{y}}, \boldsymbol{P}_M \varepsilon_h^{\boldsymbol{y}} - \varepsilon_h^{\widehat{\boldsymbol{y}}}\rangle_{\partial\mathcal{T}_h}.$$

By Lemma 3.13 and Young's inequality, we have

$$\|\varepsilon_h^{\mathbb{L}}\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|\boldsymbol{P}_M \varepsilon_h^{\boldsymbol{y}} - \varepsilon_h^{\widehat{\boldsymbol{y}}}\|_{\partial\mathcal{T}_h} \lesssim h^{s_{\mathbb{L}}}\|\mathbb{L}\|_{s_{\mathbb{L}},\Omega} + h^{s_{\boldsymbol{y}}-1}\|\boldsymbol{y}\|_{s_{\boldsymbol{y}},\Omega}.$$

Step 3: Estimate for ε_h^y by a duality argument

Next, we introduce the dual problem

$$\mathbb{A} - \nabla\boldsymbol{\Phi} = 0 \text{ in } \Omega, \quad -\nabla\cdot\mathbb{A} - \nabla\Psi = \Theta \text{ in } \Omega, \quad \nabla\cdot\boldsymbol{\Phi} = 0 \text{ in } \Omega, \quad \boldsymbol{\Phi} = 0 \text{ on } \partial\Omega. \tag{3.17}$$

Since the domain Ω is convex, we have the following regularity estimate:

$$\|\mathbb{A}\|_{1,\Omega} + \|\boldsymbol{\Phi}\|_{2,\Omega} + \|\Psi\|_{1,\Omega} \le C\|\Theta\|_{0,\Omega}. \tag{3.18}$$

Before we estimate $\varepsilon_h^{\boldsymbol{y}}$, we introduce the following notation, which is similar to the earlier notation in (3.14):

$$\delta^{\mathbb{A}} = \mathbb{A} - \boldsymbol{\Pi}_{\mathbb{K}}\mathbb{A}, \quad \delta^{\boldsymbol{\Phi}} = \boldsymbol{\Phi} - \Pi^{\mathrm{RT}}\boldsymbol{\Phi}, \quad \delta^{\Psi} = \Psi - \Pi_W\Psi, \quad \delta^{\widehat{\Psi}} = \Psi - P_Q\Psi, \quad \delta^{\widehat{\boldsymbol{\Phi}}} = \boldsymbol{\Phi} - \boldsymbol{P}_M\boldsymbol{\Phi}.$$

Since $\boldsymbol{\Phi} = 0$ on $\partial\Omega$, by using Lemma 3.10 we have the following lemma:

Lemma 3.15. *Let $(\mathbb{A}, \boldsymbol{\Phi}, \Psi)$ be the solution of (3.17), then for all $(\mathbb{T}_1, \boldsymbol{v}_1, w_1, \widehat{w}_1, \boldsymbol{\mu}_1) \in \mathbb{K}_h \times \boldsymbol{V}_h \times W_h^0 \times Q_h \times \boldsymbol{M}_h(o)$, we have*

$$\begin{aligned}&\mathscr{B}(\boldsymbol{\Pi}_{\mathbb{K}}\mathbb{A}, \Pi^{\mathrm{RT}}\boldsymbol{\Phi}, \Pi_W\Psi, P_Q\Psi, \boldsymbol{P}_M\boldsymbol{\Phi}; \mathbb{T}_1, \boldsymbol{v}_1, w_1, \widehat{w}_1, \boldsymbol{\mu}_1)\\ &= (\Theta, \boldsymbol{v}_1)_{\mathcal{T}_h} - \langle h^{-1}\boldsymbol{P}_M\delta^{\boldsymbol{\Phi}}, \boldsymbol{v}_1\rangle_{\partial\mathcal{T}_h} + \langle h^{-1}\boldsymbol{P}_M\delta^{\boldsymbol{\Phi}}, \boldsymbol{\mu}_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}\\ &\quad + \langle\delta^{\mathbb{A}}\boldsymbol{n}, \boldsymbol{v}_1\rangle_{\partial\mathcal{T}_h} - \langle\delta^{\mathbb{A}}\boldsymbol{n}, \boldsymbol{\mu}_1\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^\partial}.\end{aligned}$$

Lemma 3.16. *We have*

$$\|\varepsilon_h^{\boldsymbol{y}}\|_{\mathcal{T}_h} \lesssim h^{s_{\mathbb{L}}+1}\|\mathbb{L}\|_{s_{\mathbb{L}},\Omega} + h^{s_{\boldsymbol{y}}}\|\boldsymbol{y}\|_{s_{\boldsymbol{y}},\Omega}. \tag{3.19}$$

Proof. Consider the dual problem (3.17) and let $\Theta = \varepsilon_h^{\boldsymbol{y}}$. Since $\varepsilon_h^{\widehat{\boldsymbol{y}}} = 0$ on $\mathcal{E}_h^\partial$, it follows from Lemmas 3.6 and 3.15 that

$$\begin{aligned}&\mathscr{B}(\varepsilon_h^{\mathbb{L}}, \varepsilon_h^{\boldsymbol{y}}, \varepsilon_h^p, \varepsilon_h^{\widehat{p}}, \varepsilon_h^{\widehat{\boldsymbol{y}}}; -\boldsymbol{\Pi}_{\mathbb{K}}\mathbb{A}, \Pi^{\mathrm{RT}}\boldsymbol{\Phi}, \Pi_W\Psi, P_Q\Psi, \boldsymbol{P}_M\boldsymbol{\Phi})\\ &= \mathscr{B}(\boldsymbol{\Pi}_{\mathbb{K}}\mathbb{A}, \Pi^{\mathrm{RT}}\boldsymbol{\Phi}, -\Pi_W\Psi, -P_Q\Psi, \boldsymbol{P}_M\boldsymbol{\Phi}; -\varepsilon_h^{\mathbb{L}}, \varepsilon_h^{\boldsymbol{y}}, \varepsilon_h^p, \varepsilon_h^{\widehat{p}}, \varepsilon_h^{\widehat{\boldsymbol{y}}})\\ &= \langle\delta^{\mathbb{A}}\boldsymbol{n}, \varepsilon_h^{\boldsymbol{y}} - \varepsilon_h^{\widehat{\boldsymbol{y}}}\rangle_{\partial\mathcal{T}_h} - \langle h^{-1}\delta^{\boldsymbol{\Phi}}, \boldsymbol{P}_M\varepsilon_h^{\boldsymbol{y}} - \varepsilon_h^{\widehat{\boldsymbol{y}}}\rangle_{\partial\mathcal{T}_h} + \|\varepsilon_h^{\boldsymbol{y}}\|_{\mathcal{T}_h}^2.\end{aligned}$$

On the other hand, taking $(\mathbb{T}_1, \boldsymbol{v}_1, w_1, \widehat{w}_1, \boldsymbol{\mu}_1) = (-\boldsymbol{\Pi}_{\mathbb{K}}\mathbb{A}, \Pi^{\mathrm{RT}}\boldsymbol{\Phi}, \Pi_W\Psi, P_Q\Psi, \boldsymbol{P}_M\boldsymbol{\Phi})$ in (3.15) gives

$$\begin{aligned}&\mathscr{B}(\varepsilon_h^{\mathbb{L}}, \varepsilon_h^{\boldsymbol{y}}, \varepsilon_h^p, \varepsilon_h^{\widehat{p}}, \varepsilon_h^{\widehat{\boldsymbol{y}}}; -\boldsymbol{\Pi}_{\mathbb{K}}\mathbb{A}, \Pi^{\mathrm{RT}}\boldsymbol{\Phi}, \Pi_W\Psi, P_Q\Psi, \boldsymbol{P}_M\boldsymbol{\Phi})\\ &= \langle\delta^{\mathbb{L}}\boldsymbol{n}, \Pi^{\mathrm{RT}}\boldsymbol{\Phi} - \boldsymbol{P}_M\boldsymbol{\Phi}\rangle_{\partial\mathcal{T}_h} + \langle h^{-1}\delta^{\boldsymbol{y}}, \boldsymbol{P}_M\delta^{\boldsymbol{\Phi}}\rangle_{\partial\mathcal{T}_h}.\end{aligned}$$

Then we have

$$\begin{aligned}\|\varepsilon_h^{\boldsymbol{y}}\|_{\mathcal{T}_h}^2 &= \langle\delta^{\mathbb{L}}\boldsymbol{n}, \Pi^{\mathrm{RT}}\boldsymbol{\Phi} - \boldsymbol{P}_M\boldsymbol{\Phi}\rangle_{\partial\mathcal{T}_h} - \langle\delta^{\mathbb{A}}\boldsymbol{n}, \varepsilon_h^{\boldsymbol{y}} - \varepsilon_h^{\widehat{\boldsymbol{y}}}\rangle_{\partial\mathcal{T}_h}\\ &\quad + \langle h^{-1}\delta^{\boldsymbol{\Phi}}, \boldsymbol{P}_M\varepsilon_h^{\boldsymbol{y}} - \varepsilon_h^{\widehat{\boldsymbol{y}}}\rangle_{\partial\mathcal{T}_h} + \langle h^{-1}\delta^{\boldsymbol{y}}, \boldsymbol{P}_M\delta^{\boldsymbol{\Phi}}\rangle_{\partial\mathcal{T}_h},\end{aligned}$$

which together with the approximation properties of the L^2-orthogonal projection and the projection Π^{RT} and Lemma 3.14 gives the desired result.

