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Compact Minimal Submanifolds in a Large Class of Riemannian Manifolds

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Abstract. Through a new technique, we provide uniqueness, rigidity and non-existence results for compact minimal submanifolds of arbitrary dimension in a large class of Riemannian manifolds, which include between others, Riemannian double-twisted and warped products. Moreover, we show that our results can be applied in particular, to space forms and Cartan–Hadamard manifolds, re-obtaining several classic results in a different approach. Interesting applications to Geometric Analysis are also showed.

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1. Introduction

The importance of minimal submanifolds (and, in particular, minimal surfaces) is very well known. Among the elliptic quasi-linear PDEs, the equation of minimal hypersurfaces in Euclidean space

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=0 \tag{1}$$

has a long and fruitful history and has deserved the attention of many researchers.

More precisely, given a function $u \in C^{\infty}(\Omega)$, where Ω denotes an open domain in $\mathbb{R}^n$, the graph $\Sigma_u = \{(u(p), p) : p \in \Omega\}$ in the Euclidean space $\mathbb{R}^{n+1}$ defines a minimal hypersurface if and only if u is solution to Eq. (1). From a geometric point of view, Eq. (1) is the Euler–Lagrange equation of a classical variational problem. Specifically, for each $u \in C^{\infty}(\Omega)$, where Ω denotes an open domain in $\mathbb{R}^n$, the volume element of the induced metric from $\mathbb{R}^{n+1}$ is represented by the n-form $\sqrt{1+|Du|^2}dV$ on the graph Σ_u, where dV is the canonical volume element of $\Omega \subset \mathbb{R}^n$. The critical points of the n-volume functional $u \mapsto \int \sqrt{1+|Du|^2}dV$ are given by the Eq. (1). In 1914, Bernstein

[6], amended latter by Hopf in 1950 [18], proved his well-known uniqueness theorem for $n = 2$

The only entire solutions to the minimal surface equation in $\mathbb{R}^3$ *are the affine functions*

$$u(x, y) = ax + by + c,$$

where $a, b, c \in \mathbb{R}$.

In terms of PDEs, Bernstein proved a general Liouville type result

Any bounded solution $u \in C^\infty(\mathbb{R}^2)$ *of the PDE*

$$A\, u_{xx} + 2B\, u_{xy} + C\, u_{yy} = 0,$$

where $A, B, C \in C^\infty(\mathbb{R}^2)$ such that $AC - B^2 > 0$, *must be constant.*

This previous result is obtained as an application of the so-called Bernstein's geometric theorem

If the Gauss curvature of the graph of $u \in C^\infty(\mathbb{R}^2)$ *in* $\mathbb{R}^3$ *satisfies* $K \leq 0$ *everywhere and* $K < 0$ *at some point, then* u *cannot be bounded.*

Then, a lot of work has been made to extend the classical Bernstein result to higher dimensions (see [23] for a survey until 1984 and [13] for the case of surfaces). A notable progress was made by Moser [21] in 1961, who obtained the so-called Moser–Bernstein theorem

The only entire solutions u *to the minimal surface equation in* $\mathbb{R}^{n+1}$ *such that* $|Du| \leq C$, *for some* $C \in \mathbb{R}^+$, *are the affine functions*

$$u(x_1, \ldots, x_n) = a_1 x_1 + \ldots + a_n x_n + c,$$

where $a_i, c \in \mathbb{R}$, $1 \leq i \leq n$ and $\sum_{i=1}^n a_i^2 \leq C^2$.

It is remarkable that this result of Moser for $n = 2$ joined with a previous result of Bers [7], who proves that solutions of the minimal surface equation in $\mathbb{R}^3$ defined on the exterior of a closed disc in $\mathbb{R}^2$ have bounded gradient, provides another proof for the classical Bernstein theorem. In 1968, Simons [29], together with other results of De Giorgi [14] and Fleming [16], yields a proof of the Bernstein theorem for $n \leq 7$. Furthermore, it was found a counterexample $u \in C^\infty(\mathbb{R}^n)$ for each $n \geq 8$.

Then, much research has been made to characterize minimal submanifolds in different Riemannian ambients. For example, Rosenberg studied minimal surfaces in the product of $\mathbb{R}$ and a Riemannian surface in [27]. In a more general setting, minimal surfaces are studied in warped product manifolds in several papers (see, for instance, [1–4, 10, 26]).

Focusing in the case of compact submanifolds, it is known that the Euclidean space $\mathbb{R}^n$ does not admit any compact minimal submanifold. However, this fact does not occur in $\mathbb{S}^n$, for example. Hence, it is natural to consider the problem of obtaining characterization results for minimal submanifolds in a large class of Riemannian manifolds.

Consider a smooth 1-parametrized family of Riemannian metrics (F^{n+m}, g_t), $t \in I \subseteq \mathbb{R}$, on a differential manifold F, i.e., a smooth map $g : I \times F \to T_{0,2}(F)$, where $T_{0,2}(F)$ denotes the fiber bundle of 2-covariant tensors on F, such that $g_{_t} : F \rightarrow T_{0,2}(F)$ is a positive definite metric tensor for all $t \in I$, and a positive function $\beta \in C^\infty(I \times F)$. The product manifold $I \times F$ can be endowed with the following metric:

$$\overline{g}_{_{(t,x)}} = \beta(t,x)\, \pi_{_I}^*(\mathrm{d}t^2) + (\pi_{_F}^*(g_{_t}))_{_x} \quad (\,\beta\, \mathrm{d}t^2 + g_{_t} \quad \text{in short}), \tag{2}$$

where $\pi_{_I}$ and $\pi_{_F}$ denote the canonical projections onto I and F, respectively.

We will say that a Riemannian manifold is *orthogonally splitted* if it is isometric to a Riemannian manifold $(\overline{M}, \overline{g} = \beta \mathrm{d}t^2 + g_t)$. Note that the class of orthogonal-splitted Riemannian manifold includes the double-twisted Riemannian products where one of its factors is the real line, and, in particular, the Riemannian warped products (see [25]).

Observe that a suitable open normal neighbourhood of an arbitrary Riemannian manifold lies in this family (in this case, consider the function $\beta \equiv 1$ and the coordinate t as the geodesic distance to a fixed point of the normal neighbourhood). In particular, removing a point of a (simply connected) complete Cartan–Hadamard manifold, we have that the resulting manifold possesses this structure (see [24]).

We focus our attention on the case in which an orthogonal-splitted Riemannian manifold has an isotropic behaviour associated with the t coordinate. Given any compact subset, we desire that its volume does not increase or decrease, by the flux along the vector field ∂_t. To make clear this idea, we may introduce the following notion: an orthogonal-splitted Riemannian manifold $(\overline{M}, \overline{g} = \beta \mathrm{d}t^2 + g_t)$ is said be *non-shrinking* (resp. *expanding*) *throughout* the vector filed ∂_t if for all $x \in F$ and for all $v \in T_x F$

$$\partial_t \beta \geq 0 \qquad \text{and} \quad (\partial_t g_{_t})(\tilde{w}, \tilde{w}) \geq 0 \;\; (\text{resp.} > 0),$$

where $\tilde{w}$ is the vector filed along the curve $I \times \{x\}$ given at each point $(t,x) \in I \times \{x\}$ by the lift $\tilde{w}$ of w to (t,x). For the non-shrinking (resp. expanding) case, this definition is equivalent to the *Lie derivative* $\mathcal{L}_{\partial_t} \overline{g}$ to be a semi-definite (resp. definite) positive tensor field.

Dually, we will say that an orthogonal-splitted Riemannian manifold is *non-expanding* (resp. *contracting*) throughout ∂_t if it is non-shrinking (resp. expanding) throughout $-\partial_t$. Obviously, the geometrical interpretations also hold with their respective changes.

We may unify these two notions with the following one. We will say that the manifold $(\overline{M}, \beta\, \mathrm{d}t^2 + g_{_t})$ is *monotone* (resp. *strictly monotone*) if it is non-shrinking or non-expanding (resp. expanding or contracting) throughout ∂_t.

This paper is organized as follows. Section 2 is devoted to several preliminaries. In Sect. 3, we show several rigidity results for compact minimal Riemannian submanifolds with and without boundary. The first of them is Theorem 5,

> *In a monotone orthogonal-splitted Riemannian manifold, every compact minimal submanifold must be contained in a level hypersurface $\{t_0\} \times F$,for some $t_0 \in I$. Moreover, in the case of codimension higher than one, S must be a minimal submanifold of (F, g_{t_0}).*

For the case with boundary, we obtain Theorem 7

Let $(\overline{M}, \overline{g})$ be a non-expanding (resp. non-shrinking) orthogonal-splitted Riemannian manifold; and let S be a compact submanifold with boundary. Assume that:

(i) $(\overline{M}, \overline{g})$ *is non-expanding (resp. non-shrinking).*

(ii) *The mean curvature vector* $\vec{H}$ of S, *points in the opposite (resp. same) direction of* ∂_t *when is different from zero.*

(iii) $\partial S \subset \{t_0\} \times F$.

(iv) $\min_S(\tau) \geq t_0$ (resp. $\max_S(\tau) \leq t_0$).

Then, S is fully contained in $\{t_0\} \times F$.

We finished this section by extending our results to orthogonal-splitted Riemannian manifold with change in the monotonic behaviour (see Theorem 12, and showing an existence theorem for compact minimal hypersurfaces immersed in a orthogonal-splitted Riemannian manifold (see Theorem 15).

