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Using statistical compatibility to derive advanced probabilistic fatigue models

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Abstract

Heuristic deterministic models are often proposed to reproduce $S-N$ and $\varepsilon-N$ fields as well as crack growth curves aiming at providing basic material characterization to be used in lifetime design. Usually, they arise from the intuition of the researchers and are supported by experimental data, sometimes being complemented by micromechanical considerations based on good knowledge of physical and metallurgical properties of the material. In spite of their utility, this procedure implies serious limitations. In this work, some new methodological suggestions are presented allowing the functional structure of the problem under consideration to be derived, in an attempt of avoiding arbitrary assumptions. The relevant variables involved in the problem are recognized from experimental evidence, a reduced set of dimensionless variables is selected using dimensional analysis, a set of compatibility conditions are established and the constraints to be fulfilled by the model considered. Such models also enable planning a meaningful testing program for reliable estimation of the parameters.

Keywords: Fatigue models, statistical compatibility, functional equations

1. Introduction

Searching for fundamental information to be used in lifetime design, scientists use to resort to heuristic deterministic models to reproduce $S-N$ and $\varepsilon-N$ fields and crack growth curves. Often, the intuition of the researcher plays herewith an important role that sometimes may be complemented by some micromechanical considerations based on good knowledge of physical and metallurgical properties of the material. The suitability of the model is expected to be supported by the experimental results, at least for the particular case handled. Possible description is based on out–of plain constraint concept. Both approaches explain more or less the same effect using different methodologies.

This procedure, despite the successes being achieved up to now, is not fully satisfactory since it implies: (a) serious limitations, such as an indeterminable and uncertain degree of arbitrariness in establishing the model, (b)

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overlooking relevant, sometimes transcendental, information related to the model, (c) difficulty in recognizing internal properties of interest, (d) necessity of additional considerations in order to proceed to subsequent statistical analysis of data, (e) lack of information to optimize the experimental program, (f) limited applicability of the model and subsequent uncertainty in the extrapolation outside its definition range of application, and (g) restrictions to extend the current model aiming at achieving improved and more advanced models to meet higher degrees of suitability in more complex cases.

In this work, the advantages of considering an alternative methodology for different fatigue modelling cases based on the statistical compatibility between the participating functions are emphasized. This allows the functional structure of the problem to be derived thus avoiding or at least diminishing arbitrary assumptions. The relevant variables involved in the problem are recognized from experimental evidence, and a reduced set of dimensionless variables is selected using dimensional analysis. Thereafter, a set of compatibility conditions are imposed, and the model constraints fulfilled. Moreover, a meaningful testing program may be planned in accordance with the model proposed supplying a suitable data set from which the model parameters can be reliably estimated. Satisfying these requirements guarantees an enhancement of the model quality. Occasionally, the properties shown by the model enable even a better comprehension of the phenomenon and further development to advanced models. Extension to practical fatigue design models can be subsequently envisaged, as it happens in the case of damage accumulation.

2. Steps in modelling fatigue problems

The first step in gaining a first insight in engineering problems implies the observation of experimental results. This is mandatory to derive a model capable to reproduce the physical character of the process at a deeper level of understanding, and to allow us to explain, simulate and predict the results of the phenomenon being analyzed under various conditions. Models typically arise just based on empirical observations (as a lower case of understanding and capability of reproducing results), to which mathematical, mechanical or thermal considerations should be added. Phenomenological models sometimes involve additional or partially scientific considerations complemented by empirical corrections when the phenomenon is not completely justified according to well-established scientific criteria. Apart of this, micromechanical considerations help definitely, as could not be otherwise expected, to confirm the potential suitability of mathematical models, to enhance their quality, and to check their applicability to the experimental and practical cases near reality.