As a consequence of Lemmas 3.14 and 3.16, a simple application of the triangle inequality gives optimal convergence rates for $\|\mathbb{L} - \mathbb{L}_h(u)\|_{\mathcal{T}_h}$ and $\|\boldsymbol{y} - \boldsymbol{y}_h(u)\|_{\mathcal{T}_h}$:

Lemma 3.17. *Let* $(\mathbb{L}, \boldsymbol{y}, p)$ *and* $(\mathbb{L}_h(u), \boldsymbol{y}_h(u), p_h(u))$ *be the solution of* (1.3) *and* (3.13a)*, respectively. We have*

$$\|\mathbb{L}-\mathbb{L}_h(u)\|_{\mathcal{T}_h} \lesssim h^{s_{\mathbb{L}}}\|\mathbb{L}\|_{s_{\mathbb{L}},\Omega}+h^{s_y-1}\|\boldsymbol{y}\|_{s_y,\Omega}, \tag{3.20a}$$

$$\|\boldsymbol{y}-\boldsymbol{y}_h(u)\|_{\mathcal{T}_h} \lesssim h^{s_{\mathbb{L}}+1}\|\mathbb{L}\|_{s_{\mathbb{L}},\Omega}+h^{s_y}\|\boldsymbol{y}\|_{s_y,\Omega}. \tag{3.20b}$$

Step 4: The error equation for part 2 of the auxiliary problem (3.13b)

We continue to bound the error between the solutions of the auxiliary problem and the mixed form (2.2a)–(2.2g) of the optimality system. In steps 4–5, we focus on the dual variables, i.e., $\mathbb{G}$, z and q. We use the following notation

$$\begin{aligned}
\delta^{\mathbb{G}} &= \mathbb{G}-\boldsymbol{\Pi}_{\mathbb{K}}\mathbb{G}, & \varepsilon_h^{\mathbb{G}} &= \boldsymbol{\Pi}_{\mathbb{K}}\mathbb{G}-\mathbb{G}_h(u),\\
\delta^{z} &= \boldsymbol{z}-\Pi^{\mathrm{RT}}\boldsymbol{z}, & \varepsilon_h^{z} &= \Pi^{\mathrm{RT}}\boldsymbol{z}-\boldsymbol{z}_h(u),\\
\delta^{q} &= q-\Pi_W q, & \varepsilon_h^{q} &= \Pi_W q-q_h(u),\\
\delta^{\widehat{q}} &= q-P_Q q, & \varepsilon_h^{\widehat{q}} &= P_Q q-\widehat{q}_h(u),\\
\delta^{\widehat{z}} &= \boldsymbol{z}-\boldsymbol{P}_M\boldsymbol{z}, & \varepsilon_h^{\widehat{z}} &= \boldsymbol{P}_M\boldsymbol{z}-\widehat{\boldsymbol{z}}_h(u).
\end{aligned} \tag{3.21}$$

The derivation of the error equation for part 2 of the auxiliary problem (3.13b) is similar to the analysis for part 1 of the auxiliary problem in step 1. Therefore, we state the result and omit the proof.

Lemma 3.18. *For all* $(\mathbb{T}_2, \boldsymbol{v}_2, w_2, \widehat{w}_2, \boldsymbol{\mu}_2) \in \mathbb{K}_h \times \boldsymbol{V}_h \times W_h^0 \times Q_h \times \boldsymbol{M}_h(o)$*, we have*

$$\begin{aligned}
&\mathscr{B}(\varepsilon_h^{\mathbb{G}}, \varepsilon_h^{z}, -\varepsilon_h^{q}, -\varepsilon_h^{\widehat{q}}, \varepsilon_h^{\widehat{z}}; \mathbb{T}_2, \boldsymbol{v}_2, w_2, \widehat{w}_2, \boldsymbol{\mu}_2)\\
&= -\langle h^{-1}\boldsymbol{P}_M\delta^{z}, \boldsymbol{v}_2\rangle_{\partial\mathcal{T}_h} + \langle h^{-1}\boldsymbol{P}_M\delta^{z}, \boldsymbol{\mu}_2\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^{\partial}}\\
&\quad + \langle \delta^{\mathbb{G}}\boldsymbol{n}, \boldsymbol{v}_2\rangle_{\partial\mathcal{T}_h} - \langle \delta^{\mathbb{G}}\boldsymbol{n}, \boldsymbol{\mu}_2\rangle_{\partial\mathcal{T}_h\backslash\mathcal{E}_h^{\partial}} + (\boldsymbol{y}-\boldsymbol{y}_h(u), \boldsymbol{v}_2)_{\mathcal{T}_h}.
\end{aligned} \tag{3.22}$$

Step 5: Estimate for $\varepsilon_h^{\mathbb{G}}$

Before we estimate $\varepsilon_h^{\mathbb{G}}$, we give the following discrete Poincaré inequality from [28, Proposition A.2].

Lemma 3.19. *We have*

$$\|\varepsilon_h^{z}\|_{\mathcal{T}_h} \le C(\|\nabla\varepsilon_h^{z}\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|\varepsilon_h^{z}-\varepsilon_h^{\widehat{z}}\|_{\partial\mathcal{T}_h}). \tag{3.23}$$

Lemma 3.20. *We have*

$$\begin{aligned}
&\|\varepsilon_h^{\mathbb{G}}\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|\boldsymbol{P}_M\varepsilon_h^{z}-\varepsilon_h^{\widehat{z}}\|_{\partial\mathcal{T}_h}\\
&\lesssim h^{s_{\mathbb{L}}+1}\|\mathbb{L}\|_{s_{\mathbb{L}},\Omega}+h^{s_y}\|\boldsymbol{y}\|_{s_y,\Omega}+h^{s_{\mathbb{G}}}\|\mathbb{G}\|_{s_{\mathbb{G}},\Omega}+h^{s_z-1}\|\boldsymbol{z}\|_{s_z,\Omega},
\end{aligned} \tag{3.24a}$$

$$\|\varepsilon_h^{z}\|_{\mathcal{T}_h} \lesssim h^{s_{\mathbb{L}}+1}\|\mathbb{L}\|_{s_{\mathbb{L}},\Omega}+h^{s_y}\|\boldsymbol{y}\|_{s_y,\Omega}+h^{s_{\mathbb{G}}}\|\mathbb{G}\|_{s_{\mathbb{G}},\Omega}+h^{s_z-1}\|\boldsymbol{z}\|_{s_z,\Omega}. \tag{3.24b}$$

Proof. First, we note the key inequality in Lemma 3.13 is valid with $(\mathbb{L}, \boldsymbol{y}, \hat{\boldsymbol{y}})$ in place of $(\mathbb{G}, z, \hat{z})$. This gives

$$\|\nabla\varepsilon_h^{z}\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|\varepsilon_h^{z}-\varepsilon_h^{\widehat{z}}\|_{\partial\mathcal{T}_h} \lesssim \|\varepsilon_h^{\mathbb{G}}\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|\boldsymbol{P}_M\varepsilon_h^{z}-\varepsilon_h^{\widehat{z}}\|_{\partial\mathcal{T}_h}, \tag{3.25}$$

which we use below. Next, since $\varepsilon_h^{\widehat{z}} = 0$ on $\mathcal{E}_h^{\partial}$, the property of $\mathscr{B}$ in (3.9) gives

$$\mathscr{B}(\varepsilon_h^{\mathbb{G}}, \varepsilon_h^{z}, -\varepsilon_h^{q}, -\varepsilon_h^{\widehat{q}}, \varepsilon_h^{\widehat{z}}; \varepsilon_h^{\mathbb{G}}, \varepsilon_h^{z}, -\varepsilon_h^{q}, -\varepsilon_h^{\widehat{q}}, \varepsilon_h^{\widehat{z}}) = \|\varepsilon_h^{\mathbb{G}}\|_{\mathcal{T}_h}^2 + h^{-1}\|\boldsymbol{P}_M\varepsilon_h^{z}-\varepsilon_h^{\widehat{z}}\|_{\partial\mathcal{T}_h}^2. \tag{3.26}$$

