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Black hole chemistry, the cosmological constant and the embedding tensor

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Abstract: We study black-hole thermodynamics in theories that contain dimensionful constants such as the cosmological constant or coupling constants in Wald's formalism. The most natural way to deal with these constants is to promote them to scalar fields introducing a $(d-1)$-form Lagrange multiplier that forces them to be constant on-shell. These $(d-1)$-form potentials provide a dual description of them and, in the context of superstring/supergravity theories, a higher-dimensional origin/explanation. In the context of gauged supergravity theories, all these constants can be collected in the *embedding tensor*. We show in an explicit 4-dimensional example that the embedding tensor can also be understood as a thermodynamical variable that occurs in the Smarr formula in a duality-invariant fashion. This establishes an interesting link between black-hole thermodynamics, gaugings and compactifications in the context of superstring/supergravity theories.

Keywords: Black Holes, String Duality, Classical Theories of Gravity

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Introduction. The realization in refs. [1, 2] that the cosmological constant can be considered as a thermodynamical variable in the context of black-hole physics has led to a host of new developments encompassed in the field of *black-hole chemistry*.[1] As shown in [5], other constants defining a theory can also be seen as thermodynamical variables; in said reference, these constants occur as coefficients of higher-order curvature terms that are Lovelock densities.

Clearly, the same ideas can be applied to $f(R)$ theories. Such theories can, however, be rewritten as theories of gravity coupled to a real scalar field with a non-trivial scalar potential. The form of the potential is related to the function $f(R)$ and therefore contains the same constants as the function $f(R)$. In ref. [6], one of the authors proposed that the constants occurring in general scalar potentials can also be seen as thermodynamical variables in black-hole physics. In gauged supergravity,[2] though, these constants are related to the gauge coupling constants, which can be generically represented by the so-called *embedding tensor*.[3] This leads us to conjecture that the embedding tensor itself should also be regarded as a thermodynamical variable. Testing this conjecture is one of the main goals of this paper.

[1]For reviews with many references see refs. [3, 4].
[2]For a review with many references, see ref. [7].
[3]For a pedagogical introduction and references, see, for instance, ref. [8].

Wald's formalism [9–11] provides an efficient method to study black-hole thermodynamics, once the gauge freedoms of the fields have correctly been taken into account as explained in refs. [12–14].[4] It is only by doing this that one obtains the work terms in the first law of black-hole mechanics. However, one does not get all the work terms that are usually admitted in the literature[5] because there are no gauge symmetries associated to all of them: the gauge symmetry of the electromagnetic field gives rise to a work term of the form $\Phi\delta Q$, where Q is the electric charge and Φ is the electric potential on the event horizon, but there is no additional gauge symmetry that gives rise to the dual term $\tilde{\Phi}\delta P$, where P is the magnetic charge and $\tilde{\Phi}$ is the magnetic potential on the horizon. There is no term proportional to the variation of the moduli, either, because there are no gauge symmetries associated to them. In addition, closer to our concerns in this paper, there is no term involving variations of the cosmological constant, for the same reason.[6] One can make such a term appear as in ref. [19], but since the action of diffeomorphisms on constants is trivial, this does not happen in a natural way and the physics behind this variation is unclear. The same happens to coupling constants.

Variations of the cosmological constant are possible in the context of supergravity/superstring theories, however. In higher-dimensional supergravity/superstring theories there are higher-rank forms which give $(d-1)$-form potentials after compactification to d dimensions. These potentials are dual to constants which are determined dynamically by the equations of motion of the potentials. The main example is provided by the 3-form potential of 11-dimensional supergravity that gives rise to the cosmological constant of $\mathcal{N}=8, d=4$ supergravity [20, 21], but there are many others. Ultimately, however, it is expected that all the parameters of lower-dimensional theories (which can be collected in the embedding tensor) can be explained in a similar fashion.

The $(d-1)$-form potentials dual to the constants do have an associated gauge symmetry. This suggests that the terms proportional to the variations of those constants in the first law could arise associated to the gauge symmetry of the dual $(d-1)$-forms. This possibility was first proposed in ref. [22] for the cosmological constant. In ref. [23] the full formalism was worked out for a cosmological constant understood as the charge arising from a 3-form gauge potential in four dimensions. Later on, in ref. [24] an explicit example of thermodynamics with variable cosmological constant was first worked out for a Kerr-Newman black hole. We will explore this idea in a more systematic way here, using Wald's formalism, for the cosmological constant and for all the components of the embedding tensor in a toy model.

In section 1, we are going to review the description of the cosmological constant in terms of a $(d-1)$-form potential in the simplest setting: no matter fields. We will show that one can recover the first law of black-hole thermodynamics of refs. [1, 2] using Wald's

[4] A different approach to handle gauge charges based on a "solution phase space" has been proposed in ref. [15].

[5] See, e.g. ref. [16], in which terms proportional to the variations of the moduli are included. This inclusion has been contested in ref. [17].

[6] One may argue that, perhaps, the procedure proposed in refs. [12–14] produces a Noether-Wald charge that simply misses terms. However, as shown in ref. [18], the Noether-Wald charge found in this way leads to a Smarr formula that also contains magnetic charges and potentials in a duality-invariant form, which suggests that nothing is missing from it.

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formalism treating the gauge symmetry as in refs. [12–14].[7] Furthermore, we will derive the Smarr formula using the Komar integral as proposed in refs. [1, 5, 6, 18, 27, 28].

In section 2 we are going to consider a more general example in 4 dimensions, with two real scalars and a 1-form field coupled to gravity. The theory is invariant under constant shifts of one of the scalars and this global symmetry can be gauged using the 1-form and its dual as gauge fields introducing at the same time two coupling constants that can be combined into a 2-component embedding tensor. This provides the opportunity to test the conjectured interpretation of the embedding tensor as a thermodynamical variable in black-hole physics.[8] As is well known [29–33] the consistency of this kind of electric/magnetic gaugings demands the introduction of 2-form fields which, in their turn demand the introduction of 3-form fields etc., giving rise to the so-called *tensor hierarchy*.

Complete tensor hierarchies have been constructed only in a few cases [34–37] and they show a one-to-one relation between the $(d-2)$-forms and the global symmetries of the theory that can be gauged (just 1-dimensional in our toy model), between the $(d-1)$-forms and the deformation constants of the theory (just the 2 components of the embedding tensor in our toy model) and between the d-forms and the constraints satisfied by the deformation constants of the theory (0 in our toy model). In section 2 we will omit the construction of the tensor hierarchy of the model and we will directly introduce its fields (the 2 real scalars, the 1-form and its dual, the single 2-form dual to the Noether-Gaillard-Zumino current [38, 39] and the two 3-forms dual to the coupling constants) and a democratic action, based on the one in ref. [31], in which all of them are present and in which the components of the embedding tensor are not constants, but functions which are forced to be constant on-shell.[9]

Then, in section 2 we will study the symmetries and define the conserved charges, including the Noether-Wald charge and the Komar charge, essentially along the lines of ref. [42]. We will use the last two to prove the first law of black hole mechanics, to obtain the Smarr formula and to study the role the embedding tensor plays in both of them.

We will discuss our results and their implications in section 3.

Finally, the appendix contains a, not so successful, search for black hole solutions of our toy model to which our results can be applied. Unfortunately, it is very difficult to find charged solutions and we only managed to find an embedding of the Reissner-Nordström-(a)DS solution into the model.

[7] A derivation of the first law in the presence of a cosmological constant treated as the conserved charge associated to a $(d-1)$-form potential has been carried out in ref. [25] using the formalism proposed in ref. [17]. The Smarr formula was found in ref. [26]. Our treatment is very similar to the one in these references, but not identical because we need to work with a democratic action in which the cosmological constant and its dual occur on equal footing in order to deal with the general problem in section 2.

[8] This conjecture is clearly related to and in agreement with the conjecture put forward in ref. [26] that all dimensionful constants in the Lagrangian contribute to the Smarr formula.

[9] This democratic action is a true action, as opposed to the democratic action of ref. [40], which is a *pseudoaction* [41] whose equations of motion must be supplemented by duality constraints to reproduce the equations of motion of the theory.

1 Dualizing the cosmological constant

The action for pure gravity (described by General Relativity) coupled to a cosmological constant Λ in arbitrary dimension d is

$$S[g_{\mu\nu}] = \frac{1}{16\pi G_N^{(d)}} \int d^dx \sqrt{|g|}\, [R(g) - (d-2)\Lambda]\,, \tag{1.1}$$

and leads to the equations of motion

$$G_{\mu\nu} + \frac{(d-2)}{2} g_{\mu\nu}\Lambda = 0\,, \tag{1.2}$$

which can be reduced to just

$$R_{\mu\nu} = \Lambda g_{\mu\nu}\,. \tag{1.3}$$

The dimension-dependent factor of Λ in the action has been chosen so as to obtain this last equation. On the other hand, in the conventions that we are using, when Λ is positive (negative), the maximally symmetric solution of the above equations is the (anti-) De Sitter solution. If we interpret the cosmological term in the Einstein equations (1.2) as an energy-momentum tensor

$$T_{\mu\nu} = -\frac{(d-2)}{16\pi G_N^{(d)}} \Lambda g_{\mu\nu}\,, \tag{1.4}$$

and we compare it with that of a perfect fluid $-(\rho+p)u_\mu u_\nu + p g_{\mu\nu}$, we find that the perfect fluid is characterized by

$$\rho = -p = \frac{(d-2)}{16\pi G_N^{(d)}} \Lambda\,. \tag{1.5}$$

In differential-form language, the action (1.1) takes the form

$$S[e^a] = \frac{(-1)^{d-1}}{16\pi G_N^{(d)}} \int \Big[\star(e^a \wedge e^b) \wedge R_{ab} - (d-2) \star \Lambda\Big]\,. \tag{1.6}$$

The equations of motion that one obtains from this action, defined by the variation of the action, up to total derivatives

$$\delta S = \int \mathbf{E}_a \wedge \delta e^a\,, \tag{1.7}$$

are given by

$$16\pi G_N^{(d)} \mathbf{E}_a = \imath_a \star (e^b \wedge e^c) \wedge R_{bc} - (d-2)\imath_a \star \Lambda\,, \tag{1.8}$$

and it is not hard to see that they can be rewritten in the form

$$16\pi G_N^{(d)} \mathbf{E}_a = (-1)^d 2 \left\{ G_{ab} + \frac{(d-2)}{2} g_{ab}\Lambda \right\} \star e^b\,, \tag{1.9}$$

which provides a check of the equivalence of the actions eqs. (1.6) and (1.1).

As is well known [21], the cosmological constant Λ can be dualized into a $(d-1)$-form potential that we will denote by C. The dualization can be carried out as follows: first of all, in order to encompass both the $\Lambda > 0$ and $\Lambda < 0$ cases, we define

$$\Lambda = \text{sign}\,\Lambda\ \lambda^2\,, \tag{1.10}$$

and promote the positive constant λ to a function $\lambda(x)$ that we immediately constrain to be constant by introducing a Lagrange-multiplier term in the action

$$\begin{aligned} S[e^a] \longrightarrow S[e^a,\lambda,C] &= \frac{1}{16\pi G_N^{(d)}}\int \Big[(-1)^{d-1}\star(e^a\wedge e^b)\wedge R_{ab} \\ &\qquad +(-1)^d(d-2)\text{sign}\,\Lambda\star\lambda^2 - C\wedge d\lambda\Big]\,, \\ &\equiv \int \mathbf{L}\,, \end{aligned} \tag{1.11}$$

with the dual $(d-1)$-form C playing the role of Lagrange multiplier. A general variation of this action

$$\delta S = \int \left\{\mathbf{E}_a\wedge\delta e^a + \mathbf{E}_\lambda\delta\lambda + \mathbf{E}_C\wedge\delta C + d\mathbf{\Theta}(\varphi,\delta\varphi)\right\}\,, \tag{1.12}$$

where φ stands for the fields e^a,λ,C, gives the equations of motion and total derivative

$$16\pi G_N^{(d)}\mathbf{E}_a = \imath_a\star(e^b\wedge e^c)\wedge R_{bc} - (d-2)\,\text{sign}\,\Lambda\,\lambda\imath_a\star\lambda\,, \tag{1.13a}$$

$$16\pi G_N^{(d)}\mathbf{E}_\lambda = (-1)^{d-1}\left[dC - 2(d-2)\text{sign}\,\Lambda\,\star\lambda\right]\,, \tag{1.13b}$$

$$16\pi G_N^{(d)}\mathbf{E}_C = (-1)^d d\lambda\,, \tag{1.13c}$$

$$16\pi G_N^{(d)}\mathbf{\Theta}(\varphi,\delta\varphi) = -\star(e^a\wedge e^b)\wedge\delta\omega_{ab} + (-1)^d C\delta\lambda\,. \tag{1.13d}$$

We can use λ's equation of motion

$$\lambda = \frac{(-1)^{d-1}\text{sign}\,\Lambda}{2(d-2)}\star dC\,, \tag{1.14}$$

to replace λ by $G\equiv dC$, the d-form field strength of C, arriving at the dual action

$$S[e^a,C] = \frac{1}{16\pi G_N^{(d)}}\int\left[(-1)^{d-1}\star(e^a\wedge e^b)\wedge R_{ab} + \frac{\text{sign}\,\Lambda}{4(d-2)}G\star G\right]\,. \tag{1.15}$$

The variation of the action, up to total derivatives,

$$\delta S = \int\left\{\mathbf{E}_a\wedge\delta e^a + \mathbf{E}_C\wedge\delta C\right\}\,, \tag{1.16}$$

gives the following equations of motion

$$16\pi G_N^{(d)}\mathbf{E}_a = \imath_a\star(e^b\wedge e^c)\wedge R_{bc} + \frac{(-1)^d}{4(d-2)}\text{sign}\,\Lambda\,\imath_a G\star G\,, \tag{1.17a}$$

$$16\pi G_N^{(d)}\mathbf{E}_C = -\text{sign}\,\Lambda\,d\star G\,. \tag{1.17b}$$

$\mathbf{E}_C = 0$ is solved by a constant $\star G$. If the constant is written in the form

$$\star G = (-1)^{d-1}2(d-2)\text{sign}\,\Lambda\ \lambda\,, \tag{1.18}$$

we recover the cosmological Einstein equations eq. (1.9).