Nowadays, no one researcher would dare to deny the transcendental contribution supplied by different mathematical techniques in solving typical engineering problems, among others in fracture mechanics domain, in particular related to lifetime prediction. Every one is familiar with the application of differential equations or with the implementation of numerical solutions provided, for instance, by the finite element method, boundary element method, etc. Nevertheless, if statistical techniques are suggested for being considered as a complementary tool indispensable for deriving reliable and realistic fracture mechanics models a sceptic, if not despising, attitude is noticed, and the intruder detracted as theoretical, or strange to experimental and practice. Not to mention if resorting to functional equations (see Castillo et al. [1]) as a new mathematical tool to derive fracture mechanics models. The cases analyzed in next Sections prove that this technique, practically unknown by engineering researchers, represents an ignored but powerful potential of knowledge and scientific enrichment.

The following steps are suggested in modelling a fatigue problem:

1. Gaining experience in the fatigue problem to be dealt with as a prerequisite to deal with such complex problems.
2. Identifying the set of relevant variables involved in the problem. Mostly, former models from previous research represent a helpful benchmark for further improvements, but sometimes well-established models stemming from wrong premises steer the future development in a wrong direction delaying the appearance of unorthodox, for the status being, valid alternative models.
3. Using the Buckingham theorem [2] to obtain a set of dimensionless variables allowing us to reduce the number of parameters involved, to recognize some features of the functional structure of the model, and to gain a better insight into the problem to explore feasible parameter relations. It also contributes to discard inconsequent solutions.
4. Setting the conditions or constraints to be satisfied by the model. In particular, compatibility conditions play a central role helping to identify the structure of the model whereas the recognition of the statistical distribution families being involved in the compatibility is crucial for establishing the functional equation.
5. Solving the system of equations associated with the set of conditions to obtain the functional structure of the model without arbitrary assumptions.

6. Designing an adequate testing plan to obtain experimental data to fit the model. The test strategy to be applied in the planning of a fatigue programme is strongly related to the fatigue model used in the evaluation of results.
7. Using data to estimate the model parameters by choosing a suitable method of parameter estimation. Various methods have been proposed as the maximum likelihood and regression models.
8. Applying the model to solve the problem being analyzed.

3. Identifying the set of relevant variables in the fatigue problem

The first step in deriving a model to solve an engineering problem consists of identifying the participating variables in the problem. The relevant variables to be considered vary in accordance to the specific lifetime problem handled and the level of exigencies accepted, see [3-7] and the specific problems handled in the following Section. Their number can be progressively increased as a better insight on the model allows increasing its complexity aiming at reaching a higher validity level. Attention should be paid to guarantee the dimensional consistency using for instance the Buckingham theorem, which allows representing any existing relation between the initial set of variable in terms of another reduced set of non-dimensional variables.
From the Buckingham theorem we can learn, a) whether or not there are enough variables for a valid physical relation to be possible, b) the minimum number of dimensionless variables required to reproduce the physical relation, and c) what sets of dimensionless variables can be used. Nevertheless, as we will see, in general, additional auxiliary techniques are needed to develop more advanced models.

4. Some examples of compatibility models

In what follows the derivation of three representative cases of modelling in the fatigue domain are presented showing up how the consideration of internal requirements, particularly that of the statistical compatibility, leads to an enhancement of their quality and utility in practical applications.

4.1 Applying compatibility in the derivation of the basic $S-N$ model

In the classical case of fatigue data, an analytical description of the Wöhler field is aimed at in which the number of cycles to failure is intended to be predicted in a probabilistic way in terms of the stress range $\Delta\sigma$ for fixed stress level ($R=\sigma_{\min}/\sigma_{\max}$, $\sigma_{\min}$, etc). As stated in the previous research and experimental evidence, the stress range $\Delta\sigma$ is acknowledged as the primary parameter controlling the fatigue life, characterized as the number of cycles. For the moment being, we renounce to include the effect of the stress level as the secondary parameter in the model, which will be handled in Subsection 4.2 as an extension of the basic fatigue model. Based on the weakest link principle, and after setting physical and statistical requirements (stability, compatibility and limit conditions) it is proven that the distributions $F(N;\Delta\sigma)$ and $E(\Delta\sigma;N)$ related to the data results belong to three-parameter Weibull distribution families for minima [3]. Without resorting to an arbitrary predetermined shape of the Wöhler curves, as is advocated by most of the current $S-N$ models, we formalize the interrelation between $F(N;\Delta\sigma)$ and $E(\Delta\sigma;N)$ as a regression function defining the Wöhler field by stating that both Weibull distributions for minima must be compatible, i.e., by accomplishing the equality