In Sect. 4, we apply our previous results to the study of compact minimal submanifolds in Riemannian spaces with certain infinitesimal symmetries. Therefore, we obtain Theorem 17,

Let $(\overline{M}, \overline{g})$ be a simply connected Riemannian manifold admitting a complete, globally defined conformal Killing vector field K. Assume that the norm of K along its flow is monotonic (either non-decreasing or non-increasing), then any minimal compact submanifold has to be contained in a leaf of the foliation $K^{\perp}$, orthogonal to the conformal vector field.

In Sect. 5, we deal with several non-existence results getting interesting applications to space forms, as well as to Cartan–Hadamard manifolds, so we re-obtain the following result (see Corollary 23),

A Cartan-Hadamard Riemannian manifold admits no compact minimal submanifold.

Also in Sect. 5, we obtain a general geometric obstruction for the existence of minimal submanifolds. In particular, for the case of the Euclidean space, its application shows (in a new approach) the well-known result about non-existence of compact isometric immersion.

Finally, Sect. 6 is devoted to show new results in the field of Geometric Analysis. Specifically, we study the entire solutions to a family of quasi-linear elliptic PDEs on a compact Riemannian manifold (see, Theorems 34, 35, Corollaries 36, 37). Moreover, we managed to solve several Dirichlet problems on a general Riemannian manifold (see, Theorem 38, Corollaries 40, 41.)

2. On Orthogonally Splitted Manifolds

Let us begin by fixing some notation and definitions:

Definition 1. We will say that a Riemannian manifold is *orthogonally splitted* if it is isometric to a Riemannian manifold $(\overline{M}^{n+m+1}, \overline{g})$ where

$$\overline{M} = I \times F^{n+m}, \quad \overline{g} = \beta \pi_I^*(\mathrm{d}t^2) + \pi_F^*(g_t) \left(\equiv \beta \mathrm{d}t^2 + g_t\right), \tag{3}$$

where I is an open interval on $\mathbb{R}$, $\pi_I : \overline{M} \to I, \pi_F : \overline{M} \to F$ are the corresponding projections, $\beta : \overline{M} \to (0, \infty)$ is a smooth function, and $\{g_t\}_{t\in I}$ is a smooth 1-parametric family of Riemannian metrics defined over F.

Let $x : S^n \to \overline{M}$ be an n-dimensional connected isometric immersed submanifold. As it is well known, for all vector fields $X, Y \in \mathfrak{X}(S)$, we have the classical decomposition

$$\overline{\nabla}_X Y = \nabla_X Y + \mathrm{II}(X, Y), \tag{4}$$

where ∇ is the Levi-Civita connection on S induced from the ambient space, and $\mathrm{II}(X, Y) = \left(\overline{\nabla}_X Y\right)^{\perp}$ is the *second fundamental form*, that is, the corresponding normal component of $\overline{\nabla}_X Y$. Then, the *mean curvature vector field* $\vec{H}$ is defined as the (normalized) $\mathcal{C}_1^1$ metric contraction of the second fundamental form. Namely, for $\{E_i\}_{i=1}^n$, a local orthonormal (reference) frame on S

$$\vec{H} = \frac{1}{n}\Sigma_{i=1}^n \mathrm{II}(E_i, E_i). \tag{5}$$

A submanifold S is called *minimal* if $\vec{H} = 0$.

Consider now the function $\tau : S \to I$ given by $\tau := \pi_I \circ x$. On the one hand, it is clear that the submanifold S is contained in a slice $\{t_0\} \times F$ if and only if $\tau \equiv t_0$. On the other hand, it is not difficult to show that

$$\nabla\tau = \frac{1}{\beta}\partial_t^T, \tag{6}$$

where ∇ denotes the induced gradient on S, and ∂_t^T is the orthogonal projection of ∂_t on TS. Now, let us take a local orthonormal frame $\{E_1\}_{i=1}^n$ of S on an open set $U \subset S$, and let $\{N_j\}_{j=1}^{m+1}$ be a local orthonormal frame of the normal vector bundle of S in $\overline{M}$. Hence, we can define a local orthonormal frame $\mathcal{B} = \{E_1, \ldots, E_n, N_1, \ldots, N_{m+1}\}$ on S. Making use of standard computations, we can see that

$$\begin{aligned}
\Delta\tau &= -\overline{g}(\nabla(\ln\beta), \nabla\tau) + \frac{1}{\beta}\overline{\mathrm{div}}\,(\partial_t) - \sum_{j=1}^{m+1} \frac{1}{\beta}\overline{g}\left(\overline{\nabla}_{N_j}\partial_t, N_j\right) \\
&\quad + \frac{1}{\beta}\sum_{j=1}^{m+1}\sum_{i=1}^{n} \overline{g}\,(N_j, \partial_t)\,\overline{g}\,(N_j, \mathrm{II}(E_i, E_i)) \\
&= -\overline{g}(\nabla(\ln\beta), \nabla\tau) + \frac{1}{\beta}\overline{\mathrm{div}}\,(\partial_t) \\
&\quad - \sum_{j=1}^{m+1} \frac{1}{\beta}\overline{g}\left(\overline{\nabla}_{N_j}\partial_t, N_j\right) + \frac{1}{\beta}\sum_{j=1}^{m+1} \overline{g}\,(N_j, \partial_t)\,\overline{g}\left(N_j, n\vec{H}\right) \\
&= -\overline{g}(\nabla(\ln\beta), \nabla\tau) + \frac{1}{\beta}\overline{\mathrm{div}}\,(\partial_t) - \sum_{j=1}^{m+1} \frac{1}{\beta}\overline{g}\left(\overline{\nabla}_{N_j}\partial_t, N_j\right) + \frac{1}{\beta}\overline{g}\left(n\vec{H}, \partial_t\right).
\end{aligned} \tag{7}$$

To take advantage of the expression of the metric in (3), we will decompose each vector $N_j = \frac{1}{\beta}\overline{g}(N_j, \partial_t)\partial_t + N_j^F$, where N_j^F is the orthogonal projection of N_j onto the corresponding slice $\{t\} \times F$ and satisfies that $[N_j^F, \partial_t] = 0$. Then, taking into account Koszul formulae and the fact that ∂_t and N_j^F are orthogonal

$$\begin{aligned} \overline{g}(\overline{\nabla}_{N_j}\partial_t, N_j) &= \frac{1}{2}\overline{g}\left(\frac{\partial_t}{\sqrt{\beta}}, N_j\right)^2 \partial_t(\ln\beta) + \frac{1}{2}\left(\mathcal{L}_{\partial_t} g_t\right)(N_j^F, N_j^F) \\ &\quad + \frac{1}{\beta}\overline{g}(N_j, \partial_t)\left(\overline{g}(\overline{\nabla}_{N_j^F}\partial_t, \partial_t) - \overline{g}(\overline{\nabla}_{\partial_t} N_j^F, \partial_t)\right) \\ &= \frac{1}{2}\overline{g}\left(\frac{\partial_t}{\sqrt{\beta}}, N_j\right)^2 \partial_t(\ln\beta) + \frac{1}{2}\left(\mathcal{L}_{\partial_t} g_t\right)(N_j^F, N_j^F), \end{aligned} \tag{8}$$

where here, for convenience, we are denoting with $\mathcal{L}_{\partial_t} g_t := \partial_t g_t$. Finally, let us express $\overline{\mathrm{div}}(\partial_t)$ in a more suitable manner. For this, let us consider an orthonormal frame $\mathcal{B}' = \left\{\frac{1}{\sqrt{\beta}}\partial_t, \tilde{E}_1, \ldots, \tilde{E}_{n+m}\right\}$ for the metric $\overline{g}$. It is straightforward to see that

$$\overline{\mathrm{div}}(\partial_t) = \frac{1}{2}\left(\partial_t(\ln\beta) + \sum_{i=1}^{m+n}\left(\mathcal{L}_{\partial_t} g_t\right)(\tilde{E}_i, \tilde{E}_i)\right). \tag{9}$$

Joining both (8) and (9) together with (7), we arrive to

$$\begin{aligned} \Delta\tau &= -\overline{g}(\nabla(\ln\beta), \nabla\tau) + \frac{1}{2\beta}\left(\sum_{i=1}^{n+m}\left(\mathcal{L}_{\partial_t} g_t\right)(\tilde{E}_i, \tilde{E}_i) - \sum_{j=1}^{m+1}\left(\mathcal{L}_{\partial_t} g_t\right)(N_j^F, N_j^F)\right) \\ &\quad + \frac{1}{2\beta}\partial_t(\ln\beta)\left(1 - \sum_{j=1}^{m}\overline{g}\left(N_j, \frac{\partial_t}{\sqrt{\beta}}\right)^2\right) + \frac{1}{\beta}\overline{g}(n\vec{H}, \partial_t). \end{aligned} \tag{10}$$

Previous expression can be simplified even more by making use of the following observations: On the one hand, we can make a convenient conformal change, so the first term is *absorbed* by the Laplacian. In fact, let us recall that under a conformal change $\tilde{g} = e^{2\varphi}\overline{g}$, the Laplace operator transform as (see [8], for instance)

$$\tilde{\Delta} f = e^{-2\varphi}\left(\Delta f - (n-2)\overline{g}\left(\nabla f, \nabla\varphi\right)\right). \tag{11}$$