This proves the classical equivalence of the original and the dual formulations, although the second is slightly more general since the equation of motion of C can be solved by piecewise constant $\lambda(x)$s whose discontinuities can be associated to $(d-2)$-brane sources, which couple in a natural way to the $(d-2)$-form C.[10]

An important difference between C and λ is that the former has a gauge freedom, under which it transforms as

$$\delta_\chi C = d\chi\,, \tag{1.19}$$

where χ is an arbitrary $(d-2)$-form. These gauge transformations leave the field strength G and the dual action eq. (1.15) invariant. The action eq. (1.11) is also gauge invariant, but only up to a total derivative. In any case, this invariance is associated to a conserved charge, apparently not present in the original system. This charge can be understood in terms of the branes that source C and is directly related to λ. We study the definition of this charge in the next section.

Although the actions eqs. (1.15) and (1.11) are equivalent, in more complex cases in which we want to dualize constants that occur in multiple places in the action, one promotes the constants to fields and adds the Lagrange-multiplier terms with the dual potentials but one does not take the next step (eliminating the constants using their equations of motion) because the resulting actions are too complicated. Thus, one stays with actions similar to eq. (1.11) and, therefore, in what follows, we will work with it.

1.1 The gauge conserved charge

Under the gauge transformation eq. (1.19), the action eq. (1.11) transforms as

$$\delta_\chi S = -\frac{1}{16\pi G_N^{(d)}}\int d\chi\wedge d\lambda = \frac{(-1)^d}{16\pi G_N^{(d)}}\int d\left(\lambda d\chi\right)\,. \tag{1.20}$$

The total derivative is defined up to the total derivative of a total derivative, and we have made a choice that we will show is adequate to get a non-trivial result.

From eq. (1.12), instead, upon use of the Noether identity $d\mathbf{E}_C = 0$, we get

$$\delta_\chi S = \int -d\left(\mathbf{E}_C\wedge\chi\right) = -\frac{(-1)^d}{16\pi G_N^{(d)}}\int d\left(d\lambda\wedge\chi\right)\,, \tag{1.21}$$

which, together with the previous result leads to the off-shell identity

$$d\mathbf{J}[\chi] = 0\,,\quad \text{with}\quad \mathbf{J}[\chi] = \frac{(-1)^{d-1}}{16\pi G_N^{(d)}}\left(d\lambda\wedge\chi + \lambda d\chi\right)\,. \tag{1.22}$$

This identity implies that, locally, there must exist a $(d-2)$-form $\mathbf{Q}[\chi]$ such that

$$\mathbf{J}[\chi] = d\mathbf{Q}[\chi]\,, \tag{1.23}$$

and it is obvious that

$$\mathbf{Q}[\chi] = \frac{(-1)^{d-1}}{16\pi G_N^{(d)}}\lambda\chi\,. \tag{1.24}$$

[10] See, for instance, ref. [40].

Given a particular solution of the equations of motion $\{e^a, \lambda, C\}$, for each inequivalent $(d-2)$-form that preserves it (i.e. for each harmonic χ_h), we can get the conserved charge contained in a closed $(d-2)$-dimensional surface Σ^{d-2} with no boundary by integrating $\mathbf{Q}[\chi_h]$ over it

$$\mathcal{Q}[\chi_h] = \frac{(-1)^{d-1}\lambda}{16\pi G_N^{(d)}} \int_{\Sigma^{d-2}} \chi_h \,, \tag{1.25}$$

where we have used the fact that on-shell λ is constant.

Up to normalization constants, this charge is just the volume of Σ^{d-2} measured in terms of the volume form χ_h. Observe that the value of the charge does not change under the replacement of χ_h by $\chi_h + de$ for any $(d-3)$-form e. Thus, it only depends on the De Rahm cohomological class of χ_h, which is unique (up to normalization) on any compact, orientable Σ^{d-2} with no boundary. It is natural to use the induced volume form on $\Sigma^{(d-2)}$ that we will denote by $\Omega_\Sigma^{(d-2)}$ and, then,

$$\mathcal{Q} = \frac{(-1)^{d-1}\lambda}{16\pi G_N^{(d)}} \omega_\Sigma \,, \quad \text{where} \quad \omega_\Sigma \equiv \int_{\Sigma^{d-2}} \Omega_\Sigma^{(d-2)} \,. \tag{1.26}$$

Thus, up to numerical constants and the volume ω_Σ (not present in rationalized units) λ is the charge carried by C.[11]

1.2 The Noether-Wald charge

The action eq. (1.11) is also exactly invariant under diffeomorphisms and local Lorentz transformations.[12] We are interested in the Noether charge associated to the invariance under diffeomorphisms (Noether-Wald charge) and, therefore, we start by considering the variation of the action under diffeomorphisms generated by infinitesimal vector fields ξ

$$\delta_\xi S = \int \left\{ \mathbf{E}_a \wedge \delta_\xi e^a + \mathbf{E}_C \wedge \delta_\xi C + \mathbf{E}_\lambda \wedge \delta_\xi \lambda + d\mathbf{\Theta}(\varphi, \delta_\xi \varphi) \right\} \,. \tag{1.27}$$

Observe that λ must be treated as a scalar field and, therefore

$$\delta_\xi \lambda = -\imath_\xi d\lambda \,. \tag{1.28}$$

However, the infinitesimal transformations δ_ξ of e^a and C must take into account the gauge freedom of those fields as explained in refs. [12–14, 43] in such a way that the invariance of the fields under those transformations for a certain parameter ξ (which we will denote by k) is a gauge-invariant statement. Since, in particular, δ_k must leave invariant the metric, k is always a Killing vector. The transformations $\delta_\xi e^a$ and $\delta_\xi C$ are combinations of standard Lie derivatives and ξ-dependent "compensating" gauge transformations

$$\sigma_\xi{}^{ab} = \imath_\xi \omega^{ab} - P_\xi{}^{ab} \,, \tag{1.29a}$$

$$\chi_\xi = \imath_\xi C - P_\xi \,, \tag{1.29b}$$

[11] This is, essentially, the same result obtained in ref. [25].

[12] Under infinitesimal diffeomorphisms it is invariant only up to a total derivative that we will take into account later.

where

$$P_\xi{}^{ab} \equiv \nabla^{[a}\xi^{b]}\,, \tag{1.30}$$

is the (scalar) Lorentz momentum map, which satisfies for $\xi = k$

$$\mathcal{D}P_k{}^{ab} = -\imath_k R^{ab}\,. \tag{1.31}$$

On the other hand, P_ξ is the $(d-2)$-form momentum map associated to C, which is such that, for $\xi = k$

$$dP_k = -\imath_k G\,. \tag{1.32}$$

After some massaging, we can write the transformations in the form

$$\delta_\xi e^a = -(\mathcal{D}\xi^a + P_\xi{}^a{}_b e^b)\,, \tag{1.33a}$$

$$\delta_\xi \omega^{ab} = -(\imath_\xi R^{ab} + \mathcal{D}P_\xi{}^{ab})\,, \tag{1.33b}$$

$$\delta_\xi C = -(\imath_\xi G + dP_\xi)\,. \tag{1.33c}$$

The definitions of the momentum maps ensure that $\delta_k e^a = \delta_k \omega^{ab} = \delta_k C = 0$ in a gauge-invariant fashion.

Observe that, on-shell,

$$\imath_k G = (-1)^{d-1} 2(d-2)\,\mathrm{sign}\,\Lambda\;\lambda \star \hat{k}\,, \tag{1.34}$$

where $\hat{k} = k_\mu dx^\mu$ if $k = k^\mu\partial_\mu$.[13] Then,

$$P_k = (-1)^d 2(d-2)\mathrm{sign}\,\Lambda\;\lambda\,\omega_k\,, \tag{1.35}$$

where the $(d-2)$-form ω_k is the d-dimensional generalization of the *Killing co-potential* introduced in ref. [1] defined by

$$d\omega_k = \star\hat{k}\,. \tag{1.36}$$

Although the existence of P_k and, hence, of ω_k was initially guaranteed by the invariance of G under δ_k, we see here that it is also related to k being a Killing vector. Since on-shell G is, up to constants, the metric volume form, these two facts are obviously related.

Substituting the transformations eqs. (1.33) into eq. (1.27), using the Noether identities associated to the symmetries and performing simple manipulations we arrive at

$$\delta_\xi S = \int d\left\{\mathbf{\Theta}(\varphi,\delta_\xi\varphi) + (-1)^d\mathbf{E}_a\xi^a + \mathbf{E}_C \wedge P_\xi\right\} \equiv \int d\mathbf{\Theta}'\,. \tag{1.37}$$

Now we must take into account that the action eq. (1.11) is invariant under diffeomorphisms and gauge transformations up to total derivatives:

$$\delta_\xi S = \int d\left\{-\imath_\xi \mathbf{L} + \frac{(-1)^{d-1}}{16\pi G_N^{(d)}}\left[d\lambda \wedge \imath_\xi C + \lambda dP_\xi\right]\right\}\,, \tag{1.38}$$

[13] With our conventions,

$$\imath_k \star \mathbb{I} = (-1)^{d-1} \star \hat{k}\,.$$

where we have used the explicit form of the compensating $\delta_\chi \xi$ transformation eq. (1.29b) and the freedom that we have to add total derivatives of total derivatives to obtain a convenient expression.

We arrive at the off-shell identity

$$d\mathbf{J}[\xi] = 0\,, \tag{1.39}$$

where

$$\mathbf{J}[\xi] \equiv \mathbf{\Theta}' + \imath_\xi \mathbf{L} + \frac{(-1)^d}{16\pi G_N^{(d)}} \left[d\lambda \wedge \imath_\xi C + \lambda dP_\xi \right]\,, \tag{1.40}$$

and it is not difficult to see that

$$\mathbf{J}[\xi] = d\mathbf{Q}[\xi]\,, \tag{1.41}$$

where the Wald-Noether $(d-2)$-form $\mathbf{Q}[\xi]$ is given by

$$\mathbf{Q}[\xi] \equiv \frac{(-1)^d}{16\pi G_N^{(d)}} \left\{ \star(e^a \wedge e^b) P_{\xi\, ab} + \lambda P_\xi \right\}\,. \tag{1.42}$$

1.3 The generalized, restricted, zeroth law

A crucial ingredient in the proof of the first law of black-hole mechanics along the lines of refs. [12–14, 43] are the *generalized, restricted zeroth laws.* These laws are called "generalized" because they generalize the standard zeroth law of black-hole mechanics stating that the surface temperature κ is constant over the event horizon $\mathcal{H}$ to other thermodynamical potentials such as the electrostatic black-hole potential. On the other hand, they are called "restricted" because their validity is restricted to the bifurcation surface $\mathcal{BH}$,[14] and because, rather than stating the constancy of a scalar quantity, they just state the closedness of a given differential form over $\mathcal{BH}$.[15] This is enough for our purposes, though.