$$\left[\frac{\Delta\sigma^{*}-\lambda(N^{*})}{\delta(N^{*})}\right]^{\beta(N^{*})}=\left[\frac{N^{*}-\lambda(\Delta\sigma^{*})}{\delta(\Delta\sigma^{*})}\right]^{\beta(\Delta\sigma^{*})}, \tag{1}$$

where $\Delta\sigma^{*}$ and N^{*} are, respectively, the normalized stress range and the lifetime measured as the number of cycles to failure, and λ, δ and β are Weibull parameters in the respective distributions (see Fig. 1). This relation would become an unimportant curiosity should not exist the functional equation theory [1], which provides the structure of the regression function that, including $\Delta\sigma^{*}$, N^{*} and probability as the variables involved, ensures accomplishing of (1) without considering further arbitrary assumptions.

As shown in [2], only two possible models results from solving the functional equation (1):

$$\text{MODEL I}: F[N^*;\Delta\sigma^*] = E[\Delta\sigma^*;N^*] = 1-\exp\left[-\left(\frac{(N^*-B)(\Delta\sigma^*-C)-\lambda}{\delta}\right)^{\beta}\right]; \quad (2)$$

with the physical meaning of the parameters (see Fig. 2a):
B: threshold value of lifetime
C: endurance limit
γ: parameter defining the position of the corresponding zero-percentile hyperbola
δ: scale factor
β: Weibull shape parameter of the whole cumulative distribution function in the $S-N$ field.

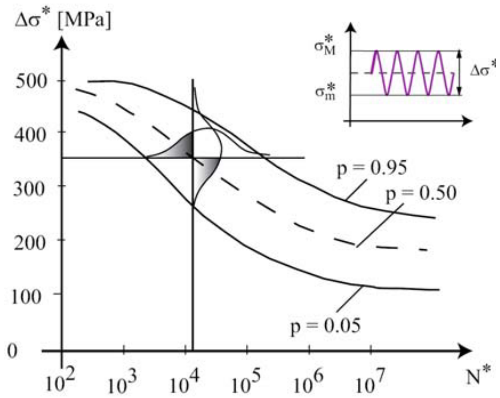

Fig. 1: Schematic illustration of the compatibility condition in the $S-N$ field

$$\text{MODEL II}: F[N^*;\Delta\sigma^*] = E[\Delta\sigma^*;N^*] = 1-\exp\left[-\left(\delta\,(N^*-B)^{\beta}(\Delta\sigma^*-C)^{\gamma}\right)\right]; \quad (3)$$

with the following physical meaning (see Fig. 2b):
B: minimum lifetime associated with all values of $\Delta\sigma^*$
C: endurance limit
β: shape parameter associated with N^*
γ: shape parameter associated with $\Delta\sigma^*$
δ: scale factor.

As soon as the five parameters in the actual chosen model are determined, the analytical expression of the whole $S-N$ field is known, which allows for the probabilistic prediction of the fatigue failure under constant amplitude loading. As can be observed, the percentile curves are represented in both models by equilateral hyperbolas.