Next, we take $(\mathbb{T}_2, \boldsymbol{v}_2, w_2, \widehat{w}_2, \boldsymbol{\mu}_2) = (\varepsilon_h^{\mathbb{G}}, \varepsilon_h^{z}, -\varepsilon_h^{q}, -\varepsilon_h^{\widehat{q}}, \varepsilon_h^{\widehat{z}})$ in (3.22) gives

$$\begin{aligned}
&\mathscr{B}(\varepsilon_h^{\mathbb{G}}, \varepsilon_h^{z}, -\varepsilon_h^{q}, -\varepsilon_h^{\widehat{q}}, \varepsilon_h^{\widehat{z}}; \varepsilon_h^{\mathbb{G}}, \varepsilon_h^{z}, -\varepsilon_h^{q}, -\varepsilon_h^{\widehat{q}}, \varepsilon_h^{\widehat{z}})\\
&= -\langle \delta^{z}, \boldsymbol{P}_M\varepsilon_h^{z}-\varepsilon_h^{\widehat{z}}\rangle_{\partial\mathcal{T}_h} + \langle \delta^{\mathbb{G}}\boldsymbol{n}, \varepsilon_h^{z}-\varepsilon_h^{\widehat{z}}\rangle_{\partial\mathcal{T}_h} + (\boldsymbol{y}-\boldsymbol{y}_h(u), \varepsilon_h^{z})_{\mathcal{T}_h}.
\end{aligned}$$

The estimate in (3.25), Lemmas 3.17 and 3.19 and Young's inequality give the desired result.

As a consequence of Lemma 3.20 and a simple application of the triangle inequality we obtain the optimal convergence rates for $\|\mathbb{G}-\mathbb{G}_h(u)\|_{\mathcal{T}_h}$ and $\|\boldsymbol{z}-\boldsymbol{z}_h(u)\|_{\mathcal{T}_h}$:

Lemma 3.21. *Let* $(\mathbb{G}, z, q)$ *and* $(\mathbb{G}_h(u), z_h(u), p_h(u))$ *be the solution of* (1.3) *and* (3.13b), *respectively. We have*

$$\|\mathbb{G} - \mathbb{G}_h(u)\|_{\mathcal{T}_h} \lesssim h^{s_{\mathbb{L}}+1} \|\mathbb{L}\|_{s_{\mathbb{L}},\Omega} + h^{s_y} \|\boldsymbol{y}\|_{s_y,\Omega} + h^{s_{\mathbb{G}}} \|\mathbb{G}\|_{s_{\mathbb{G}},\Omega}, \tag{3.27a}$$

$$\|\boldsymbol{z} - \boldsymbol{z}_h(u)\|_{\mathcal{T}_h} \lesssim h^{s_{\mathbb{L}}+1} \|\mathbb{L}\|_{s_{\mathbb{L}},\Omega} + h^{s_y} \|\boldsymbol{y}\|_{s_y,\Omega} + h^{s_{\mathbb{G}}} \|\mathbb{G}\|_{s_{\mathbb{G}},\Omega} + h^{s_z} \|\boldsymbol{z}\|_{s_z,\Omega}. \tag{3.27b}$$

Step 6: Estimates for $\|u - u_h\|_{\mathcal{E}_h^\partial}$ *and* $\|y - y_h\|_{\mathcal{T}_h}$

Next, we bound the error between the solutions of the auxiliary problem and the HDG problem (3.7). We use these error bounds and the error bounds in Lemmas 3.17 and 3.21 to obtain the main results. For the next step, we denote

$$\begin{aligned}
&\zeta_{\mathbb{L}} = \mathbb{L}_h(u) - \mathbb{L}_h, \quad \zeta_{\boldsymbol{y}} = \boldsymbol{y}_h(u) - \boldsymbol{y}_h, \quad \zeta_p = p_h(u) - p_h, \quad \zeta_{\widehat{p}} = \widehat{p}_h(u) - \widehat{p}_h,\\
&\zeta_{\mathbb{G}} = \mathbb{G}_h(u) - \mathbb{G}_h, \quad \zeta_{\boldsymbol{z}} = \boldsymbol{z}_h(u) - \boldsymbol{z}_h, \quad \zeta_q = q_h(u) - q_h, \quad \zeta_{\widehat{q}} = \widehat{q}_h(u) - \widehat{q}_h,
\end{aligned}$$

and

$$\begin{aligned}
&\zeta_{\widehat{\boldsymbol{y}}} = \widehat{\boldsymbol{y}}_h^o(u) - \widehat{\boldsymbol{y}}_h^o \ \text{ on } \ \varepsilon_h^o \ \text{ and } \ \zeta_{\widehat{\boldsymbol{y}}} = P_M u\boldsymbol{\tau} - u_h\boldsymbol{\tau} \ \text{ on } \ \mathcal{E}_h^\partial,\\
&\zeta_{\widehat{\boldsymbol{z}}} = \widehat{\boldsymbol{z}}_h^o(u) - \widehat{\boldsymbol{z}}_h^o \ \text{ on } \ \varepsilon_h^o \ \text{ and } \ \zeta_{\widehat{\boldsymbol{z}}} = 0 \ \text{ on } \ \mathcal{E}_h^\partial.
\end{aligned}$$

Subtracting the auxiliary problem and the HDG problem gives the following error equations

$$\mathscr{B}(\zeta_{\mathbb{L}}, \zeta_{\boldsymbol{y}}, \zeta_p, \zeta_{\widehat{p}}, \zeta_{\widehat{\boldsymbol{y}}}; \mathbb{T}_1, \boldsymbol{v}_1, w_1, \widehat{w}_1, \boldsymbol{\mu}_1) = \langle (P_M u - u_h)\boldsymbol{\tau}, h^{-1}\boldsymbol{v}_1 + \mathbb{T}_1\boldsymbol{n}\rangle_{\mathcal{E}_h^\partial}, \tag{3.28a}$$

$$\mathscr{B}(\zeta_{\mathbb{G}}, \zeta_{\boldsymbol{z}}, -\zeta_q, -\zeta_{\widehat{q}}, \zeta_{\widehat{\boldsymbol{z}}}; \mathbb{T}_2, \boldsymbol{v}_2, w_2, \widehat{w}_2, \boldsymbol{\mu}_2) = (\zeta_{\boldsymbol{y}}, \boldsymbol{v}_2)_{\mathcal{T}_h} \tag{3.28b}$$

for all $(\mathbb{T}_1, \boldsymbol{v}_1, w_1, \widehat{w}_1, \boldsymbol{\mu}_1; \mathbb{T}_2, \boldsymbol{v}_2, w_2, \widehat{w}_2, \boldsymbol{\mu}_2) \in [\mathbb{K}_h \times \boldsymbol{V}_h \times W_h^0 \times Q_h \times \boldsymbol{M}_h(o)]^2$.