Thus, at this point we are going to focus on solutions of the action eq. (1.11) which describe stationary black-hole spacetimes with a cosmological constant determined by the value of λ, with bifurcate event horizons that coincide with the Killing horizon associated to a certain asymptotically timelike Killing vector k. By definition, k vanishes over the bifurcation surface which we denote by $\mathcal{BH}$

$$k \stackrel{\mathcal{BH}}{=} 0\,. \tag{1.43}$$

Thus, if all fields are regular over the horizon, it is clear that the inner products of their field strengths with k must vanish on $\mathcal{BH}$:

$$\imath_k G \stackrel{\mathcal{BH}}{=} 0\,, \tag{1.44a}$$

$$\imath_k R^a{}_b \stackrel{\mathcal{BH}}{=} 0\,. \tag{1.44b}$$

[14] One could use the arguments of ref. [44] to extend their validity to the complete event horizon, though.

[15] Actually, when dealing with forms of rank higher than 1 (1 would correspond to an electromagnetic field), it is not clear which other covariant statement could play the role of zeroth law.

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Let us consider the first of these properties. According to the definition eq. (1.32), the $(d-2)$-form P_k is closed on $\mathcal{BH}$. Being a $(d-2)$-form on $\mathcal{BH}$, it must be proportional to the induced volume form of $\mathcal{BH}$, $\Omega_{\mathcal{BH}}$:

$$P_k = f\Omega_{\mathcal{BH}}\,. \tag{1.45}$$

Then, the closedness of P_k implies that the coefficient f is a constant. This statement is one of the generalized, restricted zeroth laws of black-hole mechanics that have been used in refs. [12–14, 43, 45–47] to prove the first law. If we normalize the volume form such that its integral is equal to 1, f will be proportional to the volume of $\mathcal{BH}$. This is the thermodynamical potential ("volume") associated to the thermodynamical variable λ ("pressure"). In order to make contact with the conventions of ref. [6], it is more convenient to use the Killing co-potential $(d-2)$-form ω_k, which due to eq. (1.35), must also be closed on $\mathcal{BH}$ on-shell and, therefore, proportional to the volume form. Thus, we define the volume Θ_λ by[16]

$$\frac{P_k}{16\pi G_N^{(d)}} = (-1)^{d-1}\Theta_\lambda\omega_k/V_k\,,\quad \text{with}\quad V_k \equiv \int_{\mathcal{BH}}\omega_k\,,\quad \Theta_\lambda = -\text{sign}\,\Lambda\,\frac{(d-2)\lambda V_k}{8\pi G_N^{(d)}}\,, \tag{1.46}$$

so that the volume Θ_λ is positive for aDS black holes ($\text{sign}\Lambda < 0$).

Following ref. [6], Θ_λ can written as

$$\Theta_\lambda = \frac{(-1)^d}{16\pi G_N^{(d)}}\int_{\mathcal{B}} \imath_k \star \frac{\partial\mathcal{V}}{\partial\lambda}\,,\quad \text{with}\quad \mathcal{V} \equiv (d-2)\text{sign}\,\Lambda\;\lambda^2\,, \tag{1.47}$$

where $\mathcal{B}$ is a ball whose radius is that of the horizon and whose boundary is $\mathcal{BH}$. This is an expression that we will generalize later on.

The property eq. (1.44b) is related to the standard zeroth law of black-hole mechanics because it implies

$$\mathcal{D}P_{k\,ab} \stackrel{\mathcal{BH}}{=} 0\,, \tag{1.48}$$

and because, on the bifurcation surface

$$P_{k\,ab} \stackrel{\mathcal{BH}}{=} \kappa n_{ab}\,, \tag{1.49}$$

where n_{ab} is the binormal to $\mathcal{BH}$, with the normalization $n^{ab}n_{ab} = -2$. Since κ is constant according to the zeroth law, n_{ab} must be covariantly constant on $\mathcal{BH}$. We do not have an independent proof of this property, which is of purely geometric nature. With this proof in hand, the zeroth law on $\mathcal{BH}$ ($d\kappa \stackrel{\mathcal{BH}}{=} 0$) would be a consequence of eq. (1.48). All zeroth laws (generalized or not) would follow the same pattern since they would state that the coefficients of the expansion of certain closed (or covariantly-closed) forms in a properly defined and normalized basis are constant as in refs. [12–14, 43, 45–47].

[16] Apart from the sign, there is another difference with most of the literature in black-hole chemistry: this volume is proportional to λ, which is natural for a potential, but, perhaps, not for a volume.

1.4 Komar integral and Smarr formula

Before we use the Noether-Wald charge and the restricted, generalized second laws to prove the first law of black-hole mechanics [48], it is useful to test our results constructing a Komar integral [49] following refs. [1, 5, 6, 18, 27, 28] and using it to derive a Smarr formula [50] that can be tested in actual black-hole solutions.

On-shell[17] and for a Killing vector k that generates a symmetry of the whole field configuration, the Noether-Wald current defined in eq. (1.40) satisfies

$$\mathbf{J}[k] \doteq \imath_k \mathbf{L} + \frac{(-1)^d}{16\pi G_N^{(d)}} \lambda dP_k \,. \tag{1.50}$$

On the other hand, $\mathbf{J}[k]$ satisfies eq. (1.41) off-shell with $\xi = k$, which implies that

$$d\mathbf{Q}[k] - \imath_k \mathbf{L} - \frac{(-1)^d}{16\pi G_N^{(d)}} \lambda dP_k = 0 \,. \tag{1.51}$$

We can write a Komar integral with volume terms, as in ref. [27], or we can take a step further and rewrite the last two terms as total derivatives refs. [6, 18]. This is trivial for the second term. As for the first additional term, if k generates a symmetry of the whole field configuration,

$$0 \doteq \pounds_k \mathbf{L} = d\imath_k \mathbf{L} \,, \tag{1.52}$$

and $\imath_k \mathbf{L}$ must be locally exact. Therefore, there must exist a $(d-2)$-form ϖ_k such that

$$d\varpi_k \doteq \imath_k \mathbf{L} + \frac{(-1)^d}{16\pi G_N^{(d)}} \lambda dP_k \,, \tag{1.53}$$

which leads to the identity

$$d\left\{\mathbf{Q}[k] - \varpi_k\right\} \doteq 0 \,. \tag{1.54}$$

Then, the Komar integral over the codimension-2 surface Σ^{d-2} can be defined as the integral over the *Komar charge* $-\{\mathbf{Q}[k] - \varpi_k\}$ [6]

$$\mathcal{K}(\Sigma^{d-2}) = -\int_{\Sigma^{d-2}} \left\{\mathbf{Q}[k] - \varpi_k\right\} \,. \tag{1.55}$$

In order to determine ϖ_k, we first calculate the on-shell value of the Lagrangian density: tracing over the Einstein equations (1.13a)

$$\begin{aligned} e^a \wedge \mathbf{E}_a &= (d-2) \star (e^b \wedge e^c) \wedge R_{bc} - (d-2)d \,\mathrm{sign}\,\Lambda \,\lambda \star \lambda \\ &= (-1)^{d-1}(d-2) \left\{ \mathbf{L} + \frac{(-1)^d \,\mathrm{sign}\,\Lambda}{8\pi G_N^{(d)}} \star \lambda^2 \right\} \,, \end{aligned} \tag{1.56}$$

so

$$\mathbf{L} \doteq \frac{(-1)^{d-1} \,\mathrm{sign}\,\Lambda}{8\pi G_N^{(d)}} \star \lambda^2 \,. \tag{1.57}$$

[17] Here we use the symbol $\doteq$ for identities that only hold on-shell.

Now, using the equation of motion of λ, eq. (1.13b), to replace $\star\lambda$ by G and the definition of the momentum map P_k in eq. (1.32) to replace $\imath_k G$ by $-dP_k$, we get

$$\imath_k \mathbf{L} \doteq d\left\{\frac{(-1)^d \,\lambda P_k}{(d-2)16\pi G_N^{(d)}}\right\}, \tag{1.58}$$

and

$$\varpi_k = \frac{(-1)^d (d-1)\,\lambda P_k}{(d-2)16\pi G_N^{(d)}}. \tag{1.59}$$

The Komar integral is, then

$$\mathcal{K}(\Sigma^{d-2}) = \frac{(-1)^{d-1}}{16\pi G_N^{(d)}} \int_{\Sigma^{d-2}} \left\{\star(e^a \wedge e^b) P_{k\,ab} - \frac{\lambda P_k}{(d-2)}\right\}. \tag{1.60}$$

Let us consider the anti-De Sitter case ($\operatorname{sgn}\Lambda < 0$): if we integrate the exterior derivative of the integrand over a hypersurface whose boundary is the union of a spatial section of a stationary black-hole Killing horizon (the bifurcation surface, $\mathcal{BH}$, for the sake of convenience) and spatial infinity, S^{d-2}_∞, Stokes' theorem tells us that

$$\mathcal{K}(\mathcal{BH}) = \mathcal{K}(S^{d-2}_\infty). \tag{1.61}$$

For the sake of simplicity, let us consider a static, spherically symmetric black-hole solution: the Schwarzschild-aDS-Tangherlini solution [51], whose metric is given by

$$ds^2 = W dt^2 - W^{-1} dr^2 - r^2 d\Omega^2_{(d-2)}, \quad \text{with} \quad W = 1 - \frac{2m}{r^{d-3}} + \frac{|\Lambda|}{d-1} r^2, \tag{1.62}$$

where

$$m = \frac{8\pi G_N^{(d)} M}{(d-2)\omega_{(d-2)}}, \tag{1.63}$$

$\omega_{(d-2)}$ being the volume of the unit, round, $(d-2)$-sphere, and M the ADM mass.

The event horizon of this solution is placed at some value r_h at which $W(r_h) = 0$. The Hawking temperature and Bekenstein-Hawking entropy can be expressed in terms of r_h even if its value cannot be determined explicitly. They are given, respectively, by [2]

$$T = \frac{1}{2\pi(d-1) r_h^{d-2}} \left[(d-1)(d-3)m + |\Lambda| r_h^{d-1}\right], \tag{1.64a}$$

$$S = \frac{\omega_{(d-2)} r_h^{d-2}}{4 G_N^{(d)}}, \tag{1.64b}$$

and the product ST leads to the Smarr formula

$$\frac{(d-3)}{(d-2)} M = ST - \frac{1}{8\pi G_N^{(d)}} \frac{\omega_{(d-2)} r_h^{d-1}}{(d-1)} |\Lambda|. \tag{1.65}$$

We can evaluate the Komar integral on a constant r surface S^{d-2}_r

$$\mathcal{K}(S^{d-2}_r) = \frac{\omega_{(d-2)}}{16\pi G_N^{(d)}} \left\{r^{d-2} W' - 2 r^{d-1} \frac{|\Lambda|}{(d-1)}\right\}. \tag{1.66}$$

At infinity, the second term on the r.h.s. cancels a divergent term coming from W' and we get the left-hand side of the Smarr relation eq. (1.65). At the horizon, the first term gives directly ST and we get the right-hand side of eq. (1.65) and we recover the complete Smarr relation from the Komar integral.

Observe that, since the restricted, generalized, zeroth law guarantees that, over the bifurcation surface, P_k is a constant times the volume form, in general we can take that constant outside of the Komar integral $\mathcal{K}(\mathcal{BH})$. In this simple case, also λ can be taken out of the integral, but in more general cases, only the constant defining the momentum map can be taken outside the integral. The integral of λ is also the integral of $\star G$ on-shell, which gives, up to normalization constants, the associated charge.

Finally, using the definition of Θ_λ in eq. (1.47) and

$$V_k = \frac{r_h^{d-1}\omega_{(d-2)}}{d-1}\,, \tag{1.67}$$

the Smarr formula can be written in the form

$$(d-3)M = (d-2)ST - \Theta_\lambda \lambda\,, \tag{1.68}$$

which is the form that follows from the usual scaling and homogeneity arguments [2, 26, 50].[18]

1.5 The first law and black-hole chemistry

We are ready to prove the first law of black-hole mechanics in this theory using Wald's formalism [9–11].

We consider field configurations that describe stationary, black-hole spacetimes admitting a timelike Killing vector k whose bifurcate Killing horizon coincides with the black hole's event horizon $\mathcal{H}$. k, then, will be given by a linear combination with constant coefficients Ω^n of the timelike Killing vector associated to stationarity, $t^\mu\partial_\mu$, and the $[\frac{1}{2}(d-1)]$ generators of inequivalent rotations in d spacetime dimensions $\phi_n^\mu\partial_\mu$

$$k^\mu = t^\mu + \Omega^n \phi_n^\mu\,. \tag{1.69}$$

The constant coefficients Ω^n are the angular velocities of the horizon.

The starting point of the proof is the fundamental relation [9–11]

$$d\left(\delta\mathbf{Q}[k] + \imath_k \mathbf{\Theta}'\right) = 0\,, \tag{1.70}$$

valid for on-shell field configurations φ satisfying the equations of motion and perturbations of the fields $\delta\varphi$ satisfying the linearized equations of motion.