The first model is found to be more likely applicable, and has been extensively analyzed. Compared to conventional methods its proves to be superior concerning reliability und capability to reproduce the whole Wöhler field, as the up-and-down [8], it is suitable as confirmed in a number of applications to different material and situations, including size effect [9] and extrapolation as reported in previous works of the authors, including sensitivity analysis [9] allowing the consideration of runouts. Besides, it provides knowledge that facilitates optimizing planning of testing programs by means of an adequate testing strategy. Also, fortresses and weaknesses were recognized and discussed. In particular, the non-consideration of an upper limit, physically conditioned by the existence of the yield stress, evidences its inability to reproduce adequately the low-cycle fatigue (LCF) region. This fact can currently accepted as a limitation of the model derived or it points rather out that the failure mechanism in the LCF region cannot be properly reproduced using the same statistical model as that for the long life-cycle fatigue region since they represent two competing failure mechanisms which effects should be further investigated (see Subsection 4.3).

Finally, the simple consideration of a new variable $V = (N^* - B)(\Delta\sigma^* - C)$ illustrates that as soon as B and C are known, the whole $S - N$ field reduces to a unique statistical distribution. This opens very interesting perspectives in the consideration of the damage accumulation for predicting lifetime under varying loading [3,4].

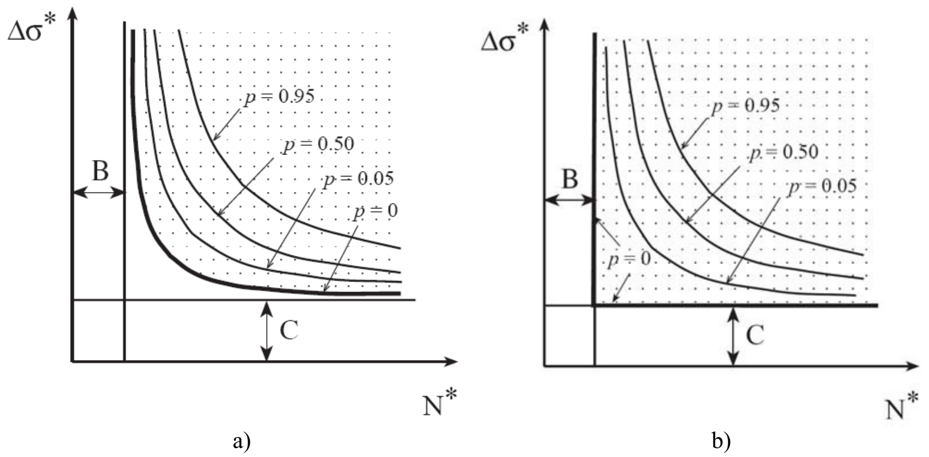

Fig. 2: Representation of the *S* – -*N* field with percentiles curves according a) to the model I, and b) to the model II [3]

4.2 Applying compatibility in the derivation of the S – N model for varying stress level.

With the aim at deriving a *S – N* model capable to include the influence of the stress level, two different families of *S – N* fields, denoted "M" and "m", are considered the former corresponding to tests carried out at fixed, but different, maximum stress, the latter corresponding to tests carried out at fixed, but different, minimum stress. Fig. 3 shows two *S – N* fields resulting from tests performed with constant maximum stress (denoted by sub-indices M_1 and M_2) and two with constant minimum stress (denoted by sub-indices m_1 and m_2). In this case, $\sigma_{\min}$ and $\sigma_{\max}$ are considered in the derivation of the model instead of the stress range $\Delta\sigma$ and the stress level σ_l as in the case of the *S – N* field.

Since a particular test belonging to the family M_i conducted at the minimum stress m_j and a particular test belonging to the family m_j conducted at the maximum stress M_i are equivalent, the statistical distribution of both must coincide. This allows us to formulate the compatibility condition between the two families of *S – N* fields as:

$$h\left(\frac{(N^* - B_m(\sigma_m^*))(\sigma_m^* - \sigma_M^* - C_m(\sigma_m^*)) - \lambda_m(\sigma_m^*)}{\delta_m(\sigma_m^*)}\right) = h\left(\frac{(N^* - B_M(\sigma_M^*))(\sigma_m^* - \sigma_M^* - C_M(\sigma_M^*)) - \lambda_M(\sigma_M^*)}{\delta_M(\sigma_M^*)}\right), \quad (4)$$

where, accepting the solution found by the basic model, h represents the same extreme value distribution for minima in both cases, N^* is the normalized non-dimensional number of cycles to failure, σ_M^* and σ_m^* are, respectively, the generic maximum and minimum stress, B and C are the asymptotes of the $S - N$ field, and λ, δ and β are, respectively, the location, scale and shape parameter of the model.