Hence, if we assume that $n > 2$ and take $e^{2\varphi} = \beta^{2/n-2}$

$$\begin{aligned} \tilde{\Delta}\tau &= \frac{1}{2}\beta^{-\frac{n}{n-2}}\left\{\sum_{i=1}^{n+m}\left(\mathcal{L}_{\partial_t} g_t\right)(\tilde{E}_i, \tilde{E}_i) - \sum_{j=1}^{m+1}\left(\mathcal{L}_{\partial_t} g_t\right)(N_j^F, N_j^F)\right. \\ &\quad \left. + \partial_t(\ln\beta)\left(1 - \sum_{j=1}^{m}\overline{g}\left(N_j, \frac{\partial_t}{\sqrt{\beta}}\right)^2\right) + 2\overline{g}(n\vec{H}, \partial_t)\right\}. \end{aligned} \tag{12}$$

On the other hand, let us define a $(0, 2)$-tensor ξ on $\overline{M}$ given by

$$\xi(V, W) = \left(\mathcal{L}_{\partial_t} g_t\right)(d\pi_F(V), d\pi_F(W)), \quad V, W \in \mathcal{X}(\overline{M}).$$

Recall that, for a given point $p \in S$, we have two different orthogonal frames, $\mathcal{B}$ and $\mathcal{B}'$. Using the latter, it follows that:

$$\operatorname{tr}(\xi) = \sum_{i=1}^{n+m} \left(\mathcal{L}_{\partial_t} g_t\right) (d\pi_F(\tilde{E}_i), d\pi_F(\tilde{E}_i)), \tag{13}$$

while with the former

$$\operatorname{tr}(\xi) = \sum_{i=i}^{n} \left(\mathcal{L}_{\partial_t} g_t\right) (d\pi_F(E_i), d\pi_F(E_i)) + \sum_{j=1}^{m+1} \left(\mathcal{L}_{\partial_t} g_t\right) (d\pi_F(N_j), d\pi_F(N_j))$$

$$= \operatorname{tr}(\xi|_{T_pS}) + \sum_{j=1}^{m+1} \left(\mathcal{L}_{\partial_t} g_t\right) (N_j^F, N_j^F). \tag{14}$$

Finally, due the fact that $\overline{g}(\partial_t/\sqrt{\beta}, \partial_t/\sqrt{\beta}) = 1$, we can find $\theta \in [0, \pi/2]$, so

$$\sin^2(\theta) = 1 - \sum_{j=1}^{m} \overline{g}\left(N_j, \frac{\partial_t}{\sqrt{\beta}}\right)^2. \tag{15}$$

Joining all together with (12), we deduce that

$$\tilde{\Delta}\tau = \frac{1}{2}\beta^{-n/(n-2)} \left(\operatorname{tr}(\xi|_{T_pS}) + \sin^2(\theta)\partial_t(ln(\beta)) + 2\overline{g}(n\vec{H}, \partial_t)\right). \tag{16}$$

Remark 2. Some remarks are in order.

(a) As we have mentioned before, for previous conformal change, we have to assume that $n > 2$. Nevertheless, we can build a 1-dimensional extension, so the dimension of the submanifold and the Riemannian manifold is increased. Indeed, if $x : S \to I \times F$ is a minimal 2-dimensional isometric immersion, we consider the Riemannian manifold $\big((I \times F) \times \mathbb{S}^1, \overline{g} + \mathrm{d}s^2\big)$, being $\mathrm{d}s^2$ the standard metric of $\mathbb{S}^1$ and the following natural 3-dimensional isometric immersion, $\hat{x} : S \times \mathbb{S}^1 \to (I \times F) \times \mathbb{S}^1$, with $\hat{x}(p, s) = (x(p), s)$, for all $p \in S$ and $s \in \mathbb{S}^1$.

Taking into account the natural identifications $T_{(p,\alpha)}(S \times \mathbb{S}^1) \equiv T_pS \oplus T_\alpha\mathbb{S}^1$, $p \in S$, $\alpha \in \mathbb{S}^1$ and $T_{(q,\alpha)}(\overline{M} \times \mathbb{S}^1) \equiv T_q\overline{M} \oplus T_\alpha\mathbb{S}^1$, $q \in \overline{M}$, $\alpha \in \mathbb{S}^1$, for each tangent vector $v \in T_pS$ (or normal vector $w \in T_pS^\perp$), there is a canonical tangent vector $\hat{v} = (v, 0) \in T_{(p,\alpha)}(S \times \mathbb{S}^1)$ (or $\hat{w} = (w, 0) \in T_{(p,\alpha)}(S \times \mathbb{S}^1)^\perp$). Moreover, it is clear that the last term in (16) is invariant under this extension. In particular, if S is minimal in $\overline{M}$, then $S \times \mathbb{S}^1$ is minimal in $\overline{M} \times \mathbb{S}^1$. Finally, note that a similar procedure can be made when the submanifold is a geodesic.

(b) If we assume that S is an hypersurface, and so, locally we only have one normal vector N to S, Eq. (15) becomes

$$\sin^2(\theta) = 1 - \overline{g}\left(N, \frac{\partial_t}{\sqrt{\beta}}\right)^2 = 1 - \cos^2(\theta).$$

Therefore, θ is the acute angle formed between the normal N and the vector field ∂_t.

3. On Monotone Orthogonally Splitted Riemannian Manifolds

Now, we are in conditions to establish the main result in this paper. For this, we will need first to introduce the following notion:

Definition 3. Let $(\overline{M}, \overline{g})$ be an Orthogonally Splitted Riemannian manifold (O-S Riemannian manifold for short). We will say that the Riemannian manifold is *non-shrinking* (resp. *expanding*) *throughout* ∂_t if

$$\partial_t \beta \geq 0 \quad \text{and} \quad (\mathcal{L}_{\partial_t} g_t)(X, X) \geq 0 \ \ (\text{resp.} > 0). \tag{17}$$

The notions of *non-expanding* and *shrinking* are given analogously with the corresponding change in the direction of the inequalities. We will say that the O-S Riemannian manifold is *monotonic* if it is either non-shrinking or non-expanding.

We will be interested specially in compact submanifolds, both with and without boundary, but our results are also applicable with little modifications for (open complete) *parabolic* submanifolds: Let us recall (see [19]) that a submanifold S is said parabolic if it is complete and it admits no function $f \in C^\infty(S)$ bounded from below and *superharmonic* (i.e., satisfying $\Delta f \leq 0$) but the constants.

Hence, our result reads as follows:

Theorem 4. *Let $(\overline{M}, \overline{g})$ be an Orthogonally Splitted manifold, and let S be a compact n-dimensional submanifold on $\overline{M}$. Assume that*

(i) *$(\overline{M}, \overline{g})$ is non-expanding (resp. non-shrinking).*
(ii) *The mean curvature vector $\vec{H}$ of S points in the opposite (resp. same) direction of ∂_t when is different from zero.*

Then, there exists $t_0 \in I$, such that $S \subset \{t_0\} \times F$.

Proof. Let us proof the non-expanding case, as the non-shrinking will be completely analogous. Observe that, from the definition of non-expanding (17), it follows that:

$$\partial_t(\ln \beta) \leq 0, \quad \operatorname{tr}(\xi|_{T_pS}) = \sum_{i=1}^{n} (\mathcal{L}_{\partial_t} \overline{g})(E_i, E_i) \leq 0, \tag{18}$$

being $\{E_i\}$ an orthonormal base of T_pS. Moreover, from (ii), it also follows that $g(\vec{H}, \partial_t) \leq 0$. Therefore, from (16), we deduce that $\Delta\tau \leq 0$. As S is compact, it follows that τ is constant and the result follows. □

Previous result is quite general, and establishes the behaviour of compact submanifolds in monotonic O-S Riemannian manifolds. The result can be seen somewhat technical, but it has interesting consequences under more restricted hypothesis. For instance:

Theorem 5. *In a monotone O-S Riemannian manifold, every compact minimal submanifold must be contained in a level hypersurface $\{t_0\} \times F$, for some $t_0 \in I$. Moreover, in the case of codimension higher than one, S must be a minimal submanifold of (F, g_{t_0}).*

Proof. Observe that, as S is a minimal compact manifold, condition (ii) is satisfied; hence, $S \subset \{t_0\} \times F$ for some $t_0 \in I$. The last assertion is just an application of the Koszul formula in this context. $\square$

Remark 6. There are some observations worth mentioning:

(a) Both previous results can be adapted for parabolic submanifolds. In fact, Theorem 4 follows for parabolic manifolds once we include the additional assumption:

(*iii*) The function $\tau : S \to I$ is bounded from below or from above.

(b) As we can see from (16), if τ is constant and $\vec{H} = 0$, then

$$0 = \mathrm{tr}(\xi|_{T_pS}) = \sum_{i=1}^{n} (\mathcal{L}_{\partial_t} g_t)(E_i, E_i),$$

for some $\{E_i\}_i$ orthonormal basis of T_pS; and all $p \in S$. As we are in a monotone O-S Riemannian manifold, it follows then that all the terms in such a sum has to be zero, and so that ξ is a degenerate $(0, 2)$-tensor over S.

(c) In particular, we have some criteria to decide whether or not a hypersurface can contain a minimal submanifold. A necessary condition for a hypersurface to contain a minimal submanifold of dimension n is to contain points p, so the tensor ξ is degenerate in p with the dimension of its radical at least n.