We are going to integrate this relation over the hypersurface Σ defined as the space bounded by infinity and the bifurcation sphere $\mathcal{BH}$ on which $k=0$. Therefore, its boundary, $\partial\Sigma$, has two disconnected pieces: a $(d-2)$-sphere at infinity, S^{d-2}_∞, and the bifurcation

[18] $1/\lambda$ has dimensions of length.

sphere $\mathcal{BH}$. Using Stokes theorem and taking into account that $k = 0$ on $\mathcal{BH}$, we obtain the relation

$$-\delta \int_{\mathcal{BH}} \mathbf{Q}[k] = -\int_{\mathrm{S}^{d-2}_{\infty}} \left(\delta \mathbf{Q}[k] + \imath_k \mathbf{\Theta}'\right) , \tag{1.71}$$

where we have added conventional minus signs that take into account the minus sign in our definitions of the variations of the fields under diffeomorphisms.

As explained in refs. [11, 46], the right-hand side can be identified with $\delta M - \Omega^n \delta J_n$, where M is the total mass of the black-hole spacetime and J_n are the independent components of the angular momentum.

Using the explicit form of the Noether-Wald charge eq. (1.42)

$$-\delta \int_{\mathcal{BH}} \mathbf{Q}[k] = \frac{(-1)^{d-1}}{16\pi G_N^{(d)}} \delta \int_{\mathcal{BH}} \star(e^a \wedge e^b) P_{k\,ab} + \frac{(-1)^{d-1}}{16\pi G_N^{(d)}} \delta\lambda \int_{\mathcal{BH}} P_k \,. \tag{1.72}$$

The right-hand side of this identity is expected to be of the form $T\delta S + \Phi \delta \mathcal{Q}$ for some charges $\mathcal{Q}$ and potentials Φ and/or "pressures" ϑ and "volumes" Θ_ϑ. In this expression λ plays the role of charge or pressure, while Θ_λ, defined in eq. (1.46), plays the role of conjugate potential or volume. Using eqs. (1.18) and (1.46) (sign $\Lambda = -1$)

$$\frac{(-1)^{d-1}}{16\pi G_N^{(d)}} \delta\lambda \int_{\mathcal{BH}} P_k = \Theta_\lambda \delta\lambda \,. \tag{1.73}$$

Using also eq. (1.49) we arrive at

$$\delta M = T\delta S + \Omega^n \delta J_n + \Theta_\lambda \delta \lambda \,. \tag{1.74}$$

The (unconventional, in black-hole chemistry literature) factor of λ present in the definition of Θ_λ can be absorbed in $\delta\lambda$. However, when the cosmological constant arises as the square of another, more fundamental constant, as in gauged supergravity, this form of the first law is more natural. Also, in these theories, the coupling constant is often associated to $(d-1)$-form potentials coming from higher dimensions [20].

As a matter of fact, it is always possible to introduce a $(d-1)$-form potential dual to the coupling constants, masses or any other parameters occurring in the action. These $(d-1)$-forms are part of what is known as the *tensor hierarchy* of the theory. At the level of the action they can always be introduced in the same way we introduced the potential dual to the cosmological constant: promoting first the parameters to fields and introducing the dual $(d-1)$-form potentials as Lagrange multipliers that constrain the fields/parameters to be constant. Intuitively, we expect terms in the first law and Smarr formula associated to all those $(d-1)$ potentials and, henceforth, to all those coupling constants, masses and other parameters.

In the next section we are going to consider a very simple model inspired by gauged supergravity, in which we can test these ideas.

2 A more general example

In this section we want to consider a more general model which essentially describes two scalars ϕ^1, ϕ^2 and a 1-form field A coupled to gravity, represented by the Vierbein e^a, in $d = 4$. In this model, the invariance under constant shifts of ϕ^2 has been gauged using a combination of the 1-form A and its dual $\tilde{A}$ as gauge fields with two coupling constants ϑ and $\tilde{\vartheta}$. By consistency, it is necessary to introduce a 2-form B which can be taken to be the dual of the Noether current j associated to the invariance under constant shifts of ϕ^2, so no additional degrees of freedom are added to the theory. Actually, one can write an action for all these fields which gives the expected equations of motion plus the duality relations between j and B and between A and $\tilde{A}$ (see ref. [31]).

One can go further, dualizing the two coupling constants into 2 3-forms C and $\tilde{C}$ as we have done with the cosmological constant in the previous section, completing the *tensor hierarchy* as in refs. [34, 35]. Again, this introduces no new local degrees of freedom.

Before introducing the action that describes this system, we introduce some notation: the coupling constants, their dual 3-forms, the 1-form and its dual and the 0- and 2-form gauge parameters are collected in symplectic vectors $\vartheta_M, C^M, A^M, \sigma^M, \chi^M$ as follows:

$$
\begin{aligned}
&(\vartheta_M) \equiv \left(\vartheta, \tilde{\vartheta}\right), \qquad && \left(A^M\right) \equiv \begin{pmatrix} A \\ \tilde{A} \end{pmatrix}, \qquad && \left(C^M\right) \equiv \begin{pmatrix} C \\ \tilde{C} \end{pmatrix}, \\
&\left(\vartheta^M\right) = \begin{pmatrix} -\tilde{\vartheta} \\ \vartheta \end{pmatrix}, \qquad && \left(\sigma^M\right) \equiv \begin{pmatrix} \sigma \\ \tilde{\sigma} \end{pmatrix}, \qquad && \left(\chi^M\right) \equiv \begin{pmatrix} \chi \\ \tilde{\chi} \end{pmatrix}.
\end{aligned}
\tag{2.1}
$$

The field strengths are defined as

$$\mathcal{D}\phi^2 \equiv d\phi^2 - \vartheta_M A^M\,, \tag{2.2a}$$

$$F^M = dA^M + \vartheta^M B\,, \tag{2.2b}$$

$$H = dB\,, \tag{2.2c}$$

$$G^M \equiv dC^M + A^M \wedge \star j + \tfrac{1}{2}\vartheta^M B \wedge B + \delta^{M,2} B \wedge (\star F - \tilde{F})\,, \tag{2.2d}$$

where

$$j \equiv (\phi^1)^2 \mathcal{D}\phi^2\,, \tag{2.3}$$

and $\delta^{M\tilde{}}$ is 1 for $\tilde{C}$ and zero for C. Under the gauge transformations

$$\delta\phi^2 = \vartheta_M \sigma^M\,, \tag{2.4a}$$

$$\delta A^M = d\sigma^M - \vartheta^M \Lambda\,, \tag{2.4b}$$

$$\delta B = d\Lambda\,, \tag{2.4c}$$

$$\delta C^M = d\chi^M - \sigma^M \star j - \vartheta^M \Lambda \wedge B - \delta^{M,2} \Lambda \wedge (\star F - \tilde{F})\,, \tag{2.4d}$$

the above field strengths are on-shell invariant only. More precisely, $\mathcal{D}\phi^2, F^M$ and H are gauge invariant up to terms proportional to $d\vartheta_M$, while the 4-form fields strengths G^M are gauge invariant up to terms proportional to $d\vartheta_M$ and to equations of motion that establish relations of duality among the fields. Nevertheless, the action that we are going to use is off-shell gauge invariant, up to total derivatives.

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Suppressing the normalization factor $(16\pi G_N^{(4)})^{-1}$, for the moment, the action takes the form

$$
\begin{aligned}
S = \int \Big\{ & - \star (e^a \wedge e^b) \wedge R_{ab} + \tfrac{1}{2} d\phi^1 \wedge \star d\phi^1 + \tfrac{1}{2}(\phi^1)^2 \mathcal{D}\phi^2 \wedge \star \mathcal{D}\phi^2 \\
& + \tfrac{1}{2} F \wedge \star F + \tilde{\vartheta} B \wedge \left(\tilde{F} - \tfrac{1}{2} \vartheta B \right) - C^M \wedge d\vartheta_M + \star V(\phi) \Big\} ,
\end{aligned} \tag{2.5}
$$

where the potential is assumed to be a function of ϕ^1 only and of the two coupling constants $\vartheta, \tilde{\vartheta}$ which are needed for dimensional reasons; for the same reason they must appear in it quadratically, so that the potential is a homogeneous function of degree two, i.e. the potential satisfies

$$
\vartheta_M \frac{\partial V}{\partial \vartheta_M} = 2\, V . \tag{2.6}
$$

The second and third terms in the second line of the action are gong to be referred to as "additional": they are topological and do not contain kinetic terms.

The equations of motion are defined by the general variation of the action

$$
\begin{aligned}
\delta S = \int \Big\{ & \mathbf{E}_a \wedge \delta e^a + \mathbf{E}_1 \delta\phi^1 + \mathbf{E}_2 \delta\phi^2 + \mathbf{E}_{A^M} \wedge \delta A^M + \mathbf{E}_B \wedge \delta B + \mathbf{E}_{C^M} \wedge \delta C^M \\
& + \mathbf{E}_{\vartheta_M} \wedge \delta\vartheta_M + d\mathbf{\Theta}(\varphi, \delta\varphi) \Big\} .
\end{aligned} \tag{2.7}
$$

The equations of the 3-forms are just

$$
\mathbf{E}_{C^M} = d\vartheta_M , \tag{2.8}
$$

so that the ϑ_M are (piecewise) constant on-shell, as intended.

The Einstein equations only involve the field strengths of the fundamental fields ϕ^1, ϕ^2 and A because the additional terms are all topological:

$$
\begin{aligned}
\mathbf{E}_a = {} & \imath_a \star (e^c \wedge e^d) \wedge R_{cd} + \tfrac{1}{2} \left(\imath_a d\phi^1 \star d\phi^1 + d\phi^1 \wedge \imath_a \star d\phi^1 \right) \\
& + \tfrac{1}{2} (\phi^1)^2 \left(\imath_a \mathcal{D}\phi^2 \star \mathcal{D}\phi^2 + \mathcal{D}\phi^2 \wedge \imath_a \star \mathcal{D}\phi^2 \right) \\
& + \tfrac{1}{2} \left(\imath_a F \wedge \star F - F \wedge \imath_a \star F \right) - \imath_a \star V .
\end{aligned} \tag{2.9}
$$

Observe that the dual fields $\tilde{A}, B$ occur in the energy-momentum tensor through the field strengths $\mathcal{D}\phi^2$ and F.

The additional terms do not involve the scalars, either, and, therefore

$$
\mathbf{E}_1 = -d \star d\phi^1 + \phi^1 \mathcal{D}\phi^2 \wedge \star \mathcal{D}\phi^2 + \star \frac{\partial V}{\partial \phi^1} , \tag{2.10a}
$$

$$
\mathbf{E}_2 = -d \star j , \tag{2.10b}
$$

Furthermore, they do not involve A and, therefore,

$$
\mathbf{E}_A = -d \star F + \vartheta \star j . \tag{2.11}
$$

Now, let us consider the equations of motion of the dual fields which give duality relations. The equation of motion of $\tilde{A}$,

$$
\mathbf{E}_{\tilde{A}} = \tilde{\vartheta} \star j - d \left(\tilde{\vartheta} B \right) , \tag{2.12}
$$

gives the duality relation between ϕ^2 (the current j) and B (its field strength H) on-shell, when $d\tilde{\vartheta} = 0$.

The equation of motion of B,

$$\mathbf{E}_B = -\tilde{\vartheta}\left(\star F - \tilde{F}\right) , \tag{2.13}$$

is the duality relation between $\tilde{A}$ and A.

The equations of motion of the components of the embedding tensor are

$$\mathbf{E}_{\vartheta_M} = -G^M + \star\frac{\partial V}{\partial \vartheta_M} . \tag{2.14}$$

On-shell these equations are the duality relations between the components of the embedding tensor ϑ_M and the 3-forms C^M as given in [34]

$$G^M = \star\frac{\partial V}{\partial \vartheta_M} . \tag{2.15}$$

In the framework of this theory, these duality relations are only non-trivial when the corresponding component of the embedding tensor occurs in the scalar potential. However, it is clear that if those parameters[19] also occur as coefficients of terms of higher order in the Riemann curvature, the duality relations will be non-trivial as well.

Once the duality relations implied by the equations of motion of the dual fields $\tilde{A}, B, C^M$ and the embedding tensor ϑ_M are taken into account, the action we are studying describes a very simple model of a vector field and two scalars, one of which is charged with respect to the vector field, its dual or a combination of both, coupled to gravity. The use of the dual vector field as a gauge field is, perhaps, unusual, and demands the presence of the 2-form B, but we can always eliminate this aspect of the model by setting $\tilde{\vartheta} = 0$.