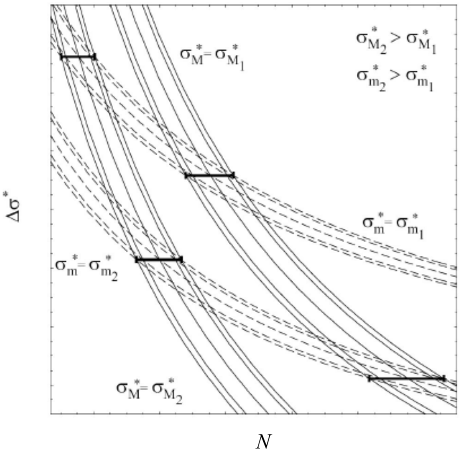

Fig. 3: Compatibility condition in the $S-N$ field for constant minimum and maximum stress level conditions [4]

Solving this functional equation, the following solution results

$$p = h\,(r\,(\sigma_m^*, \sigma_M^*) + s\,(\sigma_m^*, \sigma_M^*)\; N^*)\,, \tag{5}$$

where

$$r\,(\sigma_m^*, \sigma_M^*) = C_0 + C_1\sigma_m^* + C_2\sigma_M^* + C_3\sigma_m^*\sigma_M \tag{6}$$
$$s\,(\sigma_m^*, \sigma_M^*) = C_4 + C_5\sigma_m^* + C_6\sigma_M^* + C_7\sigma_m^*\sigma_M\,.$$

According to the extreme value theory it leads, respectively, to the Weibull model:

$$p = 1 - \exp[-(r\,(\sigma_m^*, \sigma_M^*) + s\,(\sigma_m^*, \sigma_M^*)\; N^*)^{\beta}] \tag{7}$$

and the Gumbel model:

$$p = 1 - \exp[-\exp\,(r\,(\sigma_m^*, \sigma_M^*) + s\,(\sigma_m^*, \sigma_M^*)\; N^*)]\,. \tag{8}$$

The validity of the models derived has been ascertained when applied to different experimental programs (see Fig. 4 and [5]).

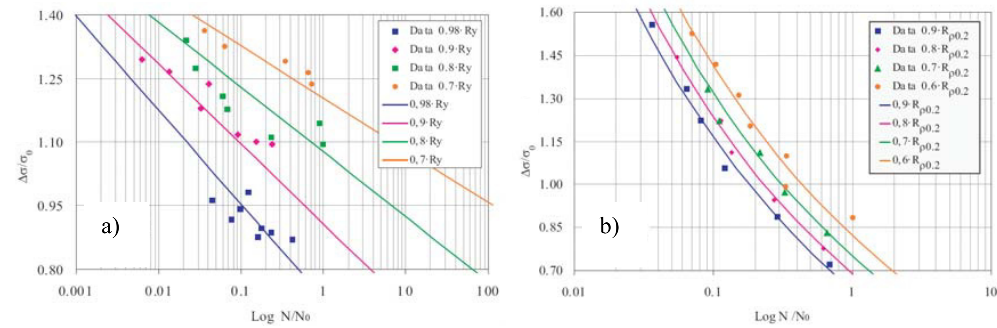

Fig. 4: Resulting $S-N$ curves from the proposed model for constant σ^*_M : a) for 42CrMo4 steel, and b) for AlMgSi1 [5]