(d) Finally, let us recall that previous result gives a complete classification of compact minimal *hypersurfaces*. In fact, to have S a compact minimal hypersurface, then necessarily $S = \{t_0\} \times F$ for some t_0 (and so, F has to be compact). Moreover, in that case, the hypersurface must be totally geodesic, as it follows from the following general formula:

$$\overline{g}\left(\overline{\nabla}_X \frac{\partial_t}{\sqrt{\beta}}, Y\right) = \frac{1}{2\sqrt{\beta}} (\mathcal{L}_t g_t)(X, Y),$$

for all X, Y tangent vectors to F.

We move now to the case where S is a compact submanifold *with boundary*. In this case, adapted arguments allow us also to obtain results in the case where the boundary ∂S belongs to a slice $\{t_0\} \times F$.

Theorem 7. *Let* $(\overline{M}, \overline{g})$ *be a non-expanding* (*resp. non-shrinking*) *O-S Riemannian manifold; and let S be a compact submanifold with boundary. Assume that*

(i) $(\overline{M}, \overline{g})$ *is non-expanding* (*resp. non-shrinking*).
(ii) The mean curvature vector $\vec{H}$ *of S points in the opposite* (*resp. same*) *direction of* ∂_t *when is different from zero.*
(iii) $\partial S \subset \{t_0\} \times F$.
(iv) $\min_S(\tau) \geq t_0$ (*resp.* $\max_S(\tau) \leq t_0$).

Then, S is fully contained in $\{t_0\} \times F$.

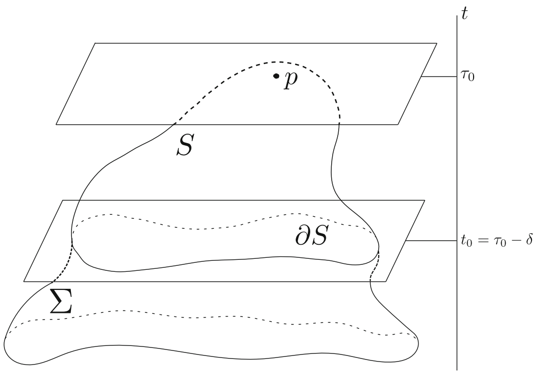

Figure 1. According to Corollary 8, there are no minimal hypersurfaces with this shape in a expanding O-S Riemannian manifold

Proof. We will consider $(\overline{M}, \tilde{g})$, where $\tilde{g}$ is the conformal metric defined for (12). Consider the vector field $V = (\tau - t_0)\tilde{\nabla}\tau$, where $\tilde{\nabla}$ denotes the gradient computed with $\tilde{g}$. Using the divergence theorem on S, and recalling that $\tau = t_0$ in ∂S, it follows that:

$$0 = \int_S \operatorname{div}\left((\tau - t_0)\tilde{\nabla}\tau\right) = \int_S \left(\left|\tilde{\nabla}\tau\right|^2 + (\tau - t_0)\tilde{\Delta}\tau\right). \tag{19}$$

Now, observe that $(\tau - t_0)\,\tilde{\Delta}\tau \geq 0$ thanks to the hypotheses (i),(ii) and (iv). Therefore, τ should be constant and the result follows. $\square$

Previous result allows us even to obtain information for the shape of minimal hypersurfaces S under some mild hypothesis. Concretely:

Corollary 8. *Assume $(\overline{M}, \overline{g})$ be an non-shrinking (resp. non-expanding) O-S Riemannian manifold. Let Σ be a minimal hypersurface. Then, the function τ on Σ does not attain a strict maximum value (resp. strict minimum value).*

Proof. We will consider the expanding case, being the shrinking one completely analogous. Assume by contradiction that there exists Σ not satisfying our conclusion. Let us say that τ_0 is a maximum value of τ at $p \in \Sigma$. We have that, for certain $\delta > 0$ small enough, there exists a simply connected compact oriented subset S of Σ containing p and whose boundary lies in $t_0 := \tau_0 - \delta$. Inside this subset, $\tau \geq t_0$ (see Fig. 1 for a visual representation). We can now apply Theorem 7 to get a contradiction. $\square$

Finally, let us recall that we can extend conclusion of Theorem 4 in the following way: Consider $(\overline{M}, \overline{g})$ a Riemannian manifold with

$$\overline{M} = I_2 \times I_1 \times F, \quad \overline{g} = \beta_2(t_1, t_2, x)\mathrm{d}t_2^2 + \beta_1(t_1, x)\mathrm{d}t_1^2 + g_{t_1,t_2}, \tag{20}$$

where I_j, with $j = 1, 2$, are two open intervals on $\mathbb{R}$, $\beta_j : \left(\Pi_{i=1}^{j} I_i\right) \times F \to \mathbb{R}$ are smooth functions and g_{t_1,t_2} is a smooth 2-parametric family of Riemannian metrics defined over F. Previous metric can be seen as a O-S Riemannian manifold if we write it like $\overline{M} = I_2 \times (I_1 \times F) = I_2 \times \tilde{F}$. Hence, under the assumptions of Theorem 4, any minimal compact submanifold S of $\overline{M}$ should be contained in $t_2^0 \times \tilde{F}$ for some constant t_2^0. In particular, we can see S as a submanifold of $\tilde{F} = I_1 \times F$.

If S has codimension bigger than 1, and assuming that the O-S Riemannian manifold $\tilde{F} = I_1 \times F$ is also in the conditions of Theorem 4, we will obtain that S should be contained in some $\{t_1^0\} \times F$ for some constant t_1^0. As a direct consequence, S is contained in $\{t_2^0\} \times \{t_1^0\} \times F$.

Observe that the only required assumption is the non-expanding/non-shrinking character of previous O-S Riemannian manifolds. In the first case, we need that both $\partial_{t_2}\beta_2$ and $\mathcal{L}_{\partial_{t_2}}\left(\mathrm{d}t_1^2 + g_{t_1,t_2}\right) = \mathcal{L}_{\partial_{t_2}} g_{t_1,t_2}$ share the same monoticity (either both are non-negative or non-positive). Analogously, we require that both $\partial_{t_1}\beta_1$ and $\left(\mathcal{L}_{\partial_{t_1}} g_{t_1,t_2}\right)$ share the same monoticity.

Summing up, and recalling that previous process can be repeated for a finite number of times, we obtain the following:

Corollary 9. *Let $(\overline{M}, \overline{g})$ be a Riemannian manifold with*

$$\overline{M} = \left(\Pi_{j=1}^{k} I_j\right) \times F, \quad \overline{g} = \left(\sum_{j=1}^{k} \beta_j \mathrm{d}t_j^2\right) \times g_{t_1,\ldots,t_k}, \tag{21}$$

where, for $j = 1, \ldots, k$, I_j is an interval of $\mathbb{R}$, $\beta_j : \left(\Pi_{i=1}^{j} I_i\right) \times F \to \mathbb{R}$ is a smooth function and $g_{t_1,\ldots,t_k}$ is a smooth k-parametric family of Riemannian metrics defined over F. Let S be a compact minimal submanifold on $\overline{M}$ of codimension bigger or equal than k. If, for all $j \in \{1, \ldots, k\}$, it follows that $\partial_{t_j}\beta_j$ and $\left(\mathcal{L}_{\partial_{t_j}} g_{t_1,\ldots,t_k}\right)$ share the same monoticity, then $S \subset \{t_1^0\} \times \cdots \times \{t_k^0\} \times F$ for some constants $t_j^0 \in I_j$.

We can particularize this corollary to the case where even F is an interval of the real line. Hence, we obtain the following generalization of the classical non-existence result for compact minimal surface in the Euclidean space.

Corollary 10. *Let $(\overline{M}, \overline{g})$ be a Riemannian manifold with $\overline{M} = \Pi_{j=1}^{n} I_j$ and $\overline{g} = \sum_{j=1}^{n} \beta_j \mathrm{d}x_j^2$, where I_j are real intervals and $\beta_j : \left(\Pi_{i=1}^{j} I_i\right) \to \mathbb{R}$ are smooth functions. If $\partial_{x_j}\beta_j$ and $\partial_{x_j}\beta_n$ have the same monotony for all j, then there exists no compact minimal submanifold $S \subset M$.*

Proof. Assume by contradiction that there exists a compact minimal submanifold S with codimension $m < n$. Then, by previous corollary, S is contained in $\{x_1^0\} \times \cdots \times \{x_m^0\} \times \mathbb{R}^{n-m+1} (\equiv \mathbb{R}^{n-m+1})$ for some constants $x_i^0 \in I_i$. However, this is a contradiction, as there exists no compact minimal submanifold in the Euclidean space. □

3.1. O–S Riemannian Manifolds with Change in the Monotonic Behaviour

Previous computations allow us even to obtain results in cases even where the monotonic behaviour is non-constant. Concretely, we are able to prove the following result.

Theorem 11. *Let $(\overline{M}, \overline{g})$ be a O-S Riemannian manifold, and let S be a compact n-dimensional submanifold. Assume that the following conditions hold (compare with Theorem 4):*

(i′) *There exists $t_1 \in I = (a, b)$, so $((a, t_1) \times F, \overline{g})$ is non-expanding and $((t_1, b) \times F, \overline{g})$ is non-shrinking.*

(ii′) *The mean curvature vector $\vec{H}$ satisfies that for all $p \in S$, $(\tau(p) - t_0)g(\vec{H}, \partial_t)|_p \geq 0$.*

Then, there exists t_0, so $S \subset \{t_0\} \times F$.