Lemma 3.22. *We have*

$$\begin{aligned}
\gamma\|u - u_h\|_{\mathcal{E}_h^\partial}^2 + \|\zeta_{\boldsymbol{y}}\|_{\mathcal{T}_h}^2 = &\langle \gamma u\boldsymbol{\tau} - \mathbb{G}_h(u)\boldsymbol{n} + h^{-1}\boldsymbol{P}_M \boldsymbol{z}_h(u), (u - u_h)\boldsymbol{\tau}\rangle_{\mathcal{E}_h^\partial}\\
&- \langle \gamma u_h\boldsymbol{\tau} - \mathbb{G}_h\boldsymbol{n} + h^{-1}\boldsymbol{P}_M \boldsymbol{z}_h, (u - u_h)\boldsymbol{\tau}\rangle_{\mathcal{E}_h^\partial}.
\end{aligned} \tag{3.29}$$

Proof. First, we have

$$\begin{aligned}
&\langle \gamma u\boldsymbol{\tau} - \mathbb{G}_h(u)\boldsymbol{n} + h^{-1}\boldsymbol{P}_M \boldsymbol{z}_h(u), (u - u_h)\boldsymbol{\tau}\rangle_{\mathcal{E}_h^\partial}\\
&- \langle \gamma u_h\boldsymbol{\tau} - \mathbb{G}_h\boldsymbol{n} + h^{-1}\boldsymbol{P}_M \boldsymbol{z}_h, (u - u_h)\boldsymbol{\tau}\rangle_{\mathcal{E}_h^\partial}\\
&= \gamma\|u - u_h\|_{\mathcal{E}_h^\partial}^2 + \langle -\zeta_{\mathbb{G}}\boldsymbol{n} + h^{-1}\boldsymbol{P}_M \zeta_{\boldsymbol{z}}, (u - u_h)\boldsymbol{\tau}\rangle_{\mathcal{E}_h^\partial}.
\end{aligned}$$

Next, Lemma 3.6 gives

$$\mathscr{B}(\zeta_{\mathbb{L}}, \zeta_{\boldsymbol{y}}, \zeta_p, \zeta_{\widehat{p}}, \zeta_{\widehat{\boldsymbol{y}}}; -\zeta_{\mathbb{G}}, \zeta_{\boldsymbol{z}}, \zeta_q, \zeta_{\widehat{q}}, \zeta_{\widehat{\boldsymbol{z}}}) = \mathscr{B}(\zeta_{\mathbb{G}}, \zeta_{\boldsymbol{z}}, -\zeta_q, -\zeta_{\widehat{q}}, \zeta_{\widehat{\boldsymbol{z}}}; -\zeta_{\mathbb{L}}, \zeta_{\boldsymbol{y}}, \zeta_p, \zeta_{\widehat{p}}, \zeta_{\widehat{\boldsymbol{y}}}).$$

On the other hand, from (3.28a) and (3.28b) we have

$$\begin{aligned}
&\mathscr{B}(\zeta_{\mathbb{L}}, \zeta_{\boldsymbol{y}}, \zeta_p, \zeta_{\widehat{p}}, \zeta_{\widehat{\boldsymbol{y}}}; -\zeta_{\mathbb{G}}, \zeta_{\boldsymbol{z}}, \zeta_q, \zeta_{\widehat{q}}, \zeta_{\widehat{\boldsymbol{z}}}) - \mathscr{B}(\zeta_{\mathbb{G}}, \zeta_{\boldsymbol{z}}, -\zeta_q, -\zeta_{\widehat{q}}, \zeta_{\widehat{\boldsymbol{z}}}; -\zeta_{\mathbb{L}}, \zeta_{\boldsymbol{y}}, \zeta_p, \zeta_{\widehat{p}}, \zeta_{\widehat{\boldsymbol{y}}})\\
&= -(\zeta_{\boldsymbol{y}}, \zeta_{\boldsymbol{y}})_{\mathcal{T}_h} + \langle P_M(u - u_h)\boldsymbol{\tau}, -\zeta_{\mathbb{G}}\boldsymbol{n} + h^{-1}\zeta_{\boldsymbol{z}}\rangle_{\mathcal{E}_h^\partial}\\
&= -(\zeta_{\boldsymbol{y}}, \zeta_{\boldsymbol{y}})_{\mathcal{T}_h} + \langle (u - u_h)\boldsymbol{\tau}, -\zeta_{\mathbb{G}}\boldsymbol{n} + h^{-1}\boldsymbol{P}_M\zeta_{\boldsymbol{z}}\rangle_{\mathcal{E}_h^\partial}.
\end{aligned}$$

Comparing the above two equalities gives

$$(\zeta_{\boldsymbol{y}}, \zeta_{\boldsymbol{y}})_{\mathcal{T}_h} = \langle (u - u_h)\boldsymbol{\tau}, -\zeta_{\mathbb{G}}\boldsymbol{n} + h^{-1}\boldsymbol{P}_M\zeta_{\boldsymbol{z}}\rangle_{\mathcal{E}_h^\partial}.$$

Theorem 3.23. *Let* $(\boldsymbol{y}, u)$ *and* $(\boldsymbol{y}_h, u_h)$ *be the solutions of* (1.3) *and* (3.7), *respectively. We have*

$$\|u - u_h\|_{\mathcal{E}_h^\partial} \lesssim h^{s_{\mathbb{L}}+\frac{1}{2}} \|\mathbb{L}\|_{s_{\mathbb{L}},\Omega} + h^{s_y-\frac{1}{2}} \|\boldsymbol{y}\|_{s_y,\Omega} + h^{s_{\mathbb{G}}-\frac{1}{2}} \|\mathbb{G}\|_{s_{\mathbb{G}},\Omega} + h^{s_z-\frac{3}{2}} \|\boldsymbol{z}\|_{s_z,\Omega}, \tag{3.30a}$$

$$\left\|\boldsymbol{y} - \boldsymbol{y}_h\right\|_{\mathcal{T}_h} \lesssim h^{s_{\mathbb{L}}+\frac{1}{2}} \|\mathbb{L}\|_{s_{\mathbb{L}},\Omega} + h^{s_y-\frac{1}{2}} \|\boldsymbol{y}\|_{s_y,\Omega} + h^{s_{\mathbb{G}}-\frac{1}{2}} \|\mathbb{G}\|_{s_{\mathbb{G}},\Omega} + h^{s_z-\frac{3}{2}} \|\boldsymbol{z}\|_{s_z,\Omega}. \tag{3.30b}$$

Proof. Since $\gamma u \boldsymbol{\tau} - \mathbb{G}\boldsymbol{n} = 0$ on $\mathcal{E}_h^\partial$ and $\gamma u_h \boldsymbol{\tau} - \mathbb{G}_h \boldsymbol{n} + h^{-1}\boldsymbol{P}_M z_h = 0$ on $\mathcal{E}_h^\partial$ we have

$$\begin{aligned}\gamma \|u - u_h\|_{\mathcal{E}_h^\partial}^2 + \left\|\zeta_{\boldsymbol{y}}\right\|_{\mathcal{T}_h}^2 &= \langle \gamma u \boldsymbol{\tau} - \mathbb{G}_h(u)\boldsymbol{n} + h^{-1}\boldsymbol{P}_M \boldsymbol{z}_h(u), (u - u_h)\boldsymbol{\tau}\rangle_{\mathcal{E}_h^\partial} \\ &= \langle (\mathbb{G} - \mathbb{G}_h(u))\boldsymbol{n} + h^{-1}\boldsymbol{P}_M \boldsymbol{z}_h(u), (u - u_h)\boldsymbol{\tau}\rangle_{\mathcal{E}_h^\partial}.\end{aligned}$$

Next, since $\widehat{\boldsymbol{z}}_h(u) = \boldsymbol{z} = \boldsymbol{0}$ on $\mathcal{E}_h^\partial$ we have

$$\begin{aligned}\|\boldsymbol{P}_M \boldsymbol{z}_h(u)\|_{\mathcal{E}_h^\partial} &= \|\boldsymbol{P}_M \boldsymbol{z}_h(u) - \boldsymbol{P}_M \Pi^{\mathrm{RT}}\boldsymbol{z} + \boldsymbol{P}_M \Pi^{\mathrm{RT}}\boldsymbol{z} - \boldsymbol{P}_M \boldsymbol{z} + \boldsymbol{P}_M \boldsymbol{z} - \widehat{\boldsymbol{z}}_h(u)\|_{\mathcal{E}_h^\partial} \\ &\le \|\boldsymbol{P}_M \varepsilon_h^{\boldsymbol{z}} - \varepsilon_h^{\widehat{\boldsymbol{z}}}\|_{\partial\mathcal{T}_h} + \|\Pi^{\mathrm{RT}}\boldsymbol{z} - \boldsymbol{z}\|_{\mathcal{E}_h^\partial}.\end{aligned}$$