4.3 Applying compatibility in the derivation of the ε – N model

As an alternative to the *S – N* field, the strain-life curves, or *ε – N* field, consider the magnitude of the varying strain from unnotched specimens to characterize the fatigue behaviour of materials proving to be a suitable method for estimating the lifetime of mechanical and structural elements in both low- and high-cycle fatigue regions.
To fit experimental results, the well-established approach proposed by Morrows [11] is extensively used:

$$\varepsilon_a = \varepsilon_a^e + \varepsilon_a^p = \frac{\sigma_f'}{E}\left(\frac{2N_f}{N_0}\right)^b + \varepsilon_f'\left(\frac{2N_f}{N_0}\right)^c, \quad (9)$$

where the superscripts *e* and *p* are used for the *elastic* and *plastic* strain, respectively. N_f is the number of cycles, σ_f' the fatigue strength coefficient, *b* the fatigue strength exponent, ε_f' the fatigue ductility coefficient, *c* the fatigue ductility exponent, *E* the Young's modulus, and N_0 a reference number of cycles to render Eq.(9) dimensionless. It gives the relation between the total strain amplitude ε_a and the fatigue life measured in cycles N_f as strain-life curves based on the former models of Basquin [12] for the elastic strain-life and Coffin–Manson [13,14] for the plastic strain-life. In view of the inherent inconsistencies evidenced by this proposal (see [6]), an alternative derivation of the model is undertaken.

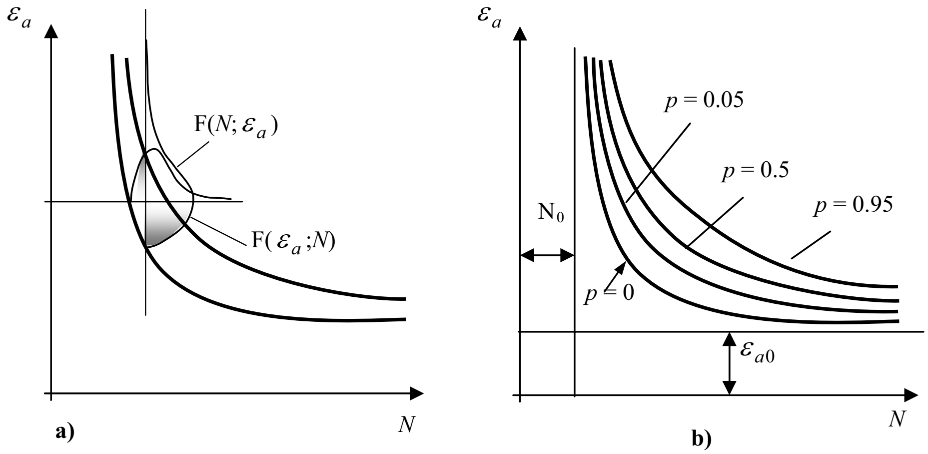

Fig. 5. a) Schematic illustration of the compatibility condition between probability distributions in the *ε – N* field, and b) *ε – N* field with percentile curves [6]

Since the basic assumptions used to derive the former Weibull *S – N* model also apply in the case of the *ε – N* field, due to a parallelism in both problems being dealt with, a Weibull regression model for the strain-life case has been proposed assuming the normalized fatigue life N_f^* and the total strain amplitude ε_a^* (the normalized stress range $\Delta\varepsilon^*$ could also be considered) to be random variables. Proceeding with non-dimensional variables, the same physical and statistical considerations as those used in the derivation of the *S – N* model apply providing the analytical probabilistic definition of the whole strain-life field as quantile curves both in the low cycle and high cycle fatigue regions, since contrary to the *S – N* field, the upper limit in the *ε – N* is practically unrestricted. A detailed description is given in [6], leading to the following model (see Fig. 5):