Proof. From the hypothesis (i) and (ii), and Eq. (16), it follows that:

$$(\tau - t_1)\,\tilde{\Delta}\tau \geq 0. \tag{22}$$

In particular, and recalling that $\tilde{\Delta}\,(\tau - t_1)^2 = 2\left(\left|\tilde{\nabla}\tau\right|^2 + (\tau - t_1)\,\tilde{\Delta}\tau\right)$, it follows that τ^2 is superharmonic. Hence, due the fact that, from hypothesis, S is compact, τ is constant, and the result follows. □

As in Theorem 4, the previous result has a much more appealing consequence once we restrict to compact minimal submanifolds.

Theorem 12. *Assume that $(\overline{M}, \overline{g})$ is a O-S Riemannian manifold satisfying the condition (i′) in the previous theorem; and assume that S is a minimal compact submanifold. Then, $S \subset \{t_0\} \times F$ for some t_0. Moreover, if S has codimension bigger than one, then S is a minimal submanifold of $\{t_0\} \times F$*

Remark 13. In analogy with previous cases, we can also obtain results for compact submanifolds with boundary for O-S Riemannian manifolds with this appropriate change on its monotonic behaviour. Concretely, we can re-obtain Theorem 7 in this context by substituting conditions (i) and (ii) with (i′) and (ii′).

3.2. A Note on Compact Hypersurfaces

It is worth mentioning that any compact hypersurface in a Riemannian manifold $(\overline{M}, \overline{g})$ can be seen as a minimal hypersurface in an appropriate conformal metric. For this, let us recall that if we consider $\tilde{g} = e^{2\varphi}\overline{g}$, we have the following relation between the mean curvatures of an hypersurface S:

$$e^{\varphi}\widetilde{H} = H + \overline{g}(\overline{\nabla}\varphi, N), \tag{23}$$

where $\widetilde{H}$ and H represent the mean curvature of S computed with $\widetilde{g}$ and $\overline{g}$, respectively. Our aim is then to find φ, so $\widetilde{H}$ is zero. For this, we will make use of the following technical lemma:

Lemma 14. *Let $(\overline{M}, \overline{g})$ be a Riemannian manifold and S a compact hypersurface. For each function $h \in C^{\infty}(S)$, there exists a function in the ambient manifold, α, such that*

$$\nabla\alpha|_S^{\perp} = hN. \tag{24}$$

Proof. The compactness of S allows to take an open interval J, such that the geodesics $\gamma_p(s)$ starting at $p \in S$ and satisfying $\gamma_p'(0) = N(p)$ are defined for all $p \in S$ and for all $s \in J$.

Therefore, we can consider $\phi : J \times S \to U$ a bijective map given by $\phi(s,p) = \gamma_p(s)$, being U a tubular neighbourhood of S. On U, we define the function $\bar{h}$ by $\bar{h} = h \circ \pi_S \circ \phi^{-1}$, being the projection $\pi_S : J \times S \to S$. That is, $\bar{h}$ is constant along the geodesics $\gamma_p(s)$, and on S, it coincides with h.

Consider now the smooth function $\left(\pi_J \circ \phi^{-1}\right)$, where π_J denotes the canonical projection of the first factor of $J \times S$, and define the function $\alpha : U \to \mathbb{R}$ given by $\alpha = \bar{h} \cdot (\pi_J \circ \phi^{-1})$. From definition, the normal component of the gradient $\overline{\nabla}\, \alpha_{|_{S^\perp}} = hN$, since $\overline{g}(N, \overline{\nabla}\alpha) = h$.

Now, let ξ be a function on $I \times F$, such that $0 \leq \phi(p) \leq 1$, for all $p \in I \times F$, and which satisfies (see Corollary in Section 1.11 of [30]),

- (i) $\xi(p) = 1$ if $p \in \{\gamma_t(p) : t \in J', p \in S\}$, being $J' \subset J$ an closed interval with $0 \in J'$.
- (ii) $\operatorname{supp} \xi \subset U$.

The function ξ can be employed to extend α on the entire $\overline{M}$. □

We can then define a function φ given by previous Lemma considering $h = -H$ the mean curvature of S computed with $\overline{g}$. Then, from (23), it follows that S is minimal for the conformal metric $\tilde{g} = e^{2\varphi}\overline{g}$. In conclusion, we have proved the following:

Theorem 15. *Let S be a compact hypersurface for a Riemannian manifold $(\overline{M}, \overline{g})$. Then, there exists a function φ, so S is a minimal compact hypersurface for the Riemannian metric $(\overline{M}, e^{2\varphi}\overline{g})$.*

4. Uniqueness Results in Riemannian Manifolds with (Infinitesimal) Symmetries

Let us assume now that $(\overline{M}, \overline{g})$ is a Riemannian manifold admitting a globally defined conformal Killing vector field K, that is, a non-zero vector field K, so

$$\mathcal{L}_K \overline{g} = \Omega\, \overline{g}, \tag{25}$$

for some function Ω. If $\Omega \equiv 0$, we will just say that K is *Killing* vector field.

As it is well established by the extensive literature on the topic, if the vector field fulfills some regular assumptions, then we can get a topological and geometrical description of the Riemannian manifold. Let us recall the main arguments here for such a description. We will follow the arguments leading to [26, Proposition 1], even so our result will be slightly more general.

It is quite straightforward to see that the conformal Killing vector field K is a *Killing* vector field for $\tilde{g} = g/g(K,K)$. Now, observe that, if we assume that the vector field K is *irrotational*, then the Frobenius theorem asserts that the orthogonal distribution of the vector field K is integrable. Hence, for certain Σ open set of an integral leaf of $K^\perp$, $\overline{M}$ is locally isometric to $\big((a,b)\times\Sigma, \mathrm{d}t^2 + g_\Sigma\big)$. Moreover, it can be easily proved that the vector field K is locally a gradient vector field.

Observe that if M is simply connected, Poincaré's lemma ensures that K is globally a gradient vector field. Let $l \in C^\infty(\overline{M})$ be a function, so $dl = \omega$, being ω the one form metrically equivalent to K.

Finally, let us denote by $\phi(t,p)$ the global flow of K. From construction, it follows that $\frac{\mathrm{d}}{\mathrm{d}t} l(\phi(t,p)) = 1$; thus, the integral curves of K cross each leaf of $K^\perp$ only one time. Hence, if we assume that the vector field K is complete, we have that the map

$$\phi : \mathbb{R} \times P \to (\overline{M}, \tilde{g}),$$

is an isometry. In conclusion, we have arrived to the following:

Proposition 16. *Let $\overline{M}$ be a Riemannian manifold which admits an irrotational nowhere zero conformal Killing vector field K. If $\overline{M}$ is simply connected and K is complete, then M is globally isometric to*

$$(\mathbb{R} \times P, h\left(\mathrm{d}t^2 + g_P\right)), \tag{26}$$

where $h = g(K,K)$, P is a leaf of the foliation $K^\perp$, and g_P is a Riemannian metric over P.

We can then deduce that, under the assumptions of previous proposition, $(\overline{M}, \overline{g})$ is an O-S Riemannian manifold with $\beta = h$ and $g_t = h\, g_P$. In particular, the variation along t of both β and g_t will depend on the evolution of the norm of K over its flow.

With all previous observations, we are in conditions to prove the following result (which is a consequence of Theorem 5):

Theorem 17. *Let $(\overline{M}, \overline{g})$ be a simply connected Riemannian manifold admitting a complete, globally defined conformal Killing vector field K. Assume that the norm of K along its flow is monotonic (either non-decreasing or non-increasing), then any minimal compact submanifold has to be contained in a leaf of the foliation $K^\perp$.*

Remark 18. Observe that we can also obtain a result under the assumption that the norm of K changes its monotonic behaviour appropriately along its flow and using Theorem 11.

In the particular case that K is Killing, the norm of K is invariant over the flow of K, and so, it is always monotonic. Hence, we have the following:

Corollary 19. *In a simply connected Riemannian manifold containing a globally defined Killing vector field K, any minimal compact submanifold is contained in a leaf of the foliation $K^\perp$ orthogonal to K.*

Let us finish this section making a final observation regarding the simply connectedness of $\overline{M}$. Assume, for instance, that $\overline{M}$ is not simply connected and consider $\tilde{M}$ its universal covering (denoting by $\pi : \tilde{M} \to \overline{M}$ the covering map). If the submanifold S is simply connected, we have that there exists a unique immersion $\tilde{x} : S \to \tilde{M}$ satisfying that S is minimal if, and only if, $\tilde{S} := \tilde{x}(S)$ is. Hence, the simply connectedness assumption can be assumed in the previous theorem, not in $\overline{M}$, but on S.

Corollary 20. *Let M be a complete Riemannian manifold admitting a complete, globally defined conformal Killing vector field K. Assume that the norm of K along its flow is monotonic. Then, any minimal simply connected submanifold must be contained in a leaf of the foliation $K^{\perp}$.*

5. Non-existence Results for Minimal Compact Submanifolds

The results in Sect. 3 allow us to obtain non-existence results in different contexts. We will focus on minimal submanifolds in this section, even so Theorem 4 allows us to obtain restrictions on the geometric behaviour of submanifolds due the information provided by its mean curvature vector.