Finally, $\mathbf{\Theta}$ receives contributions from the variations of $e^a, \phi^1, \phi^2, A^M, \vartheta_M$ but not from those of B or C^M, which occur in the action with no derivatives:

$$\begin{aligned}\mathbf{\Theta}(\varphi, \delta\varphi) = &- \star(e^a \wedge e^b) \wedge \delta\omega_{ab} + \star d\phi^1 \delta\phi^1 + \star j \delta\phi^2 + \star F \wedge \delta A \\ &+ \tilde{\vartheta} B \wedge \delta\tilde{A} + C^M \delta\vartheta_M .\end{aligned} \tag{2.16}$$

As we have mentioned, the action is invariant under gauge transformations up to a total derivative that takes the form

$$\delta_{\text{gauge}} S = \int d\left\{\tilde{\vartheta}\Lambda \wedge d\tilde{A} + \vartheta_M d\chi^M\right\} . \tag{2.17}$$

This total derivative is only defined up to total derivatives and we can make use of this freedom to obtain gauge-invariant results, if need be.

[19] Here we are referring to all dimensionful parameters of the theory, not necessarily associated to gaugings and, therefore, not conventionally included in the concept of embedding tensor.

2.1 Gauge conserved charges

We are going to study the effect of all the independent gauge transformations simultaneously. We will denote all of them by $\delta_{\rm g}$. From the general variation of the action eq. (2.7) we get

$$\delta_{\rm g} S = \int \left\{ \mathbf{E}_2 \delta_{\rm g} \phi^2 + \mathbf{E}_{A^M} \wedge \delta_{\rm g} A^M + \mathbf{E}_B \wedge \delta_{\rm g} B + \mathbf{E}_{C^M} \wedge \delta_{\rm g} C^M + d\mathbf{\Theta}(\varphi, \delta_{\rm g}\varphi) \right\} , \tag{2.18}$$

with

$$\mathbf{\Theta}(\varphi, \delta_{\rm g}\varphi) = \star j \delta_{\rm g} \phi^2 + \star F \wedge \delta_{\rm g} A + \tilde{\vartheta} B \wedge \delta_{\rm g} \tilde{A} \, . \tag{2.19}$$

Substituting the above $\delta_{\rm g}$ variations and the expressions for the equations of motion and operating, we arrive at

$$\delta_{\rm g} S = \int d\mathbf{\Theta}'(\varphi, \delta_{\rm g}\varphi) \, , \tag{2.20}$$

with

$$\begin{aligned} \mathbf{\Theta}'(\varphi, \delta_{\rm g}\varphi) &= \mathbf{\Theta}(\varphi, \delta_{\rm g}\varphi) - \star j \vartheta_M \sigma^M + \sigma d \star F + \tilde{\sigma} d \left(\tilde{\vartheta} B\right) - \tilde{\vartheta} \left(\star F - \tilde{F}\right) \wedge \Lambda - d\vartheta_M \wedge \chi^M \\ &= d \left(\sigma \star F + \tilde{\sigma} \tilde{\vartheta} B\right) + \tilde{\vartheta} d\tilde{A} \wedge \Lambda - d\vartheta_M \wedge \chi^M \, . \end{aligned} \tag{2.21}$$

On the other hand, the action is only gauge invariant up to the total derivative in eq. (2.17) that we can write, for the sake of convenience, in the form

$$\delta_{\rm g} S = \int d \left[\tilde{\vartheta} d\tilde{A} \wedge \Lambda + \vartheta_M d\chi^M - d \left(\tilde{\vartheta} \tilde{A} \wedge \Lambda \right) \right] = \int d \left[\tilde{A} \wedge d \left(\tilde{\vartheta} \Lambda \right) + \vartheta_M d\chi^M \right] , \tag{2.22}$$

and combining this result with the previous one we arrive at the off-shell identity

$$\int d\mathbf{J} = 0 \, , \quad \text{with} \quad \mathbf{J} \equiv \mathbf{\Theta}'(\varphi, \delta_{\rm g}\varphi) - \tilde{A} \wedge d \left(\tilde{\vartheta} \Lambda \right) - \vartheta_M d\chi^M \, , \tag{2.23}$$

which implies, locally

$$\mathbf{J} = d\mathbf{Q} \, , \tag{2.24}$$

where

$$\mathbf{Q} = \sigma \star F + \tilde{\sigma} \tilde{\vartheta} B + \tilde{\vartheta} \tilde{A} \wedge \Lambda - \vartheta_M \chi^M \, . \tag{2.25}$$

Now we must identify the Killing parameters $\sigma^M, \Lambda, \chi^M$ that generate transformations that leave invariant all the fields.

The 2-form B is left invariant by 1-forms which are closed, so $\Lambda = h + d\alpha$, for a harmonic 1-form h and an arbitrary function α. However, the 1-forms A^M are invariant for functions σ^M such that $d\sigma^M = \vartheta^M(h + d\alpha)$, which implies that $h = 0$ and $\sigma^M = \vartheta^M \alpha + \beta^M$, for arbitrary, constant symplectic vectors β^M. The invariance of ϕ^2 implies that $\vartheta_M \beta^M = 0$. If $\vartheta_M \neq 0$, then $\beta^M = \vartheta^M \beta$ for a single arbitrary constant β, but when $\vartheta_M = 0$, β^M is arbitrary. Finally, the invariance of C^M implies that the 2-forms $\chi^M = \vartheta^M \alpha + \beta^M B + \Omega^M + d\epsilon^M$ where ϵ^M and Ω^M are, respectively, symplectic vectors of 1-forms and harmonic 2-forms. The (pullback of the) latter are proportional to the volume

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of the 2-dimensional space on which the charge 2-form is going to be integrated and we can write, with a slight abuse of language $\omega^M = \gamma^M \Omega_{\partial V}$. Summarizing:

$$\sigma^M = \vartheta^M \alpha + \beta^M \,, \tag{2.26a}$$

$$\Lambda = d\alpha \,, \tag{2.26b}$$

$$\chi^M = \vartheta^M \alpha + \beta^M B + \gamma^M \Omega_{\partial V} + d\epsilon^M \,, \tag{2.26c}$$

with

$$\vartheta_M \beta^M = 0 \,, \qquad d \star \Omega^M = 0 \,. \tag{2.27}$$

When the fields are on-shell, these Killing parameters give rise to several independent conserved charges in the 3-volume V with compact boundary ∂V:

Electric charge: associated to β, which can set to 1:[20]

$$Q \equiv \frac{-1}{16\pi G_N^{(4)}} \int_{\partial V} (\star F - \vartheta B) \,. \tag{2.28}$$

If we deform ∂V without crossing any sources (i.e. points at which the equations of motion are not satisfied.), the difference between the charges will be, via Stokes theorem, the volume integral

$$\Delta \mathbf{Q} = \frac{-1}{16\pi G_N^{(4)}} \int_V d(\star F - \vartheta B) \,, \tag{2.29}$$

whose integrand vanishes on-shell.

2-form charge: associated to the function α

$$Q[\alpha] \equiv \frac{-1}{16\pi G_N^{(4)}} \int_{\partial V} \left[\alpha (\star F - \vartheta B) + d\alpha \wedge \tilde{A}\right] = \frac{1}{16\pi G_N^{(4)}} \int_{\partial V} d\left(\vartheta \alpha \tilde{A}\right) = 0 \,. \tag{2.30}$$

3-form charge: associated to the space which we are going to integrate over,[21] it is just its volume (surface)

$$Q[V] \equiv \int_{\partial V} \Omega_{\partial V} \,. \tag{2.31}$$

We can also define a magnetic charge

$$P \equiv \frac{-1}{16\pi G_N^{(4)}} \int_{\partial V} \left(F + \tilde{\vartheta} B\right) \,, \tag{2.32}$$

which is conserved in the same sense as the electric one thanks to the Bianchi identity instead of the equations of motion. This charge can be combined with the electric one in a symplectic vector

$$\begin{pmatrix} P \\ Q \end{pmatrix} = \left(Q^M\right) \,, \qquad Q^M = \frac{-1}{16\pi G_N^{(4)}} \int_{\partial V} \left(F^M - \vartheta^M B\right) \,. \tag{2.33}$$

[20] This is the upper component of β^M.

[21] There are no non-trivial charges associated to the 1-forms ϵ^M because the integrand is, again, a total derivative. We have normalized $\vartheta_M \gamma^M = 1$.

2.2 The Noether-Wald charge

2.2.1 Transformations of the fields

As usual, we want to define transformations δ_ξ that annihilate all the fields of a given solution in a gauge-invariant way for certain parameters $\xi = k$ which are, in particular, Killing vectors. We have to combine standard Lie derivatives and k-dependent ("compensating") gauge transformations into gauge-covariant Lie derivatives.

It is convenient to start by analyzing the 2-form B through its gauge-invariant 3-form field strength $H = dB$. Due to the Bianchi identity $dH = 0$

$$\delta_\xi H = -d\imath_\xi H\,. \tag{2.34}$$

When $\xi = k$, there must exist a momentum map 1-form $\mathbf{P}_k$ such that

$$d\mathbf{P}_k = -\imath_k H\,. \tag{2.35}$$

Now, the transformation of B is the Lie derivative plus a gauge transformation with a ξ-dependent 1-form parameter Λ_ξ

$$\delta_\xi B = -d\imath_\xi B - \imath_\xi H + d\Lambda_\xi\,. \tag{2.36}$$

When $\xi = k$ we can use the definition of the momentum map 1-form $\mathbf{P}_k$ to get

$$d\left(\imath_k B - \mathbf{P}_k - \Lambda_k\right) = 0 \tag{2.37}$$

which is solved by the choice

$$\Lambda_k = \imath_k B - \mathbf{P}_k\,. \tag{2.38}$$

Then, we define

$$\delta_\xi B = -\left(\imath_\xi H + d\mathbf{P}_\xi\right)\,, \tag{2.39}$$

where the 1-form $\mathbf{P}_\xi$ is the momentum map 1-form $\mathbf{P}_k$ when $\xi = k$. With these definitions, $\delta_k B = 0$ automatically and in a gauge-invariant fashion.

Let us now consider the gauge-invariant 2-form field strengths F^M:

$$\delta_\xi F^M = -d\imath_\xi F^M - \imath_\xi dF^M = -d\imath_\xi F^M - \imath_\xi\left(d\vartheta^M \wedge B + \vartheta^M H\right)\,. \tag{2.40}$$

On-shell and for $\xi = k$

$$\delta_k F^M = -d\imath_k F^M - \vartheta^M \imath_k H = -d\left(\imath_k F^M - \vartheta^M \mathbf{P}_k\right) = 0\,, \tag{2.41}$$

upon use of the definition of $\mathbf{P}_k$. Then, locally, there must exist momentum maps P_k^M such that

$$\imath_k F^M - \vartheta^M \mathbf{P}_k = -dP^M{}_k\,. \tag{2.42}$$

The transformation of the 1-forms A^M is (minus) their Lie derivative plus a gauge transformation with a ξ-dependent 1-form parameter Λ_ξ which has to be the same we determined before and a gauge transformation with ξ-dependent 0-form parameters σ_ξ^M

$$\begin{aligned}\delta_\xi A^M &= -d\imath_\xi A^M - \imath_\xi dA^M + d\sigma_\xi^M - \vartheta^M \Lambda_\xi \\ &= -d\imath_\xi A^M - \imath_\xi F^M + d\sigma_\xi^M + \vartheta^M \mathbf{P}_\xi\,.\end{aligned} \tag{2.43}$$

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When $\xi = k$

$$\delta_k A^M = -d\left(\imath_k A^M - P_k^M - \sigma_k^M\right) = 0\,, \tag{2.44}$$

which is solved by the choice

$$\sigma_k^M = \imath_k A^M - P_k^M\,. \tag{2.45}$$

Therefore, we define

$$\sigma_\xi^M \equiv \imath_\xi A^M - P_\xi^M\,, \tag{2.46}$$

and

$$\delta_\xi A^M = -\left(\imath_\xi F^M + dP_\xi^M - \vartheta^M \mathbf{P}_\xi\right)\,, \tag{2.47}$$

where, when $\xi = k$, P_ξ^M and $\mathbf{P}_\xi$ are, respectively, the momentum map 0- and 1-forms. Again, $\delta_k A^M = 0$ automatically and in a gauge-invariant form.

ϕ^1 is a scalar, and transforms in the standard way

$$\delta_\xi \phi^1 = -\pounds_\xi \phi^1 = -\imath_\xi d\phi^1\,. \tag{2.48}$$

This transformation is assumed to vanish for $\xi = k$.