$$p = F(N_f^*;\varepsilon_a^*) = 1 - \exp\left[-\left(\frac{(N_f^* - B)(\varepsilon_a^* - C) - \lambda}{\delta}\right)^\beta\right]; \quad (N_f^* - B)(\varepsilon_a^* - C) \ge \lambda, \quad (10)$$

where $N_f^* = \log(N_f / N_0)$ and $\varepsilon_a^* = \log(\varepsilon_a / \varepsilon_{a0})$. Note that the probability of failure p depends only on the product $N_f^* \varepsilon_a^*$ this having a Weibull distribution $N_f^* \varepsilon_a^* \sim W(\lambda, \delta, \beta)$. Accordingly, the proposed method is an adequate alternative candidate to the traditional model (10), to represent the $\varepsilon - N$ field.

The five model parameters $N_0, \varepsilon_{a0}, \lambda, \delta$ and β in (10) can be estimated using different well established methods as proposed in the fatigue literature [1,6]. In particular, the maximum likelihood and the two-stage methods provide adequate estimates based on the set of experiments, whereas run outs can be also dealt with.

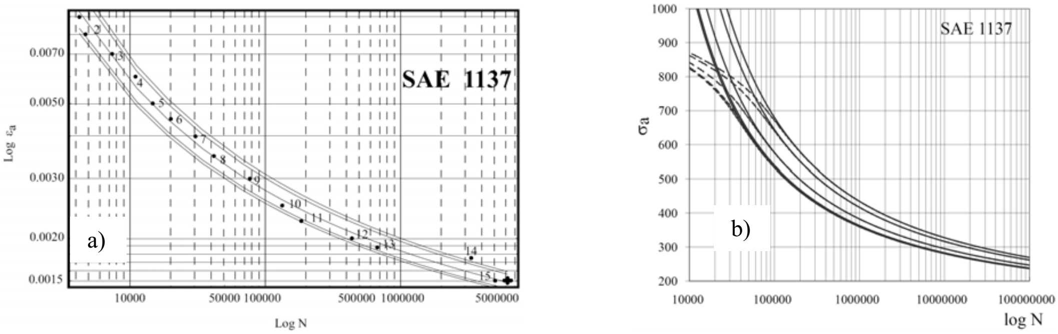

Fig. 6: a) $\varepsilon - N$ field for SAE1137 steel using the proposed model and b) Resulting $S - N$ field considering a Ramberg-Osgood model [4,9]

The testing program, due to the similitude to the $S - N$ case, can be planned with greater flexibility as the one usually applied to collecting data according to Morrows model. The proposed $\varepsilon - N$ model can be applied for probabilistic lifetime prediction under varying loading. Using the $\sigma - \varepsilon$ cyclic curves, it is possible to derive the corresponding stress life curves, i.e., the probabilistic $S - N$ field, which in this case presents the typical change in curvature in the low-cycle fatigue region (see Fig. 6b) [15].

5. Applying compatibility in the derivation of the crack growth models under constant loading.

Crack growth models allow relating the current crack size to the number of cycles applied. Apparently, their study is more complex than those related to $S - N$ and $\varepsilon - N$ fields, because physical process realizations rather than mere failure data, are handled. The observation of the experimental data results permits recognizing the random character of the crack growth process and dependency on the stress range $\Delta\sigma$ and stress level σ_l along with the following features:

- The different crack growth curves follow similar patterns with concavity from above.
- The stress range and stress level and/or the initial crack size are parameters which variation lead to different sets of curves are obtained.
- The initial crack size of the specimens is random.
- Failure occurs as unstable crack propagation at different crack sizes, which depend on the stress range and level.

In what follows, the statistical compatibility is applied to the study of the shape, provides information about the characteristics of the shape of the crack growth curves under constant loading conditions.