First, observe that, as we have mentioned in Remark 6, we require that $(\mathcal{L}_{\partial_t} g_t)$ to be singular in F to ensure the existence of minimal compact submanifolds. Hence, if we assume that the O-S Riemannian manifold is *strictly* monotone (either expanding or shrinking), we know from definition that $(\mathcal{L}_{\partial_t} g_t)$ is definite positive or negative. Therefore:

Corollary 21. *In a strictly monotone O-S Riemannian manifold, there exist no compact minimal submanifolds.*

Moreover, under the assumptions of Theorem 5, we ensure that any minimal compact submanifold in a O-S Riemannian manifold $(\overline{M}, \overline{g})$ should be contained in a hypersurface orthogonal to some vector field (represented in (1) by ∂_t). If $(\overline{M}, \overline{g})$ admits a second orthogonal splitting with respect another vector field (let us denote this vector field by $\partial_{t'}$) which also satisfies the assumptions of such a theorem, then any minimal compact submanifold on $\overline{M}$ should also be contained in the hypersurface orthogonal to $\partial_{t'}$. If both ∂_t and $\partial_{t'}$ are not collinear in an open set U, the intersection of any minimal compact submanifold with U should be contained in the intersection of two hypersurfaces in $\overline{M}$; hence, we obtain a restriction over the dimension of such a manifold.

In particular, and due to the fact that previous arguments can be done for a finite number of orthogonal decompositions, we arrive to the following result:

Corollary 22. *Let $(\overline{M}, \overline{g})$ be a Riemannian manifold admitting an orthogonal decomposition as in (1) through q vector fields. Assume that such decompositions are monotonic and that there exists an open set U where all q vector fields are non-collinear. Then, there exist no compact minimal submanifolds of dimension bigger than $n + m - (q - 1)$ intersecting the open set U.*

Now, we will move to a different scenario by considering $(\overline{M}, \overline{g})$ a Cartan–Hadamard manifold. Let us recall that a Cartan–Hadamard manifold (see for instance [24]) is a simply connected complete Riemannian manifold with non-positive sectional curvature. As it is well known, in a Cartan–Hadamard manifold, for a given point $p \in \overline{M}$, the exponential map $\exp_p$ is a global diffeomorphism, and we can define ∂_r the *radial polar* vector field defined over $\overline{M} \setminus \{p\}$.

For any two non-collinear vectors $v, w \in T_p\overline{M}$, let us define

$$\Pi(v,w) := \left\{\exp_p(u) : u \in \mathrm{Span}\left\{v,w\right\}, u \neq 0\right\} \subset \overline{M} \setminus \{p\}. \tag{27}$$

Observe that $\Pi(v,w)$ has non-positive Gauss curvature when endowed with the induced metric from $\overline{g}$. Let us represent, by an abuse of notation, $(\Pi(v,w), \overline{g}_{\Pi(v,w)})$ as $((0,\infty), \mathrm{d}r^2 + f(r,\theta)^2 d\theta^2)$. Let us consider a point $q \in \Pi(v,w)$ obtained as $q = \exp_p(u)$. Now, let us take $T_q\Pi(v,w) = \mathrm{Span}(u_1,u_2)$, and recall that the Gauss curvature K of $\Pi(v,w)$ and the sectional curvature $\tilde{K}$ of $(\overline{M},\overline{g})$ for the plane $T_q\Pi(v,w)$ are related by

$$K = \tilde{K} - \frac{\overline{g}(\mathrm{II}(u_1,u_2),\mathrm{II}(u_1,u_2)) - \overline{g}(\mathrm{II}(u_1,u_1),\mathrm{II}(u_2,u_2))}{|u_1|^2 |u_2|^2 - \overline{g}(u_1,u_2)^2}. \tag{28}$$

Now, observe that we can always take u_1 so $\mathrm{II}(u_1,u_1) = 0$, so $K \leq \tilde{K}$. Recalling then the expression for $\overline{g}_{\Pi(v,w)}$, we have that

$$-\frac{\partial_{rr} f(r,\theta)}{f} = K \leq \tilde{K} \leq 0, \tag{29}$$

and so, $\partial_{rr} f(r,\theta) > 0$. Hence, if $\partial_r f(r_0,\theta) \geq 0$ for some r_0, then $\partial_r f(r,\theta) > 0$ for all $r > r_0$. Moreover, as $f(r,\theta) > 0$ and $\lim_{r\to 0} f(r,\theta) = 0$, it cannot exist $\epsilon > 0$, so $\partial_r f(r,\theta) < 0$ for all $r \in (0,\epsilon)$; thus, $\partial_r f(r,\theta) > 0$ for all r.

Therefore, $(\Pi(v,w), \overline{g}_{\Pi(v,w)})$ is an expanding O-S Riemannian manifold. As this happens for any $v, w \in T_pM$, it follows that the entire $(\overline{M},\overline{g})$ is expanding, so by virtue of Corollary 21, we have proved the following.

Corollary 23. *A Cartan–Hadamard Riemannian manifold admits no compact minimal submanifold.*

Moreover, reasoning as the paragraph leading to Corollary 20, we can move the simply connectedness from the ambient space to the submanifold, obtaining also the following result:

Corollary 24. *A Riemannian manifold with non-positive curvature admits no simply connected compact minimal submanifold.*

Remark 25. Observe that the previous results admit local versions. In particular, reasoning as before, we have that no minimal compact submanifold can be contained in a neighbourhood of a Riemannian manifold where all the points have non-positive sectional curvature.

Finally, let us observe that our results are also applicable for Riemannian manifolds with constant sectional curvature (non-necessarily non-positive). Let us take a simply connected Riemannian manifold $(\overline{M},\overline{g})$ with constant sectional curvature and a point $p \in \overline{M}$. As it is well known, the Riemannian manifold obtained by removing p from $\overline{M}$ is isometric to one of the following models (for k a positive scalar):

(i) $\mathbb{R}^{n+m+1} - \{p\} \equiv \left((0,\infty) \times \mathbb{S}^{n+m}, \mathrm{d}r^2 + r^2 g_{\mathbb{S}^{n+m}}\right)$, for zero sectional curvature.

(ii) $\mathbb{H}^{n+m+1}(-k)-\{q\} = \left((0,\infty)\times\mathbb{S}^{n+m}, \mathrm{d}r^2 + \frac{1}{\sqrt{k}}\cosh^2(\sqrt{k}r)g_{\mathbb{S}^{n+m}}\right)$, for negative sectional curvature.

(iii) $\mathbb{S}^{n+m+1}(k)-\{n,s\} = \left((0,\frac{1}{\sqrt{k}})\times\mathbb{S}^{n+m}, \mathrm{d}r^2 + \frac{1}{\sqrt{k}}\sin^2(\sqrt{k}r)g_{\mathbb{S}^{n+m}}\right)$ for positive sectional curvature, where n, s denotes the north and south pole on the sphere.

As we can see, both (i) and (ii) are expanding O-S Riemannian manifolds. Hence, they fall under Corollary 21, and so, they contain no compact submanifolds.

Now, take $(\overline{M},\overline{g})$ a manifold with negative or zero constant sectional curvature and a compact minimal submanifold S on it. By removing from $\overline{M}$ a point $p\notin S$, we can isometrically embed S onto either (i) or (ii), which is a contradiction.

The situation with (iii) is quite different as such a O-S Riemannian manifold is not monotonic. However, if we restrict ourselves to the *half* of such a manifold, then we will be again in the hypothesis of Corollary 21. Moreover, as this manifold is compact, we will focus in this case only on compact submanifolds.

In conclusion, we can obtain the following result (compare with [6])

Corollary 26. *No simply connected Riemannian manifold with non-positive constant sectional curvature admits a compact minimal submanifold. In a round sphere, there exists no compact minimal submanifold contained in an open ball of radius the half of its diameter.*

Proof. The first assertion has been proved in previous paragraphs, so we will focus on the second one. Let S be a minimal compact submanifold on $\overline{M}$, and take $p\in\overline{M}$ a point such $S\subset B(p,\mathrm{diam}(\overline{M})/2)$, where $\mathrm{diam}(\overline{M})$ denotes the diameter of $(\overline{M},\overline{g})$.

Observe that we can always consider p, so it is not contained in S. In fact, if $p\in S$, we can consider a sequence $\{q_i\}_i\to p$ with all $q_i\notin S$. As S is compact, there will be some i_0 big enough, so $S\subset B(q_{i_0},\mathrm{diam}(\overline{M})/2)$.

Once we have that $p\notin S$, reasoning again as in previous paragraphs, we can isometrically embed S onto a half sphere, where Corollary 21 is applicable and the result follows. □

Remark 27. *Let us make some observations regarding this last result:*

(a) For a round 2-sphere, observe that any closed geodesic is a minimal hypersurface of the sphere. These geodesics, seen as minimal submanifolds of the sphere, are nice counterexamples to see that our kind of assumptions are needed.

(b) On the other hand, some topological assumption is necessary, as the simply connectedness. In fact, in the torus T^3, there are compact minimal submanifolds.

5.1. A Geometric Obstruction for the Existence of Minimal Submanifolds

The results on previous sections depend essentially on the expanding or shrinking condition for the O-S Riemannian manifold. In particular, we require that $\mathcal{L}_{\partial_t}g_t$ is either positive or negative definite. We can, however, relax this condition to just semi-definite and obtain a geometric restriction in the acute angle associated with any compact minimal submanifold.