This together with Lemma 3.21 gives

$$\begin{aligned}\|u - u_h\|_{\mathcal{E}_h^\partial} + \|\zeta_{\boldsymbol{y}}\|_{\mathcal{T}_h} &\lesssim h^{-\frac{1}{2}} \left\|\varepsilon_h^{\mathbb{G}}\right\|_{\mathcal{T}_h} + h^{s_{\mathbb{G}}-\frac{1}{2}} \|\mathbb{G}\|_{s_{\mathbb{G}},\Omega} \\ &\quad + h^{-1}\|\boldsymbol{P}_M \varepsilon_h^{\boldsymbol{z}} - \varepsilon_h^{\widehat{\boldsymbol{z}}}\|_{\partial\mathcal{T}_h} + h^{-1}\|\Pi^{\mathrm{RT}}\boldsymbol{z} - \boldsymbol{z}\|_{\mathcal{E}_h^\partial}.\end{aligned}$$

By Lemma 3.20 and properties of the L^2 projection, we have

$$\|u - u_h\|_{\mathcal{E}_h^\partial} + \|\zeta_{\boldsymbol{y}}\|_{\mathcal{T}_h} \lesssim h^{s_{\mathbb{L}}+\frac{1}{2}} \|\mathbb{L}\|_{s_{\mathbb{L}},\Omega} + h^{s_{\boldsymbol{y}}-\frac{1}{2}} \|\boldsymbol{y}\|_{s_{\boldsymbol{y}},\Omega} + h^{s_{\mathbb{G}}-\frac{1}{2}} \|\mathbb{G}\|_{s_{\mathbb{G}},\Omega} + h^{s_{\boldsymbol{z}}-\frac{3}{2}} \|\boldsymbol{z}\|_{s_{\boldsymbol{z}},\Omega}.$$

Then, by the triangle inequality and Lemma 3.17 we obtain

$$\|\boldsymbol{y} - \boldsymbol{y}_h\|_{\mathcal{T}_h} \lesssim h^{s_{\mathbb{L}}+\frac{1}{2}} \|\mathbb{L}\|_{s_{\mathbb{L}},\Omega} + h^{s_{\boldsymbol{y}}-\frac{1}{2}} \|\boldsymbol{y}\|_{s_{\boldsymbol{y}},\Omega} + h^{s_{\mathbb{G}}-\frac{1}{2}} \|\mathbb{G}\|_{s_{\mathbb{G}},\Omega} + h^{s_{\boldsymbol{z}}-\frac{3}{2}} \|\boldsymbol{z}\|_{s_{\boldsymbol{z}},\Omega}.$$

Step 7: Estimates for $\|\mathbb{G} - \mathbb{G}_h\|_{\mathcal{T}_h}$ *and* $\|z - z_h\|_{\mathcal{T}_h}$

Lemma 3.24. *We have*

$$\begin{aligned}\|\zeta_{\mathbb{G}}\|_{\mathcal{T}_h} &\lesssim h^{s_{\mathbb{L}}+\frac{1}{2}} \|\mathbb{L}\|_{s_{\mathbb{L}},\Omega} + h^{s_p+\frac{1}{2}} \|p\|_{s_p,\Omega} + h^{s_{\boldsymbol{y}}-\frac{1}{2}} \|\boldsymbol{y}\|_{s_{\boldsymbol{y}},\Omega} \\ &\quad + h^{s_{\mathbb{G}}-\frac{1}{2}} \|\mathbb{G}\|_{s_{\mathbb{G}},\Omega} + h^{s_q-\frac{1}{2}} \|q\|_{s_q,\Omega} + h^{s_{\boldsymbol{z}}-\frac{3}{2}} \|\boldsymbol{z}\|_{s_{\boldsymbol{z}},\Omega}, \qquad (3.31a)\\ \|\zeta_{\boldsymbol{z}}\|_{\mathcal{T}_h} &\lesssim h^{s_{\mathbb{L}}+\frac{1}{2}} \|\mathbb{L}\|_{s_{\mathbb{L}},\Omega} + h^{s_p+\frac{1}{2}} \|p\|_{s_p,\Omega} + h^{s_{\boldsymbol{y}}-\frac{1}{2}} \|\boldsymbol{y}\|_{s_{\boldsymbol{y}},\Omega} \\ &\quad + h^{s_{\mathbb{G}}-\frac{1}{2}} \|\mathbb{G}\|_{s_{\mathbb{G}},\Omega} + h^{s_q-\frac{1}{2}} \|q\|_{s_q,\Omega} + h^{s_{\boldsymbol{z}}-\frac{3}{2}} \|\boldsymbol{z}\|_{s_{\boldsymbol{z}},\Omega}. \qquad (3.31b)\end{aligned}$$

Proof. By Lemma 3.5, the error equation (3.28b), and since $\zeta_{\widehat{\boldsymbol{z}}} = 0$ on $\mathcal{E}_h^\partial$, we have

$$\begin{aligned}&(\zeta_{\mathbb{G}}, \zeta_{\mathbb{G}})_{\mathcal{T}_h} + h^{-1}\|\boldsymbol{P}_M \zeta_{\boldsymbol{z}} - \zeta_{\widehat{\boldsymbol{z}}}\|_{\partial\mathcal{T}_h}^2 \\ &= \mathscr{B}(\zeta_{\mathbb{G}}, \zeta_{\boldsymbol{z}}, -\zeta_q, -\zeta_{\widehat{q}}, \zeta_{\widehat{\boldsymbol{z}}}; \zeta_{\mathbb{G}}, \zeta_{\boldsymbol{z}}, -\zeta_q, -\zeta_{\widehat{q}}, \zeta_{\widehat{\boldsymbol{z}}}) \\ &= (\zeta_{\boldsymbol{y}}, \zeta_{\boldsymbol{z}})_{\mathcal{T}_h} \\ &\le \left\|\zeta_{\boldsymbol{y}}\right\|_{\mathcal{T}_h} \|\zeta_{\boldsymbol{z}}\|_{\mathcal{T}_h} \\ &\lesssim \left\|\zeta_{\boldsymbol{y}}\right\|_{\mathcal{T}_h} (\|\nabla\zeta_{\boldsymbol{z}}\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|\zeta_{\boldsymbol{z}} - \zeta_{\widehat{\boldsymbol{z}}}\|_{\partial\mathcal{T}_h}) \\ &\lesssim \left\|\zeta_{\boldsymbol{y}}\right\|_{\mathcal{T}_h} (\|\zeta_{\mathbb{G}}\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|\boldsymbol{P}_M \zeta_{\boldsymbol{z}} - \zeta_{\widehat{\boldsymbol{z}}}\|_{\partial\mathcal{T}_h}),\end{aligned}$$

where we used the discrete Poincaré inequality in Lemma 3.19 and also (3.16). This implies

$$\begin{aligned}\|\zeta_{\mathbb{G}}\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|\boldsymbol{P}_M \zeta_{\boldsymbol{z}} - \zeta_{\widehat{\boldsymbol{z}}}\|_{\partial\mathcal{T}_h} &\lesssim \left\|\zeta_{\boldsymbol{y}}\right\|_{\mathcal{T}_h} \\ &\lesssim h^{s_{\mathbb{L}}+\frac{1}{2}} \|\mathbb{L}\|_{s_{\mathbb{L}},\Omega} + h^{s_{\boldsymbol{y}}-\frac{1}{2}} \|\boldsymbol{y}\|_{s_{\boldsymbol{y}},\Omega} \\ &\quad + h^{s_{\mathbb{G}}-\frac{1}{2}} \|\mathbb{G}\|_{s_{\mathbb{G}},\Omega} + h^{s_{\boldsymbol{z}}-\frac{3}{2}} \|\boldsymbol{z}\|_{s_{\boldsymbol{z}},\Omega}.\end{aligned}$$

The discrete Poincaré inequality in Lemma 3.19 also gives

$$\begin{aligned}\|\zeta_{\boldsymbol{z}}\|_{\mathcal{T}_h} &\lesssim \|\nabla\zeta_{\boldsymbol{z}}\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|\zeta_{\boldsymbol{z}} - \zeta_{\widehat{\boldsymbol{z}}}\|_{\partial\mathcal{T}_h} \\ &\lesssim \|\zeta_{\mathbb{G}}\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|\boldsymbol{P}_M \zeta_{\boldsymbol{z}} - \zeta_{\widehat{\boldsymbol{z}}}\|_{\partial\mathcal{T}_h} \\ &\lesssim h^{s_{\mathbb{L}}+\frac{1}{2}} \|\mathbb{L}\|_{s_{\mathbb{L}},\Omega} + h^{s_{\boldsymbol{y}}-\frac{1}{2}} \|\boldsymbol{y}\|_{s_{\boldsymbol{y}},\Omega} + h^{s_{\mathbb{G}}-\frac{1}{2}} \|\mathbb{G}\|_{s_{\mathbb{G}},\Omega} + h^{s_{\boldsymbol{z}}-\frac{3}{2}} \|\boldsymbol{z}\|_{s_{\boldsymbol{z}},\Omega}.\end{aligned}$$