The scalar ϕ^2, however, transforms non-trivially under gauge transformations. It is convenient to analyze, first, its covariant derivative, which is, actually, gauge-invariant.

$$\delta_\xi \mathcal{D}\phi^2 = -d\imath_\xi \mathcal{D}\phi^2 - \imath_\xi d\mathcal{D}\phi^2 = -d\imath_\xi \mathcal{D}\phi^2 + \vartheta_M \imath_\xi F^M\,. \tag{2.49}$$

On-shell and for $\xi = k$ the following identity must hold

$$-\,d\left(\imath_k \mathcal{D}\phi^2 + \vartheta_M P_k^M\right) = 0\,, \quad \Rightarrow \quad \imath_k \mathcal{D}\phi^2 = -\vartheta_M P_k^M\,, \tag{2.50}$$

where σ_ξ^M defined in eq. (2.46). The transformation of ϕ^2 is a combination of (minus) the Lie derivative and a gauge transformation with parameter σ_ξ^M

$$\delta_\xi \phi^2 = -\imath_\xi d\phi^2 + \vartheta_M \sigma_\xi^M = -\imath_\xi \mathcal{D}\phi^2 - \vartheta_M P_\xi^M\,, \tag{2.51}$$

and $\delta_k \phi^2$ vanishes identically by virtue of eq. (2.50).

Let us consider, finally, the 3-forms. As usual, it is convenient to study their 4-form field strengths first. They are gauge-invariant on-shell only. By assumption, and because these are 4-forms in 4 dimensions

$$0 = \delta_k G^M = -d\imath_k G^M\,, \quad \Rightarrow dP_{G\,k}^M = -\imath_k G^M\,, \tag{2.52}$$

defining the momentum map 2-forms $P_{G\,k}^M$. The transformations of the 3-forms C^M must be a combination of their Lie derivatives and gauge transformations with the parameters $\sigma_\xi^M, \Lambda_\xi$ that we have already determined and, possibly χ_ξ^M:

$$\begin{aligned}
\delta_\xi C^M &= -d\imath_\xi C^M - \imath_\xi dC^M + d\chi_\xi^M - \sigma_\xi^M \star j - \vartheta^M \Lambda_\xi \wedge B - \delta^{M\tilde{\cdot}} \Lambda_\xi \wedge (\star F - \tilde{F}) \\
&= -d\left(\imath_\xi C^M - \chi_\xi^M\right) - \imath_\xi G^M + P_\xi^M \star j - A^M \wedge \imath_\xi \star j \\
&\quad + \vartheta^M \mathbf{P}_\xi \wedge B + \delta^{M\tilde{\cdot}}\left[\mathbf{P}_\xi \wedge (\star F - \tilde{F}) + B \wedge \imath_\xi (\star F - \tilde{F})\right]\,.
\end{aligned} \tag{2.53}$$

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On-shell and for $\xi = k$

$$\delta_k C^M = -d\left(\imath_k C^M - \chi_k^M - P^M_{G\,k} - \mathbf{P}_k \wedge A^M\right) + P_k^M H + \mathbf{P}_k \wedge F^M \,. \tag{2.54}$$

We can show that the last two terms are, locally, a total derivative:

$$\begin{aligned} d\left(P_k^M H + \mathbf{P}_k \wedge F^M\right) &= dP_k^M H + d\mathbf{P}_k \wedge F^M - \vartheta^M \mathbf{P}_k \wedge H \\ &= -\imath_k F \wedge H - \imath_k H \wedge F^M = -\imath_k\left(F \wedge H\right) = 0\,. \end{aligned} \tag{2.55}$$

Thus, we define the 2-form $X^M_{2\,k}$ by

$$P_k^M H + \mathbf{P}_k \wedge F^M \equiv dX^M_{2\,k}\,. \tag{2.56}$$

Absorbing it in the definition of $P^M_{G\,k}$, which now satisfies

$$dP^M_{G\,k} = -\imath_k G^M + P_k^M H + \mathbf{P}_k \wedge F^M\,, \tag{2.57}$$

we conclude that

$$\delta_k C^M = -d\left(\imath_k C^M - \chi_k^M - P^M_{G\,k} - \mathbf{P}_k \wedge A^M\right) = 0\,, \tag{2.58}$$

which is solved by

$$\chi_k^M = \imath_k C^M - P^M_{G\,k} - \mathbf{P}_k \wedge A^M\,. \tag{2.59}$$

Then, we arrive at the definition

$$\begin{aligned} \delta_\xi C^M = &-\imath_\xi G^M - dP^M_{G\,\xi} + \mathbf{P}_\xi \wedge F^M + P_\xi^M \star j - A^M \wedge (\imath_\xi \star j + d\mathbf{P}_\xi) \\ &+ \delta^{M\tilde{\cdot}}\left[\mathbf{P}_\xi \wedge (\star F - \tilde{F}) + B \wedge \imath_\xi(\star F - \tilde{F})\right]\,, \end{aligned} \tag{2.60}$$

which vanishes automatically for $\xi = k$.

Summarizing, the transformations that we are going to consider are

$$\delta_\xi e^a = -(\mathcal{D}\xi^a + P_\xi{}^a{}_b e^b)\,, \tag{2.61a}$$

$$\delta_\xi \omega^{ab} = -(\imath_\xi R^{ab} + \mathcal{D}P_\xi{}^{ab})\,, \tag{2.61b}$$

$$\delta_\xi \phi^1 = -\imath_\xi d\phi^1\,, \tag{2.61c}$$

$$\delta_\xi \phi^2 = -\left(\imath_\xi \mathcal{D}\phi^2 + \vartheta_M P_\xi^M\right)\,, \tag{2.61d}$$

$$\delta_\xi A^M = -\left(\imath_\xi F^M + dP_\xi^M - \vartheta^M \mathbf{P}_\xi\right)\,, \tag{2.61e}$$

$$\delta_\xi B = -\left(\imath_\xi H + d\mathbf{P}_\xi\right)\,, \tag{2.61f}$$

$$\begin{aligned} \delta_\xi C^M = &-\left(\imath_\xi G^M + dP^M_{G\,\xi} - \mathbf{P}_\xi \wedge F^M - P_\xi^M \star j\right) - A^M \wedge (\imath_\xi \star j + d\mathbf{P}_\xi) \\ &+ \delta^{M\tilde{\cdot}}\left[\mathbf{P}_\xi \wedge (\star F - \tilde{F}) + B \wedge \imath_\xi(\star F - \tilde{F})\right]\,, \end{aligned} \tag{2.61g}$$

$$\delta_\xi \vartheta_M = -\imath_\xi d\vartheta_M\,, \tag{2.61h}$$

and the momentum maps 0-, 1-, and 2-forms satisfy

$$\mathcal{D}P_k{}^{ab} = -\imath_k R^{ab}\,, \tag{2.62a}$$

$$dP^M{}_k = -\imath_k F^M + \vartheta^M \mathbf{P}_k\,, \tag{2.62b}$$

$$d\mathbf{P}_k = -\imath_k H\,, \tag{2.62c}$$

$$dP^M_{G\,k} = -\imath_k G^M + P^M_k H + \mathbf{P}_k \wedge F^M\,. \tag{2.62d}$$

Furthermore, when $\xi = k$

$$\imath_k \mathcal{D}\phi^2 = -\vartheta_M P^M_k\,. \tag{2.63}$$

2.2.2 Transformation of the action

Substituting the above transformations of the fields in

$$\delta_\xi S = \int \Big\{\mathbf{E}_a \wedge \delta_\xi e^a + \mathbf{E}_1 \delta_\xi \phi^1 + \mathbf{E}_2 \delta_\xi \phi^2 + \mathbf{E}_{A^M} \wedge \delta_\xi A^M + \mathbf{E}_B \wedge \delta_\xi B + \mathbf{E}_{C^M} \wedge \delta_\xi C^M + \mathbf{E}_{\vartheta_M} \wedge \delta_\xi \vartheta_M + d\mathbf{\Theta}(\varphi, \delta_\xi\varphi)\Big\}\,, \tag{2.64}$$

integrating by parts and using the Noether identities we are left with

$$\delta_\xi S = \int d\mathbf{\Theta}'(\varphi, \delta_\xi\varphi)\,, \tag{2.65}$$

with

$$\mathbf{\Theta}'(\varphi, \delta_\xi\varphi) \equiv \mathbf{\Theta}(\varphi, \delta_\xi\varphi) + \xi^a \mathbf{E}_a + P^M_\xi \mathbf{E}_{A^M} - \mathbf{P}_\xi \wedge \left(\mathbf{E}_B + \mathbf{E}_{C^M} \wedge A^M\right) + P^M_{G\,\xi} \wedge \mathbf{E}_{C^M}\,. \tag{2.66}$$

Under these transformations, the action transforms into the integral of a total derivative, that we have chosen so as to obtain a final gauge-invariant result:

$$\delta_\xi S = \int d\left\{-\imath_\xi \mathbf{L} + \tilde{\vartheta}\imath_\xi B \wedge d\tilde{A} + \tilde{A} \wedge d\left(\tilde{\vartheta}\mathbf{P}_\xi\right) - d\vartheta_M \wedge \left(\imath_\xi C^M - \mathbf{P}_\xi \wedge A^M\right) - \vartheta_M dP^M_{G\,\xi}\right\}\,. \tag{2.67}$$

Equating this result for $\delta_\xi S$ with the one in eq. (2.65) we arrive to the identity

$$\int d\mathbf{J}[\xi] = 0\,, \tag{2.68}$$

with

$$\mathbf{J}[\xi] = \mathbf{\Theta}'(\varphi, \delta_\xi\varphi) + \imath_\xi \mathbf{L} - \tilde{\vartheta}\imath_\xi B \wedge d\tilde{A} - \tilde{A} \wedge d\left(\tilde{\vartheta}\mathbf{P}_\xi\right) + d\vartheta_M \wedge \left(\imath_\xi C^M - \mathbf{P}_\xi \wedge A^M\right) + \vartheta_M dP^M_{G\,\xi}\,. \tag{2.69}$$

Simplifying this expression we get

$$\mathbf{J}[\xi] = d\mathbf{Q}[\xi]\,, \tag{2.70a}$$

$$\mathbf{Q}[\xi] = \star(e^a \wedge e^b) \wedge P_{\xi\,ab} - \left(P_\xi \star F + \tilde{\vartheta}\tilde{P}_\xi B + \tilde{\vartheta}\tilde{A} \wedge \mathbf{P}_\xi - \vartheta_M P^M_{G\,\xi}\right)\,. \tag{2.70b}$$

The second term in this formula, the one in parenthesis, should be compared with the 2-form charge associated to gauge transformations eq. (2.25).

2.3 Generalized, restricted, zeroth laws

We just need to adapt the discussion in section 1.3 to the model at hand, which has more fields. On the bifurcation surface $\mathcal{BH}$ we have

$$dP_k^M - \vartheta^M \mathbf{P}_k \stackrel{\mathcal{BH}}{=} 0\,, \tag{2.71a}$$

$$d\mathbf{P}_k \stackrel{\mathcal{BH}}{=} 0\,, \tag{2.71b}$$

$$d\mathbf{P}^M_{G\,k} - P_k^M H - \mathbf{P}_k \wedge F^M \stackrel{\mathcal{BH}}{=} 0\,, \tag{2.71c}$$

$$\vartheta_M P_k^M \stackrel{\mathcal{BH}}{=} 0\,. \tag{2.71d}$$

These equations are equivalent to the equations that the Killing parameters discussed on page 18 must satisfy: first of all, the second equation implies that $\mathbf{P}_k = h + d\alpha$, where h is a harmonic 1-form on the bifurcation surface and α and arbitrary function. However, the first equation tells us that h has to be removed from that identity and $P_k^M = \vartheta^M\alpha + \beta^M$ for β^M which is constant over the bifurcation surface. The last equation implies that $\beta^M = \vartheta^M\beta$ if $\vartheta_M \neq 0$, but it is arbitrary when $\vartheta_M = 0$. The third equation takes the form

$$d\mathbf{P}^M_{G\,k} - \left(\vartheta^M\alpha + \beta^M\right) dB - d\alpha \wedge F^M = d\left(\mathbf{P}^M_{G\,k} - \alpha F^M - \beta^M B\right) \stackrel{\mathcal{BH}}{=} 0\,, \tag{2.72}$$

and, summarizing, we have[22]

$$P_k^M \stackrel{\mathcal{BH}}{=} \vartheta^M\alpha + \beta^M\,, \tag{2.73a}$$

$$\mathbf{P}_k \stackrel{\mathcal{BH}}{=} d\alpha\,, \tag{2.73b}$$

$$\mathbf{P}^M_{G\,k} \stackrel{\mathcal{BH}}{=} \alpha F^M + \beta^M B + \gamma^M \Omega_{\mathcal{BH}}\,, \tag{2.73c}$$

where γ^M is a constant symplectic vector and $\Omega_{\mathcal{BH}}$ is the volume 2-form of the bifurcation surface; from eq. (2.71d) we have

$$\vartheta_M \beta^M = 0\,. \tag{2.74}$$

The components of the constant vector β^M can be interpreted as the electric and magnetic potentials over the bifurcation surface and the fact that they are constant is the generalized zeroth law restricted to the bifurcation surface. It is unclear whether this property can be extended to the whole event horizon in this particularly complex model or, at least, it is unclear how to prove it. However, we will not need this proof. As the β^M are the thermodynamical potentials associated to the electric and magnetic charges, we will denote them by

$$\Phi^M \equiv \beta^M\,, \quad \text{with} \quad (\Phi^M) = \begin{pmatrix} \Phi \\ \tilde{\Phi} \end{pmatrix}\,. \tag{2.75}$$

Observe that eq. (2.74) becomes a constraint on these potentials:

$$\vartheta_M \Phi^M = 0\,. \tag{2.76}$$

[22]Compare with eqs. (2.26).