Provided the same loading conditions expressed as Q^*, implying a unique material specimen subject to the same stress range $\Delta\sigma$ and stress level σ_l, two normalized random variables are considered, the crack size given the number of cycles $a^* | N^*$ and the number of cycles given the crack size $N^* | a^*$. Assuming the weakest link principle that justify location and scale families of distributions, we stand in front of a similar compatibility condition to that handled in Subsections 3.1 and 3.3 with a remarkable difference: The observation of the nature of the variables involved let know that $N^*|a^*$ is associated with a minimum law, whereas $a^*|N^*$ is associated with a maximum law (see Fig. 7). Thus, the compatibility condition can be expressed as:

$$p = q_{\max}\left(\frac{a^* - \mu_1^*(N^*)}{\sigma_1^*(N^*)}, Q^*\right) = 1 - q_{\min}\left(\frac{N^* - \mu_2^*(a^*)}{\sigma_2^*(a^*)}, Q^*\right), \quad (12)$$

where $q_{\max}$ and $q_{\min}$ are distributions for maxima and minima, and μ and σ, the location and scale parameters, respectively, depending on a^* and N^*.

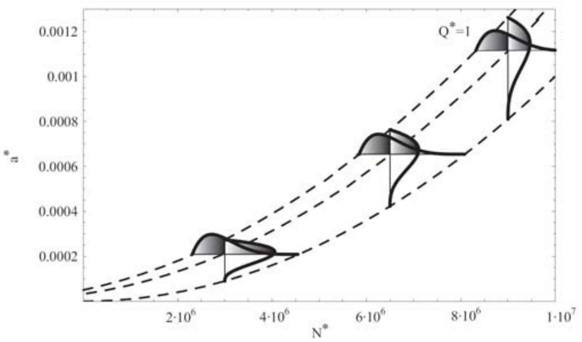

Fig. 7: Illustration of the compatibility condition based on the a* density for the same loading factor Q.

Taking into account that $q_{\max}(x) = 1 - q_{\min}(-x)$ from the extreme value theory (see [7]), this leads to the functional equation:

$$\frac{a^* - \mu_1^*(N^*)}{\sigma_1^*(N^*)} = \frac{\mu_2^*(a^*) - N^*}{\sigma_2^*(a^*)} \quad (13)$$

from which, based on minimal extreme value distributions, the following two only possible Weibull models are derived:

MODEL 1:

$$F_{a^*|N^*}(a^*, N^*) = \exp\left[-\left(-\frac{a^* - \gamma^* N^* - \eta^*}{\rho^*}\right)^{\beta^*}\right] \quad (14)$$

that representing a family of straight lines in which γ^*, η^* and ρ^* are parameters and

MODEL 2:

$$F_{a^*|N^*}(a^*, N^*) = \exp\left[-\left(-\frac{(M^* - N^*)(a^* - a_{th}^*) - \eta^*}{\rho^*}\right)^{\beta^*}\right] \quad (15)$$

that representing a family of hyperbolas in which M^*, a_{th}^*, η^* and ρ^* are parameters.
Gumbel instead of Weibull distributions can be also derived applying a parallel procedure to that shown above.

6. Conclusions

The main conclusions from the paper are the following:
- Deterministic models in fatigue are very limited and should be replaced by probabilistic models arising from compatibility conditions based on physical and statistical knowledge of the specific fatigue problem without resorting to arbitrary assumptions.
- Functional equations, as a mathematical tool being newly applied on fracture mechanics applications and, in particular, on lifetime modelling, allow establishing functional relations between the variables involved. Solving the functional equations fulfilling the set of constraints defining the fatigue problem provide the only possible models compatible with those constraints.
- $S-N$ and $\varepsilon-N$ fields and crack growth curves being three ways of representing the same fatigue characterization of a material should lead to related models. The consequent use of the proposed compatibility methodology contributes to establish this relation, and to a better comprehension of the fatigue problem implying a new way of facing the fatigue problem as an integration of the stress, strain and fracture mechanics based approaches.
- Further examples could be added related to crack growth curves for varying loading conditions as well as to crack growth rate curves. Unfortunately, we have to renounce to it here due to space limitations.

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