The above lemma along with the triangle inequality and Lemmas 3.17 and 3.21 gives the next part of the main result:

Theorem 3.25. *Let* $(\mathbb{G}, \boldsymbol{z})$ *and* $(\mathbb{G}_h, \boldsymbol{z}_h)$ *be the solutions of* (1.3) *and* (3.7)*, respectively. We have*

$$\|\mathbb{G}-\mathbb{G}_h\|_{\mathcal{T}_h} \lesssim h^{s_{\mathbb{L}}+\frac{1}{2}}\|\mathbb{L}\|_{s_{\mathbb{L}},\Omega} + h^{s_{\boldsymbol{y}}-\frac{1}{2}}\|\boldsymbol{y}\|_{s_{\boldsymbol{y}},\Omega} + h^{s_{\mathbb{G}}-\frac{1}{2}}\|\mathbb{G}\|_{s_{\mathbb{G}},\Omega} + h^{s_{\boldsymbol{z}}-\frac{3}{2}}\|\boldsymbol{z}\|_{s_{\boldsymbol{z}},\Omega}, \tag{3.32a}$$

$$\|\boldsymbol{z}-\boldsymbol{z}_h\|_{\mathcal{T}_h} \lesssim h^{s_{\mathbb{L}}+\frac{1}{2}}\|\mathbb{L}\|_{s_{\mathbb{L}},\Omega} + h^{s_{\boldsymbol{y}}-\frac{1}{2}}\|\boldsymbol{y}\|_{s_{\boldsymbol{y}},\Omega} + h^{s_{\mathbb{G}}-\frac{1}{2}}\|\mathbb{G}\|_{s_{\mathbb{G}},\Omega} + h^{s_{\boldsymbol{z}}-\frac{3}{2}}\|\boldsymbol{z}\|_{s_{\boldsymbol{z}},\Omega}. \tag{3.32b}$$

Step 8: Estimate for $\|\mathbb{L}-\mathbb{L}_h\|_{\mathcal{T}_h}$

Lemma 3.26. *If* $k \geq 1$ *holds, then*

$$\|\zeta_{\mathbb{L}}\|_{\mathcal{T}_h} \lesssim h^{s_{\mathbb{L}}}\|\mathbb{L}\|_{s_{\mathbb{L}},\Omega} + h^{s_{\boldsymbol{y}}-1}\|\boldsymbol{y}\|_{s_{\boldsymbol{y}},\Omega} + h^{s_{\mathbb{G}}-1}\|\mathbb{G}\|_{s_{\mathbb{G}},\Omega} + h^{s_{\boldsymbol{z}}-2}\|\boldsymbol{z}\|_{s_{\boldsymbol{z}},\Omega}. \tag{3.33}$$

Proof. By Lemma 3.5 and the error equation (3.28a), we have

$$\begin{aligned}
&(\zeta_{\mathbb{L}}, \zeta_{\mathbb{L}})_{\mathcal{T}_h} + \langle h^{-1}(\boldsymbol{P}_M\zeta_{\boldsymbol{y}} - \zeta_{\widehat{\boldsymbol{y}}}), \zeta_{\boldsymbol{y}} - \zeta_{\widehat{\boldsymbol{y}}}\rangle_{\partial\mathcal{T}_h\setminus\mathcal{E}_h^\partial} + \langle h^{-1}\boldsymbol{P}_M\zeta_{\boldsymbol{y}}, \boldsymbol{P}_M\zeta_{\boldsymbol{y}}\rangle_{\mathcal{E}_h^\partial}\\
&= \mathscr{B}(\zeta_{\mathbb{L}}, \zeta_{\boldsymbol{y}}, \zeta_p, \zeta_{\widehat{\boldsymbol{y}}}; \zeta_{\mathbb{L}}, \zeta_{\boldsymbol{y}}, \zeta_p, \zeta_{\widehat{\boldsymbol{y}}})\\
&= \langle (P_M u - u_h)\boldsymbol{\tau}, \zeta_{\mathbb{L}}\cdot\boldsymbol{n} + h^{-1}\zeta_{\boldsymbol{y}}\rangle_{\mathcal{E}_h^\partial}\\
&= \langle (u - u_h)\boldsymbol{\tau}, \zeta_{\mathbb{L}}\cdot\boldsymbol{n} + h^{-1}\boldsymbol{P}_M\zeta_{\boldsymbol{y}}\rangle_{\mathcal{E}_h^\partial}\\
&\lesssim \|u-u_h\|_{\mathcal{E}_h^\partial}(\|\zeta_{\mathbb{L}}\|_{\mathcal{E}_h^\partial} + h^{-1}\|\boldsymbol{P}_M\zeta_{\boldsymbol{y}}\|_{\mathcal{E}_h^\partial})\\
&\lesssim h^{-\frac{1}{2}}\|u-u_h\|_{\mathcal{E}_h^\partial}(\|\zeta_{\mathbb{L}}\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|\boldsymbol{P}_M\zeta_{\boldsymbol{y}}\|_{\mathcal{E}_h^\partial}),
\end{aligned}$$

which gives

$$\|\zeta_{\mathbb{L}}\|_{\mathcal{T}_h} \lesssim h^{-\frac{1}{2}}\|u-u_h\|_{\mathcal{E}_h^\partial} \lesssim h^{s_{\mathbb{L}}}\|\mathbb{L}\|_{s_{\mathbb{L}},\Omega} + h^{s_{\boldsymbol{y}}-1}\|\boldsymbol{y}\|_{s_{\boldsymbol{y}},\Omega} + h^{s_{\mathbb{G}}-1}\|\mathbb{G}\|_{s_{\mathbb{G}},\Omega} + h^{s_{\boldsymbol{z}}-2}\|\boldsymbol{z}\|_{s_{\boldsymbol{z}},\Omega}.$$

The above lemma along with the triangle inequality and Lemmas 3.17 and 3.21 completes the proof of the main result:

Theorem 3.27. *Let* $\mathbb{L}$ *and* $\mathbb{L}_h$ *be the solutions of* (1.3) *and* (3.7)*, respectively. If* $k \geq 1$ *holds, then*

$$\|\mathbb{L}-\mathbb{L}_h\|_{\mathcal{T}_h} \lesssim h^{s_{\mathbb{L}}}\|\mathbb{L}\|_{s_{\mathbb{L}},\Omega} + h^{s_{\boldsymbol{y}}-1}\|\boldsymbol{y}\|_{s_{\boldsymbol{y}},\Omega} + h^{s_{\mathbb{G}}-1}\|\mathbb{G}\|_{s_{\mathbb{G}},\Omega} + h^{s_{\boldsymbol{z}}-2}\|\boldsymbol{z}\|_{s_{\boldsymbol{z}},\Omega}.$$

4. Numerical experiments

In this section, we present some numerical experiments to illustrate our theoretical results (see Theorem 3.1). We use uniform triangular meshes and define

$$\operatorname{div}(\boldsymbol{y}_h) = \max_{K\in\mathcal{T}_h}\frac{1}{|K|}\int_K |\nabla\cdot\boldsymbol{y}_h|\,\mathrm{d}\boldsymbol{x}.$$

Example 4.1. We begin with an example which has an analytical solution. The domain is the unit square $\Omega = (0,1)^2$ and the data is chosen as

$$\begin{aligned}
y_1 &= -2\pi^2\sin^2(\pi x_1)\cos(\pi x_2) - 2\pi^2\sin(\pi x_1)\sin(2\pi x_2),\\
y_2 &= 2\pi^2\cos(\pi x_1)\sin^2(\pi x_2) + 2\pi^2\sin(\pi x_2)\sin(2\pi x_1),\\
z_1 &= \pi\sin^2(\pi x_1)\sin(2\pi x_2), \quad z_2 = -\pi\sin^2(\pi x_2)\sin(2\pi x_1),\\
p &= 10^n\cos(\pi x_1), \quad q = 10^n\cos(\pi x_1), \quad \gamma = 1.
\end{aligned}$$

Here n is a parameter.