Even though this constraint follows directly from our definitions, it is nonetheless surprising and the following interpretation may be helpful: observe that the gauge transformations (2.4a) and (2.4b) imply that, on-shell, the field ϕ^2 is a Stückelberg field for the combination $\vartheta_M A^M$, which therefore behaves, on-shell, as a massive vector field. The constraint (2.76) then states that the electric part of the (massive) combination $\vartheta_M A^M$ cannot give contributions on bifurcate surfaces, which is in accordance with previous results on massive vector fields in black hole spacetimes, see e.g. refs. [52–54].

The components of the constant vector γ^M are the thermodynamical potentials ("volumes") associated to the thermodynamical variables ϑ_M ("pressures"). Again, in order to make contact with the conventions of ref. [6], we can define the potentials Θ^M

$$\frac{\gamma^M}{16\pi G_N^{(4)}} \equiv -\Theta^M/V_{\mathcal{BH}}\,, \tag{2.77}$$

where $V_{\mathcal{BH}}$ is the volume of the bifurcation surface

$$V_{\mathcal{BH}} = \int_{\mathcal{BH}} \Omega_{\mathcal{BH}}\,. \tag{2.78}$$

The fact that the vector Θ^M is constant over the bifurcation surface is another generalized, restricted, zeroth law

There is no role for the function α: there are no conserved charges associated to the gauge transformations δ_Λ and α will also drop out of the Smarr formula.

2.4 Komar integral and Smarr formula

We are now ready to construct the Komar integral for this theory along the lines explained in section 1.4. It provides a highly non-trivial check of the Noether-Wald charge.

Let us consider a field configuration that satisfies the equations of motion and an infinitesimal diffeomorphism $\xi = k$ that generates a symmetry of the whole field configuration. Then, since $\mathbf{\Theta}(\varphi, \delta_\xi\varphi)$ is linear in $\delta_\xi\varphi$, it vanishes when $\xi = k$ and, since the equations of motion are satisfied, so does $\mathbf{\Theta}(\varphi, \delta_\xi\varphi)$. Then, from the definition eq. (2.69) we find that[23]

$$\mathbf{J}[k] \doteq \imath_k\mathbf{L} - \tilde{\vartheta}\imath_\xi B\wedge d\tilde{A} - \tilde{A}\wedge d\left(\tilde{\vartheta}\mathbf{P}_\xi\right) + d\vartheta_M\wedge\left(\imath_\xi C^M - \mathbf{P}_\xi\wedge A^M\right) + \vartheta_M dP^M_{G\,\xi}\,. \tag{2.79}$$

On the other hand, by construction,

$$\delta_k S = 0\,, \tag{2.80}$$

and, thus, the total derivative in eq. (2.67) evaluated for $\xi = k$, which coincides with the on-shell value of $\mathbf{J}[k]$ in eq. (2.79), must vanish identically and, locally, there is a 2-form ϖ_k such that

$$d\varpi_k \doteq \imath_k\mathbf{L} - \tilde{\vartheta}\imath_\xi B\wedge d\tilde{A} - \tilde{A}\wedge d\left(\tilde{\vartheta}\mathbf{P}_\xi\right) + d\vartheta_M\wedge\left(\imath_k C^M - \mathbf{P}_k\wedge A^M\right) + \vartheta_M dP^M_{G\,k} = \mathbf{J}[k]\,. \tag{2.81}$$

[23] As before, we use $\doteq$ for identities that only hold on-shell.

Since we also have $\mathbf{J}[k] = d\mathbf{Q}[k]$, we conclude that

$$d\left\{\mathbf{Q}[k] - \varpi_k\right\} \doteq 0\,. \tag{2.82}$$

In order to compute the Komar charge $-\{\mathbf{Q}[k] - \varpi_k\}$ we proceed as before, taking the trace of the Einstein equations (2.9)

$$e^a \wedge E_a = -2\mathbf{L} + F \wedge \star F + 2\tilde{\vartheta} B \wedge \left(\tilde{F} - \tfrac{1}{2}\vartheta B\right) - 2C^M \wedge d\vartheta_M - 2 \star V\,. \tag{2.83}$$

and, on-shell ($\star F = \tilde{F}$ and $d\vartheta_M = 0$)

$$\imath_k \mathbf{L} \doteq \tfrac{1}{2}\left(\imath_k F + 2\tilde{\vartheta}\imath_k B\right) \wedge \tilde{F} + \tfrac{1}{2}\left(F + 2\tilde{\vartheta}B\right) \wedge \imath_k \tilde{F} - \tilde{\vartheta}\vartheta \imath_k B \wedge B - \imath_k \star V\,. \tag{2.84}$$

Combining this result with the other terms and operating, we get

$$\begin{aligned}
&\imath_k \mathbf{L} - \tilde{\vartheta}\imath_\xi B \wedge d\tilde{A} - \tilde{A} \wedge d\left(\tilde{\vartheta}\mathbf{P}_\xi\right) + d\vartheta_M \wedge \left(\imath_k C^M - \mathbf{P}_k \wedge A^M\right) + \vartheta_M dP_{G\,k}^M \\
&\doteq \tfrac{1}{2}\vartheta_M \left(\mathbf{P}_k \wedge dA^M - B \wedge dP_k^M\right) - \imath_k \star V + d\left(\vartheta_M P_{G\,k}^M - \tfrac{1}{2}P_k d\tilde{A} - \tfrac{1}{2}\tilde{P}_k dA - \tilde{\vartheta}\tilde{A} \wedge \mathbf{P}_k\right)\,.
\end{aligned} \tag{2.85}$$

Let us now consider the term involving the scalar potential: using the fact that the potential is a homogeneous function of the embedding tensor, eq. (2.6), and the on-shell ϑ_M equation of motion, eq. (2.14), we obtain

$$\imath_k \star V = \tfrac{1}{2}\vartheta_M \imath_k G^M = \tfrac{1}{2}\vartheta_M \left(-dP_{G\,k}^M + P_k^M H + \mathbf{P}_k \wedge F^M\right)\,. \tag{2.86}$$

After use of the definition of the 2-form momentum map $P_{G\,k}^M$ in eq. (2.62d), we arrive at

$$\begin{aligned}
d\varpi_k &= \imath_k \mathbf{L} - \tilde{\vartheta}\imath_\xi B \wedge d\tilde{A} - \tilde{A} \wedge d\left(\tilde{\vartheta}\mathbf{P}_\xi\right) + d\vartheta_M \wedge \left(\imath_k C^M - \mathbf{P}_k \wedge A^M\right) + \vartheta_M dP_{G\,k}^M \\
&\doteq d\left(\tfrac{3}{2}\vartheta_M P_{G\,k}^M - \tfrac{1}{2}P_k d\tilde{A} - \tfrac{1}{2}\tilde{P}_k dA - \tfrac{1}{2}\vartheta_M P_k^M B + \tilde{\vartheta}\tilde{A} \wedge \mathbf{P}_k\right)\,.
\end{aligned} \tag{2.87}$$

Combining this result with eq. (2.70b) we obtain the Komar charge

$$-\left\{\mathbf{Q}[k] - \varpi_k\right\} \doteq \frac{-1}{16\pi G_N^{(4)}}\left\{\star(e^a \wedge e^b) \wedge P_{k\,ab} + \tfrac{1}{2}P_{k\,M}F^M - \tfrac{1}{2}\vartheta_M P_{G\,k}^M\right\}\,. \tag{2.88}$$

This is a manifestly (formally) symplectic-invariant result [18] that reduces to the result obtained in section 1.4 if we eliminate the scalar and 1-form fields; it reduces to the Einstein-Maxwell result upon setting the embedding tensor to zero.

We can now proceed as in section 1.4 to derive the Smarr formula through the identity between the Komar integrals over the bifurcation surface and at spatial infinity eq. (1.61). At infinity

$$\mathcal{K}(S^2_\infty) = \frac{-1}{16\pi G_N^{(4)}}\int_{S^2_\infty}\left\{\star(e^a \wedge e^b) \wedge P_{k\,ab} + \tfrac{1}{2}P_{k\,M}F^M - \tfrac{1}{2}\vartheta_M P_{G\,k}^M\right\} = \tfrac{1}{2}\left(M - \Omega J\right)\,, \tag{2.89}$$

essentially by definition. Over the bifurcation surface, using the generalized, restricted, zeroth laws eqs. (2.73), the definitions of the potentials eqs. (2.75) (2.77) and of the definitions of electric and magnetic charges eq. (2.33), we get

$$\mathcal{K}(\mathcal{BH}) = ST + \tfrac{1}{2}\Phi_M Q^M - \tfrac{1}{2}\vartheta_M \Theta^M\,, \tag{2.90}$$

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and we obtain the Smarr formula

$$M = 2ST + \Omega J + \Phi_M Q^M - \vartheta_M \Theta^M\,. \tag{2.91}$$

In the derivation of this formula we have implicitly assumed that the momentum maps which give the electrostatic and magnetostatic potentials vanish at spatial infinity and, thus, only their values at the horizon (or, more precisely, at the bifurcations surface $\mathcal{BH}$) occur in the Smarr formula. Otherwise, Φ_M would be replaced by the differences between the potentials at $\mathcal{BH}$ and at infinity. Similar boundary conditions can be demanded to higher-rank momentum maps in asymptotically-flat spaces and, obviously, they will be assumed in the derivation of the first law as well. Imposing these boundary conditions is possible because the restricted, generalized, zeroth laws only indicate the closedness of certain differential forms which, therefore, are only defined up to the addition of exact forms (constants for scalar momentum maps). We would like to stress that this ambiguity, which has no effect on the final results, does not indicate dependence on the gauge transformations of the fields of the theory even it is similar to a gauge freedom.

In order to check the Smarr formula eq. (2.91) we need explicit analytic solutions of the equations of motion of this model, but this is quite difficult, as the attempt made in the appendix shows. However, it is clear that, had we considered an additional cosmological-constant parameter λ in the theory, we would simply have obtained an additional term $-\lambda\Theta_\lambda$ in the above formula. That formula should remain valid when the embedding tensor is set to zero, in which case the theory reduces to the cosmological Einstein-Maxwell theory.

Observe that, due to the constraint eq. (2.76), only one combination of the electric and magnetic potentials occurs in the above formula and, henceforth, only one combination of the electric and magnetic charges does.

2.5 The first law and black-hole chemistry

It is not necessary to repeat here all the steps that lead to the first law

$$\delta M = T\delta S + \Omega\delta J + \Phi\delta Q + \Theta^M \delta\vartheta_M\,. \tag{2.92}$$

Observe that, as usual, only the variation of the electric charge and its associated electric potential occur in the first law. This could be due to a limitation of the techniques that we are using. Nevertheless, if the magnetic counterpart of the $\Phi\delta Q$ term was present, due to the constraint eq. (2.76), there would be a combination of electric and magnetic charges the mass of the black whole would be independent of. We will comment upon this point in the discussion section.

3 Discussion

In this paper we have shown how the variations of the cosmological constant and other dimensionful constants occurring in a theory of gravity can be consistently dealt with and understood in the framework of Wald's formalism and how they enter the first law of black-hole thermodynamics and the Smarr formula. In the example that we have completely

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worked out in section 2, the constants that we have considered can be seen as components of the embedding tensor (a very simple one since there is only a 1-dimensional symmetry to be gauged) and our result proves the conjectured role of the embedding tensor as a thermodynamical variable.