Let us consider $(\overline{M}, \overline{g})$ a O-S Riemannian manifold with function $\beta \equiv 1$, i.e., $\overline{M} = I \times F$ and $\overline{g} = \mathrm{d}t^2 + g_t$. Consider also S a compact minimal submanifold on S; and define as $\eta = \overline{\mathrm{div}}(\partial_t)$. Observe that, from Eq. (9), if $\mathcal{L}_{\partial_t} g_t$ is semi-definite, then η is signed (and both with the same sign).

We are now in conditions to prove the following:

Theorem 28. *Let $(\overline{M}, \overline{g})$ be a O-S Riemannian manifold with $\beta \equiv 1$. Assume that* (i) *$\mathcal{L}_{\partial_t} g_t$ is semi-definite and* (ii) *η satisfies $\partial_t \eta \geq \sigma \left|\nabla \eta\right|_{g_t}$, for some $\sigma \in \mathbb{R}^+$. Then, there exists no compact minimal submanifold with acute angle function satisfying* $\tan(\theta) \geq \sigma^{-1}$.

Proof. Assume by contradiction that there exists S a compact minimal submanifold with acute angle satisfying that $\tan(\theta) \geq \sigma^{-1}$, and define η as before. Let us define a vector field $Y = \eta \partial_t^T$, which satisfies (recall (6)

$$\overline{\mathrm{div}}(Y) = \partial_t^T \eta + \eta \Delta \tau. \tag{30}$$

Now, let us observe that the vector field ∂_t^T can be orthogonally splitted between ∂_t and a unitary vector u satisfying that $\overline{g}(u, \partial_t) = 0$. In particular, and using the acute angle θ as defined in (15)

$$\partial_t^T = \sin^2 \theta \, \partial_t + \sin \theta \cos \theta \, u. \tag{31}$$

Hence, previous expression of the divergence can be expressed as

$$\overline{\mathrm{div}}(Y) = \sin^2 \theta \left(\partial_t \eta + \frac{1}{\tan(\theta)} u(\eta) \right) + \eta \Delta \tau.$$

As $\tan(\theta) \geq \sigma^{-1}$, the (ii) hypothesis ensures that the first term on the right is positive. Moreover, when (i) is satisfied both $\Delta \tau$ and η have the same sign (recall that $\beta \equiv 1$ and Eqs. (9) and (10)). In conclusion, $\overline{\mathrm{div}}(Y) \geq 0$. However, if we apply the divergence theorem, it follows then that $\eta \Delta \tau = 0$. Observe that, as $\mathcal{L}_{\partial_t} g_t$ is semi-definite, then $\eta \neq 0$. Hence, $\Delta \tau = 0$ and τ is an harmonic function over a compact set. But then, τ is constant and $\theta = 0$, a contradiction with the assumption for the acute angle of S. □

Remark 29. Observe that in the Euclidean case, $\eta = \overline{\mathrm{div}}(\partial_t) = 0$, so the hypothesis of previous theorem are satisfied for all $\sigma > 0$ and no restriction on the acute angle is imposed. In particular, the previous theorem is the extension of the classical well-known result ensuring that there is no compact minimal submanifold in the Euclidean case.

6. Application to Geometric Analysis

6.1. Setting Up

Let us focus from now on the study of submanifolds S of an O-S Riemannian manifold $(\overline{M}, \overline{g})$ that can be seen as the graph of certain function $u \in C^\infty(U)$, being $U \subset F$ some suitable subset. That is

$$S \equiv S_u := \left\{ (u(p), p) : x \in U \subset F \right\}. \tag{32}$$

The graph of u inherits a metric (of the ambient), represented on F , by

$$g_u(p) = \beta(u(p), p)\mathrm{d}u^2 + g_{u(p)}.$$

Note that $t(u(x), x) = \pi_I(u(x), x) = u(x)$, for all $x \in U$, and so, on the graph, τ and u can be naturally identified.

First, we will assume that U is an open subset of F. A normal vector N to S_u can be explicitly computed, obtaining

$$N = \frac{1}{\sqrt{\frac{1}{\beta} + |\nabla u|^2}} \left(-\nabla u + \frac{1}{\beta}\partial_t \right) \equiv N^F + \overline{g}\left(N, \frac{\partial_t}{\sqrt{\beta}} \right) \frac{\partial_t}{\sqrt{\beta}},$$

$$N^F = -\frac{1}{\sqrt{\frac{1}{\beta} + |\nabla u|^2}} \nabla u, \tag{33}$$

where ∇u denotes here the gradient of u in F with the induced metric. The mean curvature $H = \overline{g}(\vec{H}, N)$ associated to S_u can be computed, obtaining

$$\begin{aligned}(nH(u) \equiv)\ \ nH &= \overline{\mathrm{div}}\, N = \mathrm{div}\left(N^F\right) + \overline{g}\left(\overline{\nabla}_{\frac{\partial_t}{\sqrt{\beta}}} N^F, \tfrac{\partial_t}{\sqrt{\beta}} \right) + \overline{\mathrm{div}}\left(\overline{g}(N, \tfrac{\partial_t}{\sqrt{\beta}}) \tfrac{\partial_t}{\sqrt{\beta}} \right) \\ &= \mathrm{div}\left(\frac{\nabla u}{\sqrt{\frac{1}{\beta}+|\nabla u|^2}} \right) + \overline{g}\left(\frac{\nabla u}{\sqrt{\frac{1}{\beta}+|\nabla u|^2}}, \tfrac{1}{2}\nabla \log \beta \right) \\ &\quad + \overline{g}(N, \tfrac{\partial_t}{\sqrt{\beta}}) \tfrac{1}{\sqrt{\beta}} \partial_t \log \mathrm{vol}_{slice}.\end{aligned} \tag{34}$$

Previous expression motivates the following definitions.

Definition 30. Let F be a manifold and consider $\overline{g}$ be a O-S Riemannian metric associated with the product manifold $\mathbb{R} \times F$. Consider an open set $U \subseteq F$ and let us define the following non-linear elliptic differential operator:

$$\begin{aligned}(\Theta(u) =)\Theta(u, \overline{g}) := \mathrm{div}\left(\frac{\nabla u}{\sqrt{\frac{1}{\beta} + |\nabla u|^2}} \right) + g_{u(p)}\left(\frac{\nabla u}{\sqrt{\frac{1}{\beta} + |\nabla u|^2}}, \frac{1}{2}\nabla \log \beta \right) \\ + \frac{1}{\sqrt{1 + \beta\, |\nabla u|^2}} \frac{1}{\sqrt{\beta}} \partial_t \log \mathrm{vol}(t),\end{aligned} \tag{35}$$

where $\mathrm{vol}(t)$ represents the volume of F computed with the metric g_t, and ∇ is the gradient computed in F with the induced metric from $\overline{g}$. We will say that a function u is $\Theta-$*harmonic* if $\Theta(u) = 0$.

As it is clear from previous computation, the graph of a Θ-harmonic function will be minimal, and so, the results on previous sections are applicable. Moreover, and as we will see later, we will be able to extend our results even further by considering an appropriate conformal change.

Let us consider some well-known examples where Θ can be explicitly obtained:

Example 31. (See, for instance, [26]). Let $I \times_f F^n$ be a warped product, where $f \in C^\infty(I)$. The *minimal hypersurface equation* on F, that is, $\Theta(u, \overline{g}) = 0$

where $\overline{g} = \mathrm{d}t^2 + f(t)g_F$ is

$$\Theta(u) = \operatorname{div}\left(\frac{\nabla^F u}{f(u)\sqrt{f(u)^2 + |\nabla^F u|^2}}\right) - \frac{f'(u)}{\sqrt{f(u)^2 + |\nabla^F u|^2}}\left\{n - \frac{|\nabla^F u|^2}{f(u)^2}\right\},$$

where ∇^F is the Levi-Civita connection of the Riemannian manifold $(F, g_{_F})$.

Example 32. Let $(I \times F, h^2\mathrm{d}t^2 + g_{_F})$, where $h \in C^\infty(I \times F)$ is a positive function. Then, the minimal hypersurface equation on F is given by

$$\Theta(u) = \operatorname{div}\left(\frac{h\,\nabla^F u}{\sqrt{1 + h^2\,|\nabla^F u|^2}}\right) + \frac{1}{\sqrt{1 + h^2\,|\nabla^F u|^2}}\, g_F\left(\nabla^F u, \nabla^F h\right).$$

Example 33. Let $(I \times F_{_1}^{n_1} \times F_{_2}^{n_2}, \mathrm{d}t^2 + f_{_1}^2 g_{F_1} + f_{_2}^2 g_{F_2})$, where $f_i : I \longrightarrow \mathbb{R}^+$ are two smooth functions. Following previous considerations, $\nabla u = \sum_{i=1}^2 \frac{1}{f_{_i}^2}\nabla^{F_i} u$, where ∇^{F_i} is the Levi-Civita connection of $(F_i, g_{_{F_i}})$, for $i = 1, 2$. Then, we have that the minimal hypersurface equation, on $F_1 \times F_2$, is

$$\sum_{i,j=1}^{2} \operatorname{div}_{F_j}\left(\phi \frac{1}{f_{_i}^2}\nabla^{F_i} u\right) = -\phi\left\{n_1(\log f_{_1})'(u) + n_2(\log f_{_2})'(u)\right\},$$

where

$$\phi^{-1} = \sqrt{1 + f_{_1}^2|\nabla^{F_1}u|^2 + f_{_2}^2|\nabla^{F_2}u|^2}.$$

The extension to a finite family of Riemannian manifolds follows easily.