To make a comparison, we first solve the optimality system (1.3) by using the HDG method proposed in [10], with $n = 2, 4, 6$ and $k = 0$. The errors for all variables are shown in Tables 1 and 2. Although the convergence

Table 1
Example 4.1: Lack of pressure-robustness: Errors and observed convergence orders for the control u, pressure p, state $\boldsymbol{y}$, and its flux $\mathbb{L}$ by using the HDG method in [10].

k	n	$\frac{\sqrt{2}}{h}$	$\mathrm{div}(\boldsymbol{y}_h)$	$\|\boldsymbol{y}-\boldsymbol{y}_h\|_{L^2(\Omega)}$		$\|\mathbb{L}-\mathbb{L}_h\|_{L^2(\Omega)}$		$\|p-p_h\|_{L^2(\Omega)}$		$\|u-u_h\|_{L^2(\Gamma)}$	
				Error	Rate	Error	Rate	Error	Rate	Error	Rate
		4	7.52E+01	1.18E+01		5.58E+01		3.63E+01		1.77E+01	
		8	3.36E+01	3.58E+00	1.72	2.80E+01	0.99	2.45E+01	0.56	6.33E+00	1.48
0	2	16	1.59E+01	1.51E+00	1.24	1.39E+01	0.99	1.43E+01	0.78	1.89E+00	1.74
		32	7.79E+00	1.03E+00	0.54	8.92E+00	0.64	1.14E+01	0.32	5.65E−01	1.74
		64	3.87E+00	9.31E−01	0.15	7.49E+00	0.25	1.10E+01	0.04	2.69E−01	1.06
		4	2.74E+03	6.17E+02		4.49E+03		3.43E+03		1.74E+03	
		8	1.17E+03	1.69E+02	1.86	2.21E+03	1.09	2.19E+03	0.64	6.24E+02	1.48
0	4	16	4.78E+02	4.56E+01	1.89	8.30E+02	1.34	9.17E+02	1.25	1.83E+02	1.76
		32	2.14E+02	1.19E+01	1.93	3.33E+02	1.31	3.14E+02	1.54	4.95E+01	1.89
		64	1.03E+02	3.15E+00	1.91	1.46E+02	1.18	1.00E+02	1.65	1.28E+01	1.94
		4	2.74E+05	6.17E+04		4.49E+05		3.43E+05		1.74E+05	
		8	1.17E+05	1.69E+04	1.86	2.21E+05	1.09	2.19E+05	0.64	6.24E+04	1.48
0	6	16	4.78E+04	4.56E+03	1.89	8.30E+04	1.34	9.17E+04	1.25	1.83E+04	1.76
		32	2.14E+04	1.19E+03	1.93	3.33E+04	1.31	3.14E+04	1.54	4.95E+03	1.89
		64	1.03E+04	3.15E+02	1.91	1.46E+04	1.18	1.00E+04	1.65	1.28E+03	1.94

Table 2
Example 4.1: Lack of pressure-robustness: Errors and observed convergence orders for the dual pressure q, dual state $\boldsymbol{z}$, and its flux $\mathbb{G}$ by using the HDG method in [10].

k	n	$\frac{\sqrt{2}}{h}$	$\mathrm{div}(\boldsymbol{z}_h)$	$\|\boldsymbol{z}-\boldsymbol{z}_h\|_{L^2(\Omega)}$		$\|\mathbb{G}-\mathbb{G}_h\|_{L^2(\Omega)}$		$\|q-q_h\|_{L^2(\Omega)}$	
				Error	Rate	Error	Rate	Error	Rate
		4	1.19E+01	3.21E+00		1.35E+01		9.28E+00	
		8	5.62E+00	8.98E−01	1.83	7.80E+00	0.79	3.31E+00	1.48
0	2	16	2.76E+00	2.30E−01	1.96	4.06E+00	0.94	1.00E+00	1.71
		32	1.38E+00	5.74E−02	2.00	2.05E+00	0.98	3.16E−01	1.67
		64	6.89E−01	1.96E−02	1.54	1.03E+00	0.98	1.20E−01	1.39
		4	1.12E+03	3.08E+02		1.31E+03		9.15E+02	
		8	5.11E+02	8.66E+01	1.83	7.59E+02	0.79	3.21E+02	1.50
0	4	16	2.51E+02	2.25E+01	1.94	3.95E+02	0.94	9.27E+01	1.79
		32	1.25E+02	5.68E+00	1.98	1.99E+02	0.98	2.45E+01	1.91
		64	6.26E+01	1.42E+00	1.99	1.00E+02	0.99	6.25E+00	1.97
		4	1.12E+05	3.08E+04		1.31E+05		9.15E+04	
		8	5.11E+04	8.66E+03	1.83	7.59E+04	0.79	3.21E+04	1.50
0	6	16	2.51E+04	2.25E+03	1.94	3.95E+04	0.94	9.27E+03	1.79
		32	1.25E+04	5.68E+02	1.98	1.99E+04	0.98	2.45E+03	1.91
		64	6.26E+03	1.42E+02	1.99	1.00E+04	0.99	6.25E+02	1.97

rates are optimal and consistent with the error analysis in [10] for $n = 4, 6$, the magnitude of the errors strongly depend on the pressures. This shows that the algorithm proposed and analyzed in [10] is not pressure-robust.

Now we use the new HDG method (see the formulation (2.4)) to test the same problem. The errors for all variables are shown in Tables 3 and 4. We see that the error magnitudes of the state $\boldsymbol{y}$, dual state $\boldsymbol{z}$ and control u are *independent* of the pressure p and the dual pressure q. We also notice that the convergence rates are higher than predicted by our error analysis; a similar phenomena has been observed for other numerical methods for Dirichlet boundary control problems involving elliptic equations [25,29,30] and Stokes equations [10,31]. To the best of our knowledge, only one work explained the above phenomena: May, Rannacher, and Vexler in [32] used a duality argument to obtain improved convergence rates for the state and dual state with the standard finite element method. It is not clear how to apply this technique to the HDG methods.

Table 3
Example 4.1: Pressure-robustness: Errors and observed convergence orders for the control u, pressure p, state $\boldsymbol{y}$, and its flux $\mathbb{L}$ by using the new HDG formulation (2.4).

k	n	$\frac{\sqrt{2}}{h}$	$\mathrm{div}(\boldsymbol{y}_h)$	$\Vert\boldsymbol{y}-\boldsymbol{y}_h\Vert_{L^2(\Omega)}$		$\Vert\mathbb{L}-\mathbb{L}_h\Vert_{L^2(\Omega)}$		$\Vert p-p_h\Vert_{L^2(\Omega)}$		$\Vert u-u_h\Vert_{L^2(\Gamma)}$	
				Error	Rate	Error	Rate	Error	Rate	Error	Rate
		4	8.88E−16	8.76E+00		5.33E+01		1.47E+01		6.30E+00	
		8	6.66E−16	2.20E+00	2.00	2.79E+01	0.93	7.20E+00	1.03	3.15E+00	1.00
0	2	16	3.33E−16	5.41E−01	2.02	1.41E+01	0.99	3.64E+00	0.99	1.64E+00	0.94
		32	3.05E−16	1.34E−01	2.01	7.05E+00	1.00	1.80E+00	1.02	8.06E−01	1.03
		64	1.94E−16	3.34E−02	2.00	3.52E+00	1.00	8.86E−01	1.02	3.98E−01	1.02
		4	1.78E−15	8.76E+00		5.33E+01		1.30E+03		6.40E+00	
		8	6.66E−16	2.20E+00	2.00	2.79E+01	0.93	6.53E+02	0.99	3.42E+00	0.91
0	4	16	3.87E−16	5.41E−01	2.02	1.41E+01	0.99	3.27E+02	1.00	1.58E+00	1.11
		32	2.91E−16	1.34E−01	2.01	7.05E+00	1.00	1.64E+02	1.00	7.53E−01	1.07
		64	1.87E−16	3.34E−02	2.00	3.52E+00	1.00	8.18E+01	1.00	3.96E−01	0.93
		4	1.78E−15	8.76E+00		5.33E+01		1.30E+05		6.49E+00	
0		8	6.66E−16	2.20E+00	2.00	2.79E+01	0.93	6.53E+04	0.99	3.42E+00	0.93
	6	16	4.44E−16	5.41E−01	2.02	1.41E+01	0.99	3.27E+04	1.00	1.66E+00	1.04
		32	3.19E−16	1.34E−01	2.01	7.05E+00	1.00	1.64E+04	1.00	7.90E−01	1.07
		64	1.87E−16	3.34E−02	2.00	3.52E+00	1.00	8.18E+03	1.00	3.98E−01	0.99
		4	1.88E−15	1.18E+00		1.40E+01		5.53E+00		1.61E+00	
		8	1.79E−15	1.52E−01	2.96	3.78E+00	1.89	1.19E+00	2.22	4.37E−01	1.89
1	2	16	1.05E−15	1.94E−02	2.97	1.03E+00	1.88	2.74E−01	2.12	1.11E−01	1.98
		32	8.90E−16	2.45E−03	2.98	2.91E−01	1.82	6.89E−02	1.99	2.77E−02	2.00
		64	4.72E−16	3.12E−04	2.98	8.73E−02	1.74	1.86E−02	1.89	6.98E−03	1.99
		4	1.72E−15	1.18E+00		1.40E+01		1.25E+02		1.65E+00	
		8	1.78E−15	1.52E−01	2.96	3.78E+00	1.89	3.14E+01	1.99	4.37E−01	1.92
1	4	16	1.07E−15	1.94E−02	2.97	1.03E+00	1.88	7.87E+00	2.00	1.11E−01	1.98
		32	8.90E−16	2.45E−03	2.98	2.91E−01	1.82	1.97E+00	2.00	2.79E−02	1.99
		64	4.55E−16	3.12E−04	2.98	8.73E−02	1.74	4.92E−01	2.00	6.98E−03	2.00
		4	1.65E−15	1.18E+00		1.40E+01		1.25E+04		1.65E+00	
1		8	1.78E−15	1.52E−01	2.96	3.78E+00	1.89	3.14E+03	1.99	4.37E−01	1.92
	6	16	1.03E−15	1.94E−02	2.97	1.03E+00	1.88	7.87E+02	2.00	1.11E−01	1.98
		32	8.92E−16	2.45E−03	2.98	2.91E−01	1.82	1.97E+02	2.00	2.79E−02	1.99
		64	4.58E−16	3.12E−04	2.98	8.73E−02	1.74	4.92E+01	2.00	6.98E−03	2.00