A very interesting aspect of the Smarr formula is that, if it is general enough and it includes all the charges a black hole can carry and all the moduli of the theory under consideration, then it has to be invariant under all the duality transformations. Observe that duality transformations act on the moduli and charges but leave the mass, temperature and entropy invariant because the Einstein metric is left invariant by them. In ref. [18] we showed that, in the context of pure $\mathcal{N} = 4, d = 4$ supergravity, indeed, the term involving the electric and magnetic potentials and charges is formally symplectic invariant. This automatically implies its invariance under the $\mathrm{SO}(6)\times\mathrm{SL}(2,\mathbb{R})$ duality group of $\mathcal{N} = 4, d = 4$ supergravity since all the 4-dimensional duality groups act on the 1-form fields as a subgroup of the symplectic group [38]. The same happens in the very simple example that we have considered here but we have also seen that the term involving the embedding tensor and its conjugate thermodynamical potential is also electric-magnetic duality invariant as it should, according to the general arguments given above. In more general models the embedding tensor is denoted by $\vartheta_A{}^M$, where the index A runs over the Lie algebra of the symmetry group of the theory. The terms that must occur in the first law and in the Smarr formula must be, respectively, of the form

$$-\Theta^A{}_M \delta\vartheta_A{}^M\,, \quad \text{and} \quad +\Theta^A{}_M \vartheta_A{}^M\,. \tag{3.1}$$

Thus, in general 4-dimensional theories with an arbitrary number of 1-form fields labeled by I, we expect the first law and the Smarr formula to take the general form[24]

$$\delta M = T\delta S + \Omega\delta J + \Phi_I \delta Q^I - \Theta^A{}_M \delta\vartheta_A{}^M\,, \tag{3.2a}$$

$$M = 2ST + \Omega J + \Phi_M Q^M + \Theta^A{}_M \vartheta_A{}^M\,. \tag{3.2b}$$

We expect to verify the validity of this general formula in more general models of gauge supergravity in forthcoming works.

Concerning the particular model that we have constructed and studied in section 2 to test these ideas, as we pointed out before, only one combination of the electric and magnetic potentials may occur in the first law. Therefore, there would be a combination of electric and magnetic charges the mass of the black holes of this theory would not depend on. In order to check this quite unusual property it is necessary to find the most general black-hole solutions of the theory. This is a very complicated problem. In the appendix we have managed to find solutions with one charge (the embedding of the Reissner-Nordström-(A)DS black hole in this theory) for a particularly simple choice of embedding tensor, but these solutions are not general enough to check whether this property, predicted by the first law, that is true.[25] Further work in this direction is necessary and under way.

[24] $\Phi_M Q^M = \Phi^I Q_I - \tilde{\Phi}_I P^I$.

[25] Here we are assuming that the complete first law should include a term proportional to the variation of the magnetic charge that we still do not know how to incorporate in our formalism. This is a problem on which we hope to report in forthcoming work.

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A Searching for solutions

We would like to have a black-hole solution of the theory introduced in section 2 in order to test the general results that we have derived. For the sake of simplicity, we set $\tilde{\vartheta} = 0$ (electric gauging) and we set $\vartheta = g$, constant. We can set $B = 0$ (so $F = dA$) and ignore C. The equations of motion that remain to be solved are

$$\begin{aligned}
\mathbf{E}_a &= \imath_a \star (e^c \wedge e^d) \wedge R_{cd} + \tfrac{1}{2}\left(\imath_a d\phi^1 \star d\phi^1 + d\phi^1 \wedge \imath_a \star d\phi^1\right) \\
&\quad + \tfrac{1}{2}\left(\imath_a \mathcal{D}\phi^2 \star j + \mathcal{D}\phi^2 \wedge \imath_a \star j\right) \\
&\quad + \tfrac{1}{2}\left(\imath_a F \wedge \star F - F \wedge \imath_a \star F\right) - \imath_a \star V\,, && \text{(A.1a)} \\
\mathbf{E}_1 &= -d \star d\phi^1 + \phi^1 \mathcal{D}\phi^2 \wedge \star \mathcal{D}\phi^2 + \star \frac{\partial V}{\partial \phi^1}\,, && \text{(A.1b)} \\
\mathbf{E}_2 &= -d \star j\,, && \text{(A.1c)} \\
\mathbf{E}_A &= -d \star F + \vartheta \star j\,, && \text{(A.1d)}
\end{aligned}$$

equated to zero.

In the search for solutions, it is convenient to express these equations in component language:

$$\begin{aligned}
G_{\mu\nu} + \tfrac{1}{2}\left(\partial_\mu \phi^1 \partial_\nu \phi^1 - \tfrac{1}{2} g_{\mu\nu} (\partial \phi^1)^2\right) + \tfrac{1}{2}\left(\mathcal{D}_\mu \phi^2 j_\nu - \tfrac{1}{2} g_{\mu\nu} \mathcal{D}^\rho \phi^2 j_\rho\right) \\
-\tfrac{1}{2}\left(F_\mu{}^\rho F_{\nu\rho} - \tfrac{1}{4} g_{\mu\nu} F^2\right) + \tfrac{1}{2} g_{\mu\nu} V = 0\,, && \text{(A.2a)} \\
-\nabla^2 \phi^1 + \phi^1 \left(\mathcal{D}\phi^2\right)^2 - \partial_{\phi^1} V = 0\,, && \text{(A.2b)} \\
-\nabla_\mu j^\mu = 0\,, && \text{(A.2c)} \\
\nabla_\mu F^{\mu\nu} - g j^\nu = 0\,. && \text{(A.2d)}
\end{aligned}$$

We are interested in static, spherically-symmetric solutions with a metric of the form

$$ds^2 = \lambda dt^2 - \lambda^{-1} dr^2 - R^2 d\Omega^2_{(2)}\,, \tag{A.3}$$

where λ and R are functions of r to be determined and

$$d\Omega^2_{(2)} = d\theta^2 + \sin^2\theta d\varphi^2\,. \tag{A.4}$$

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The timelike Killing vector is $k = \partial_t$ and we assume that it generates a diffeomorphism that leaves invariant all the fields. This means that

$$\partial_t \phi^1 = 0\,, \tag{A.5a}$$
$$\mathcal{D}_t \phi^2 = \partial_t \phi^2 - gA_t = -gP_k\,, \tag{A.5b}$$
$$F_{t\mu} = -\partial_\mu P_k\,. \tag{A.5c}$$

If we assume that the electromagnetic field is electric and we work in the gauge in which the only non-trivial component is A_t and it is only a function of r, then the scalars $\phi^{1,2}$ only depend on r as well and

$$P_k = A_t\,. \tag{A.6}$$

The r component of the Maxwell equation (A.2d), tells us that

$$j^r = 0\,, \quad \Rightarrow \quad \phi^2 = \text{constant}\,. \tag{A.7}$$

This automatically solves eq. (A.2c) and simplifies eq. (A.2b), which can be written in the form upon use of eq. (A.6)

$$\frac{1}{R^2}\left(R^2\lambda\phi^{1\,\prime}\right)' + g^2\phi^1\lambda^{-1}P_k^2 - \partial_{\phi^1}V = 0\,. \tag{A.8}$$

The t component of the Maxwell equation takes the form

$$-\frac{1}{R^2}\left(R^2 P_k'\right)' + g^2\lambda^{-1}(\phi^1)^2 P_k = 0\,. \tag{A.9}$$

Now it is the turn of the Einstein equations. We can, first, take the trace

$$R + \tfrac{1}{2}(\partial\phi^1)^2 + \tfrac{1}{2}(\phi^1)^2(\mathcal{D}\phi^2)^2 - 2V = 0\,, \tag{A.10}$$

and use it in the original equations to simplify them

$$R_{\mu\nu} + \tfrac{1}{2}\partial_\mu\phi^1\partial_\nu\phi^1 + \tfrac{1}{2}(\phi^1)^2\mathcal{D}_\mu\phi^2\mathcal{D}_\nu\phi^2 - \tfrac{1}{2}\left(F_\mu{}^\rho F_{\nu\rho} - \tfrac{1}{4}g_{\mu\nu}F^2\right) - \tfrac{1}{2}g_{\mu\nu}V = 0\,, \tag{A.11a}$$

The components of the Ricci tensor for the above metric are

$$\begin{aligned} R_{tt} &= -\tfrac{1}{2}\lambda R^{-2}\left(R^2\lambda'\right)'\,, & R_{rr} &= -\lambda^{-2}R_{tt} + 2R''/R\,, \\ R_{\theta\theta} &= \tfrac{1}{2}\left[\lambda(R^2)'\right]' - 1 & R_{\varphi\varphi} &= \sin^2\theta R_{\theta\theta}\,. \end{aligned} \tag{A.12}$$

We only need to consider the $\theta\theta, tt, rr$, components. In this order, they are

$$\tfrac{1}{2}\left[\lambda(R^2)'\right]' - 1 + \tfrac{1}{4}R^2\left(P_k'\right)^2 + \tfrac{1}{2}R^2 V = 0\,, \tag{A.13a}$$
$$-\tfrac{1}{2}\lambda R^{-2}\left(R^2\lambda'\right)' + \tfrac{1}{2}g^2(\phi^1)^2 P_k^2 + \tfrac{1}{4}\lambda\left(P_k'\right)^2 - \tfrac{1}{2}\lambda V = 0\,, \tag{A.13b}$$
$$\tfrac{1}{2}\lambda^{-1}R^{-2}\left(R^2\lambda'\right)' + 2R''/R + \tfrac{1}{2}(\phi^{1\,\prime})^2 - \tfrac{1}{4}\lambda^{-1}\left(P_k'\right)^2 + \tfrac{1}{2}\lambda^{-1}V = 0\,. \tag{A.13c}$$

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Eliminating common factors etc.

$$\left[\lambda(R^2)'\right]' - 2 + \tfrac{1}{2}R^2\left(P_k'\right)^2 + R^2V = 0\,, \tag{A.14a}$$

$$\left(R^2\lambda'\right)' - g^2R^2\lambda^{-1}(\phi^1)^2P_k^2 - \tfrac{1}{2}R^2\left(P_k'\right)^2 + R^2V = 0\,, \tag{A.14b}$$

$$\left(R^2\lambda'\right)' + 2RR'' + \lambda R^2(\phi^1{}')^2 - \tfrac{1}{2}R^2\left(P_k'\right)^2 + R^2V = 0\,. \tag{A.14c}$$

The difference between the last two equations is

$$-g^2\lambda^{-1}(\phi^1)^2P_k^2 - 2R''/R - \lambda(\phi^1{}')^2 = 0\,. \tag{A.15}$$

These equations are very difficult to solve in general. We are going to make a simplifying assumptions that $\phi^1 = 0$ and

$$\partial_{\phi^1}V\Big|_{\phi^1=0} = 0\,, \quad \text{and} \quad V(\phi^1 = 0) \equiv 2\Lambda\,. \tag{A.16}$$

Eq. (A.8) is solved automatically and the combination eq. (A.15) is solved by

$$R = ar\,, \tag{A.17}$$

where we have eliminated an integration constant through a shift of r and where the integration constant a will be set to 1 to give metric at spatial infinity the standard normalization. We are left with the following equations:

$$\left(r^2P_k'\right)' = 0\,, \tag{A.18a}$$

$$2\,(\lambda r)' - 2 + \tfrac{1}{2}r^2\left(P_k'\right)^2 + 2\Lambda r^2 = 0\,, \tag{A.18b}$$

$$\left(r^2\lambda'\right)' - \tfrac{1}{2}r^2\left(P_k'\right)^2 + 2\Lambda r^2 = 0\,. \tag{A.18c}$$

The first equation is solved by

$$P_k' = \frac{a}{r^2}\,, \tag{A.19}$$

for some other integration constant that we call, again, a. Then the other two equations take the form

$$2\,(\lambda r)' - 2 + \tfrac{1}{2}a^2r^{-2} + 2\Lambda r^2 = 0\,, \tag{A.20a}$$

$$\left(r^2\lambda'\right)' - \tfrac{1}{2}a^2r^{-2} + 2\Lambda r^2 = 0\,. \tag{A.20b}$$

Combining these two equations we can eliminate the terms that depend on a:

$$\left(r^2\lambda\right)'' - 2 + 4\Lambda r^2 = 0\,. \tag{A.21}$$

We can integrate it immediately:

$$\lambda = 1 + \frac{b}{r} + \frac{c}{r^2} - \frac{\Lambda}{3}r^2\,, \tag{A.22}$$

which corresponds to the Reissner-Nordström-(anti-)De Sitter (RN(A)DS) metric [51]. As a matter of fact, substituting the above value of λ in either of the previous equations, we find that

$$c = a^2/4\,, \tag{A.23}$$

and, since a is, up to constants, the electric charge, the identification of the solution with the RN(A)DS solution is confirmed.

The amount of charges and fields active in this solution are clearly unsufficient to test the results obtained in the main body of this paper. However, it has proven impossible for us to obtain more general solutions using more general ansatzs and more careful gauge-fixing procedures. Finding black-hole solutions of systems like the one at hands is usally a very difficult problem (see, e.g. ref. [54]). Other methods or (much better) other, more interesting and richer models should be used and work in this direction is already under way.

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