Our aim is to use the relation between the mean curvature of the submanifolds S_u and the differential operator $\Theta(u, \overline{g})$ to obtain results for non-existence or uniqueness for solutions of the corresponding PDE problem on the latter. We will focus on two main cases: entire solutions, i.e., functions $u \in C^\infty(F)$ where F is assumed to be compact; and functions $u \in C^\infty(U)$ where U is the topological closure of a open pre-compact set in F.

6.2. First Case: *F* Compact and Without Boundary

Let us begin by assuming that F is compact, and u is a function defined over F. Theorem 5 applied to the compact hypersurface S_u is translated in this context as follows.

Theorem 34. *Consider F a smooth compact manifold and take $\overline{g}$ an O-S Riemannian metric defined over $\overline{M} = \mathbb{R} \times F$. Assume one of the following conditions:*

(i) *Either $(\overline{M}, \overline{g})$ is monotonic,*
(ii) *or there exists $t_1 \in \mathbb{R}$, so $\overline{g}$ is non-expanding in $(-\infty, t_1) \times F$ and non-shrinking in $(t_1, \infty) \times F$.*

Then, there exists no Θ-harmonic non-constant function $u \in C^\infty(F)$.

Proof. Let us assume that u is a $\Theta-$harmonic function. Consider S_u the hypersurface obtained as the graph of u on $\overline{M}$.

From (34) and the hypothesis, it follows that S_u satisfies that $\Theta(u) = H(u) = 0$. Hence, we are in conditions to apply or Theorem 5 if (i) is satisfied, or Theorem 11 in the other case. In both cases, it is ensured that $S_u \subset \{t_0\}\times F$ for some t_0, and so that the function u should be constant. □

The study of $\Theta-$harmonic functions is not the only application that we can obtain from the relation between the geometry of submanifolds on $(\overline{M}, \overline{g})$ and PDEs. In fact, we can also extend previous result in the following way:

Theorem 35. *Let F be a smooth manifold, and consider $\overline{g}$ a O-S Riemannian metric defined over $\overline{M} = \mathbb{R} \times F$. Take also $\alpha \in C^\infty(F)$ a smooth function defined over F. If $(\overline{M}, \overline{g})$ is monotonic, then the only solutions of the equation*

$$\Theta(u) = -g_u(N^F, \nabla\alpha) \tag{36}$$

are the constant functions.

Proof. Using the relation between Θ and the mean curvature of the submanifold S_u (34), it follows that the mean curvature of S_u should satisfy that:

$$H(u) = -g_u(N^F, \nabla\alpha). \tag{37}$$

Let us consider $\overline{\alpha}$ an extension of the function α to $\overline{M} = \mathbb{R} \times F$ given by $\overline{\alpha}(t, x) = \alpha(x)$. Let us now consider the metric $\widetilde{g} = e^{\overline{\alpha}}\overline{g}$, conformal to $\overline{g}$.

Now, if we denote by $\widetilde{H(u)}$ the mean curvature of S_u computed with $\tilde{g}$, it follows from (23) that:

$$e^{\overline{\alpha}}\widetilde{H(u)} = H(u) + \overline{g}(\overline{\nabla}\overline{\alpha}, N) = H(u) + g_u(\nabla\alpha, N^F), \tag{38}$$

being as usual $\overline{\nabla}$ the Levi-Civita connection of $\overline{g}$ and N the unitary normal to S_u. Hence, from (37), it follows that S_u is a minimal compact hypersurface in $(\overline{M}, \widetilde{g})$. Moreover, this O-S Riemannian manifold is also monotonic, from both the monotonic character of $\overline{g}$ and how the function α is extended. In conclusion, from Theorem (5), it follows that S_u is contained in a slice $\{t_0\} \times F$, and so that u is constant. □

Previous theorem allows us to obtain several important generalizations for the classical Bernstein result [6]. For instance, recalling Examples 31 and 32, we obtain the following two corollaries.

Corollary 36. *Let (F, g_F) be a Riemannian manifold and two smooth functions $f : I \to \mathbb{R}^+$ and $\alpha \in C^\infty(F)$. Assume that f is either monotone (non-decreasing or non-increasing) or there exists $t_1 \in \mathbb{R}$, so f is non-increasing in $(-\infty, t_1)$ and non-decreasing in (t_1, ∞). If $u \in C^\infty(F)$ is a solution for the equation*

$$\operatorname{div}\left(\frac{\nabla^F u}{f(u)\sqrt{f(u)^2 + |\nabla^F u|^2}}\right) - \frac{f'(u)}{\sqrt{f(u)^2 + |\nabla^F u|^2}}\left\{n - \frac{|\nabla^F u|^2}{f(u)^2}\right\}$$
$$= -\frac{1}{\sqrt{f(u)^2 + |\nabla^F u|^2 f(u)}} g_F(\nabla^F u, \nabla^F \alpha),$$

then $u \equiv u_0$ for some $u_0 \in I$ and $f'(u_0) = 0$.

Corollary 37. *Let (F, g_F) be a Riemannian manifold and consider two functions $h : F \to \mathbb{R}^+$ and $\alpha \in C^\infty(F)$. Then, the only solutions for the equation*

$$\operatorname{div}\left(\frac{h\,\nabla^F u}{\sqrt{1+h^2\,|\nabla^F u|^2}}\right) + \frac{1}{\sqrt{1+h^2\,|\nabla^F u|^2}}\, g_F\left(\nabla^F u, \nabla^F h\right) = -g_F(\nabla^F u, \nabla^F h)$$

are the constant functions.

6.3. Second Case: *U* Compact Neighbourhood with Boundary

We can also obtain results regarding a Dirichlet problem

$$\begin{cases} \mathfrak{F}(u, \overline{g}) = 0, & p \in U \\ u(p) \geq t_0, & p \in U, \\ u(p) = t_0, & p \in \partial U, \end{cases} \tag{39}$$

being U the topological closure of pre-compact open set in F. For instance, if we take $\mathfrak{F} = \Theta$, the graph S_u of a solution u of (39) is a minimal hypersurface with boundary contained in $\{t_0\} \times F$ (see Fig. 1). Hence, we are in conditions to apply Theorem 7. Moreover, if we take $\mathfrak{F}(u, \overline{g}) = \Theta(u, \overline{g}) + g_u(N^F, \nabla\alpha)$ for some function $\alpha : F \to \mathbb{R}$, we can also apply the conformal change as in the proof of Theorem 35, and Theorem 7 will also be applicable.

Concretely, we can prove that:

Theorem 38. *Let us consider F a manifold and $\overline{g}$ an O-S Riemannian metric defined over the product manifold $\overline{M} = \mathbb{R} \times F$. If $(\overline{M}, \overline{g})$ is non-expanding, then the only possible solutions for (39) with*

$$\mathfrak{F}(u, \overline{g}) = \Theta(u, \overline{g}) + g_u(N^F, \nabla u)$$

are the constants.

Remark 39. An analogous result can be obtained for the non-shrinking case by substituting $u(p) \geq t_0$ with $u(p) \leq t_0$ in (39).

We can particularize then to the cases we have analyzed in corollaries 36 and 37, obtaining:

Corollary 40. *Let U be a compact domain with boundary inside a Riemannian manifold (F, g_F), and consider two smooth functions $f : I \to \mathbb{R}^+$ and $\alpha \in C^\infty(F)$. Assume that f is monotone (either non-decreasing or non-increasing). If u is a solution for the Dirichlet problem (39) with*

$$\mathfrak{F}(u, \overline{g}) = \operatorname{div}\left(\frac{\nabla^F u}{f(u)\sqrt{f(u)^2 + |\nabla^F u|^2}}\right) - \frac{f'(u)}{\sqrt{f(u)^2 + |\nabla^F u|^2}}\left\{n - \frac{|\nabla^F u|^2}{f(u)^2}\right\}$$
$$+ \frac{1}{f(u)\sqrt{f(u)^2 + |\nabla^F u|^2}} g_F(\nabla^F u, \nabla^F \alpha),$$

then $u \equiv u_0$ for some constant $u_0 \in \mathbb{R}$ and $f'(u_0) = 0$.

Corollary 41. *Let U be a compact domain with boundary inside a Riemannian (F, g_F). Consider two smooth functions $h : F \to \mathbb{R}^+$ and $\alpha \in C^\infty(F)$. Then,* *the only solutions for the Dirichlet problem (39) with*

$$\mathfrak{F}(u,\overline{g}) = \operatorname{div}\left(\frac{h\,\nabla^F u}{\sqrt{1+h^2\,|\nabla^F u|^2}}\right) + \frac{1}{\sqrt{1+h^2\,|\nabla^F u|^2}}\,g_F\left(\nabla^F u, \nabla^F h\right) + g_F(\nabla^F u, \nabla^F h)$$

are the constant functions.

Remark 42. Reasoning as in Sect. 5 (recall also Remark 6), if we harden the hypothesis by assuming that $(\overline{M}, \overline{g})$ is either expanding or shrinking, we can also obtain results for non-existence.

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