Example 4.2. Next, we test the problem with unknown true solutions. We use the same data from [10, Example 5.1]. We set $\Omega = (0, 0.125)^2$, $\boldsymbol{f} = \boldsymbol{0}$, and $\gamma = 1$. To show that our HDG method is pressure-robust, we perturb the target state $\boldsymbol{y}_d$ by a large gradient field. We take

$$\boldsymbol{y}_d = 200 \times 8^3[x^2(1-8x)^2y(1-8y)(1-16y), -x(1-8x)(1-16x)y^2(1-y)^2]^\top,$$
$$\widetilde{\boldsymbol{y}}_d = \boldsymbol{y}_d + 10^6[1,1]^\top.$$

We denote the corresponding velocity by $\boldsymbol{y}$ and $\widetilde{\boldsymbol{y}}$. We know the fact that perturbing the external force by a gradient field affects only the pressure, and not the velocity; this was shown in [12]. Hence, $\boldsymbol{y} = \widetilde{\boldsymbol{y}}$.

We first solve the optimality system (1.3) by using the HDG method proposed in [10] with $h = \frac{\sqrt{2}}{1024}$ and $k = 1$ for both $\boldsymbol{y}_d$ and $\widetilde{\boldsymbol{y}}_d$, we compute the difference of $\boldsymbol{y}_h$ and $\widetilde{\boldsymbol{y}}_h$:

$$\Vert\boldsymbol{y}_h - \widetilde{\boldsymbol{y}}_h\Vert_{L^2(\Omega)} = 214.$$

Next, we use the HDG formulation (2.4) in this paper, and we have

$$\Vert\boldsymbol{y}_h - \widetilde{\boldsymbol{y}}_h\Vert_{L^2(\Omega)} = 6.94 \times 10^{-7}.$$

We see that the algorithm proposed and analyzed in [10] is not pressure-robust; while the algorithm (2.4) is pressure-robust.

Table 4
Example 4.1: Pressure-robustness: Errors and observed convergence orders for the dual pressure q, dual state z, and its flux $\mathbb{G}$ by using the new HDG formulation (2.4).

k	n	$\frac{\sqrt{2}}{h}$	$\mathrm{div}(z_h)$	$\|z - z_h\|_{L^2(\Omega)}$		$\|\mathbb{G} - \mathbb{G}_h\|_{L^2(\Omega)}$		$\|q - q_h\|_{L^2(\Omega)}$	
				Error	Rate	Error	Rate	Error	Rate
		4	1.11E−16	8.51E−01		6.27E+00		1.31E+01	
		8	5.55E−17	2.47E−01	1.78	3.37E+00	0.90	6.59E+00	0.99
0	2	16	3.23E−17	6.55E−02	1.92	1.71E+00	0.97	3.29E+00	1.00
		32	2.31E−17	1.68E−02	1.96	8.60E−01	1.00	1.64E+00	1.00
		64	1.61E−17	4.24E−03	1.98	4.30E−01	1.00	8.20E−01	1.00
		4	1.11E−16	8.51E−01		6.27E+00		1.30E+03	
		8	5.55E−17	2.47E−01	1.78	3.37E+00	0.90	6.53E+02	0.99
0	4	16	4.36E−17	6.55E−02	1.92	1.71E+00	0.97	3.27E+02	1.00
		32	2.29E−17	1.68E−02	1.96	8.60E−01	1.00	1.64E+02	1.00
		64	1.39E−17	4.24E−03	1.98	4.30E−01	1.00	8.18E+01	1.00
		4	1.11E−16	8.51E−01		6.27E+00		1.30E+05	
0		8	5.55E−17	2.47E−01	1.78	3.37E+00	0.90	6.53E+04	0.99
	6	16	3.72E−17	6.55E−02	1.92	1.71E+00	0.97	3.27E+04	1.00
		32	2.08E−17	1.68E−02	1.96	8.60E−01	1.00	1.64E+04	1.00
		64	1.39E−17	4.24E−03	1.98	4.30E−01	1.00	8.18E+03	1.00
		4	1.59E−16	1.62E−01		1.93E+00		1.49E+00	
		8	1.34E−16	2.12E−02	2.93	5.06E−01	1.93	3.60E−01	2.05
1	2	16	9.26E−17	2.70E−03	2.97	1.28E−01	1.99	8.76E−02	2.04
		32	6.39E−17	3.41E−04	2.99	3.20E−02	2.00	2.17E−02	2.02
		64	3.92E−17	4.27E−05	3.00	8.01E−03	2.00	5.39E−03	2.01
		4	1.64E−16	1.62E−01		1.93E+00		1.25E+02	
		8	1.30E−16	2.12E−02	2.93	5.06E−01	1.93	3.14E+01	1.99
1	4	16	8.68E−17	2.70E−03	2.97	1.28E−01	1.99	7.87E+00	2.00
		32	6.72E−17	3.41E−04	2.99	3.20E−02	2.00	1.97E+00	2.00
		64	3.84E−17	4.27E−05	3.00	8.01E−03	2.00	4.92E−01	2.00
		4	1.64E−16	1.62E−01		1.93E+00		1.25E+04	
1		8	1.26E−16	2.12E−02	2.93	5.06E−01	1.93	3.14E+03	1.99
	6	16	8.52E−17	2.70E−03	2.97	1.28E−01	1.99	7.87E+02	2.00
		32	6.37E−17	3.41E−04	2.99	3.20E−02	2.00	1.97E+02	2.00
		64	3.95E−17	4.27E−05	3.00	8.01E−03	2.00	4.92E+01	2.00

5. Conclusion

In [10], we used an existing HDG method to approximate the solution of a tangential Dirichlet boundary control problem for the Stokes system. The velocities were not in $\boldsymbol{H}(\mathrm{div};\Omega)$ and the error estimates depended on the pressures. In this work, we devised a new globally divergence free and pressure-robust HDG method for solving this problem. We proved that the discrete velocity belongs to $\boldsymbol{H}(\mathrm{div};\Omega)$ and is globally divergence free. Furthermore, our error estimates show that the errors for the control and velocities do not depend on the pressures.

As far as we are aware, this is the first work to obtain a global divergence free and pressure-robust numerical method for an optimal boundary control problem involving Stokes equations. In the future, we will consider devising pressure-robust numerical methods when using an energy space for the control [31]. Besides that, we plan to devise divergence free and pressure-robust HDG schemes for more complicated PDEs, such as the Oseen and Navier–Stokes equations; and apply the methods to other PDE optimal control problems.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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