The d_θ-depth-based interval trimmed mean

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Abstract Central tendency of interval-valued random elements has been mainly described in terms of different notions of medians and location M-estimators in the literature, whereas the approach consisting of medians and trimmed means based on a depth function has been rarely considered. Recently, depth-based trimmed means have been adapted to the more general framework of fuzzy number-valued data in terms of the so-called D_θ-depth. The aim of this work is to study the empirical behaviour of the particularization of such a location measure when data are interval-valued.

1 Introduction

Interval-valued data arise in numerous real-life experiments when imprecise information is handled. For instance, we could refer to surveys that collect opinions, judgements or perceptions; fluctuations or ranges of a characteristic along certain period of time; imprecise observations of a real-valued random variable due to measurement errors; interval-type censoring data; aggregated information, etc. Different statistical techniques have already been adapted to deal with interval-valued data, such as regression analysis, hypotheses testing procedures, clustering... Concerning central tendency measures, the Aumann mean is the most frequently used in these techniques, despite its excessive sensitivity to outliers or data changes. Other alternatives from the literature present a more robust behaviour, which makes them more suitable for summarizing datasets that contain some contaminated data: the median based on the generalized Hausdorff metric (Sinova et al., 2010), the 1-norm median (Sinova and Van Aelst, 2013), the spatial-type interval-valued median

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(Sinova and Van Aelst, 2018) and interval-valued M-estimators of location (Sinova, 2016). Apart from these measures, which were specifically proposed for the interval-valued setting, medians and trimmed means based on the halfspace and simplicial depths have also been considered in Sinova (2016) by identifying each interval-valued datum with the point whose coordinates are the mid-point and the spread of the interval.

Recently, Sinova (2022) studied trimmed means based on a depth function in the more general framework of fuzzy number-valued data. Since they have shown very promising empirical results, this work is focused on their particularization to the interval-valued setting (the d_θ-depth-based interval trimmed mean) and the empirical comparison with the central tendency measures mentioned before. Sections 2 and 3 recall the preliminaries related to the space of (compact) intervals and interval-valued central tendency measures, respectively. Section 4 presents the d_θ-depth-based interval trimmed mean and the comparative simulation study carried out to analyze its empirical performance. Finally, Section 5 contains some concluding remarks.

2 The space $\mathcal{K}_c(\mathbb{R})$

Let $\mathcal{K}_c(\mathbb{R})$ denote the class of nonempty compact intervals of $\mathbb{R}$. Any interval $K \in \mathcal{K}_c(\mathbb{R})$ can be characterized in terms of either its extremes, $K = [\inf K, \sup K]$, or its mid-point (centre) $\operatorname{mid} K = (\inf K + \sup K)/2$ and spread (radius) $\operatorname{spr} K = (\sup K - \inf K)/2$, $K = [\operatorname{mid} K - \operatorname{spr} K, \operatorname{mid} K + \operatorname{spr} K]$.

For the statistical analysis of interval-valued data, the usual interval arithmetic will be considered. Given two intervals $K, K' \in \mathcal{K}_c(\mathbb{R})$, their *Minkowski sum* is defined as $K + K' = [\inf K + \inf K', \sup K + \sup K']$ or, in terms of the second characterization, $\operatorname{mid}(K + K') = \operatorname{mid} K + \operatorname{mid} K'$ and $\operatorname{spr}(K+K') = \operatorname{spr} K + \operatorname{spr} K'$. Analogously, the *product of an interval* $K \in \mathcal{K}_c(\mathbb{R})$ *by a scalar* $\gamma \in \mathbb{R}$ can be introduced in terms of any of the two characterizations:

$$\gamma \cdot K = \begin{cases} [\gamma \inf K, \gamma \sup K] & \text{if } \gamma \geq 0 \\ [\gamma \sup K, \gamma \inf K] & \text{if } \gamma < 0 \end{cases} = [\gamma \operatorname{mid} K - |\gamma| \operatorname{spr} K, \gamma \operatorname{mid} K + |\gamma| \operatorname{spr} K].$$

Metrics play a relevant role in the development of statistical techniques for this kind of data due to the lack of linearity of the space $(\mathcal{K}_c(\mathbb{R}), +, \cdot)$. The location measures in Sect. 3 will be based on the following metrics:

- Given $\theta \in (0, \infty)$, the *generalized Hausdorff metric* (Sinova et al., 2010) between two intervals $K, K' \in \mathcal{K}_c(\mathbb{R})$ is defined as

$$d_{H,\theta}(K, K') = |\operatorname{mid} K - \operatorname{mid} K'| + \theta |\operatorname{spr} K - \operatorname{spr} K'|.$$

- The *1-norm metric* (Vitale, 1985) between any two intervals $K, K' \in \mathcal{K}_c(\mathbb{R})$ is given by

$$\rho_1(K, K') = \frac{1}{2}|\inf K - \inf K'| + \frac{1}{2}|\sup K - \sup K'|.$$

- Given $\theta \in (0, \infty)$, the d_θ *metric* (Gil et al., 2002) between any two intervals $K, K' \in \mathcal{K}_c(\mathbb{R})$ is defined as

$$d_\theta(K, K') = \sqrt{(\operatorname{mid} K - \operatorname{mid} K')^2 + \theta(\operatorname{spr} K - \operatorname{spr} K')^2}.$$

The value of the parameter is usually $\theta \in (0, 1]$ not to weigh the deviation in location less than the deviation in shape/imprecision. All the metrics $d_{H,\theta}$, ρ_1 and d_θ are strongly equivalent.

Interval-valued data are assumed to come from a (compact) random interval, that is, a Borel measurable mapping $X : \Omega \longrightarrow \mathcal{K}_c(\mathbb{R})$, where $(\Omega, \mathcal{A}, P)$ is a probability space and the Borel σ-field on $\mathcal{K}_c(\mathbb{R})$ is generated by the topology induced by any of the previous metrics. Therefore, this notion models the random mechanism that repeatedly produces interval-valued observations. It holds that X is a random interval if, and only if, $\inf X$ and $\sup X$ are real-valued random variables such that $\inf X \leq \sup X$ (or, alternatively, $\operatorname{mid} X$ and $\operatorname{spr} X \geq 0$ are real-valued random variables).

3 Central tendency measures for random intervals

One of the best known central tendency measures for random intervals is the Aumann mean (Aumann, 1965), which generalizes the mean of a real-valued random variable as follows.

Definition 1 The *Aumann mean* of a random interval X is the interval $E[X] = [E(\inf X), E(\sup X)]$ (whenever these expectations exist) or, equivalently, $E[X] = [E(\operatorname{mid} X) - E(\operatorname{spr} X), E(\operatorname{mid} X) + E(\operatorname{spr} X)]$.

The Aumann mean is the Fréchet expectation with respect to the d_θ metric, that is, $E[X]$ is the unique solution of $\min\limits_{K \in \mathcal{K}_c(\mathbb{R})} E[(d_\theta(X, K))^2]$.

It also fulfills very convenient statistical and probabilistic properties, but, unfortunately, it is highly sensitive to outliers and data changes. The following concepts from the literature (see Sinova et al., 2010; Sinova and Van Aelst, 2013; Sinova, 2016; Sinova and Van Aelst, 2018, for more details) provide us with more robust central tendency measures for random intervals.

Definition 2 The *Hausdorff-type median* of a random interval X is the interval(s) $\operatorname{Med}[X] \in \mathcal{K}_c(\mathbb{R})$ such that minimizes $E[d_{H,\theta}(X, K)]$ over $K \in \mathcal{K}_c(\mathbb{R})$ (whenever this expectation exists).

In particular, any interval $[\mathrm{Me}(\mathrm{mid}\, X) - \mathrm{Me}(\mathrm{spr}\, X), \mathrm{Me}(\mathrm{mid}\, X) + \mathrm{Me}(\mathrm{spr}\, X)]$ is a Hausdorff-type median (and it does not depend on the value θ). In case the medians of real-valued random variables $\mathrm{Me}(\mathrm{mid}\, X)$ and/or $\mathrm{Me}(\mathrm{spr}\, X)$ are not unique, a convention such as choosing the mid-point of the interval of possible medians makes the Hausdorff-type median become unique.

Definition 3 The *1-norm median* of a random interval X is the interval(s) $\mathrm{Me}[X] \in \mathcal{K}_c(\mathbb{R})$ such that minimizes $E[\rho_1(X, K)]$ over $K \in \mathcal{K}_c(\mathbb{R})$ (whenever this expectation exists).

In particular, the interval $[\mathrm{Me}(\inf X), \mathrm{Me}(\sup X)]$ is a 1-norm median. In this case, the convention of choosing the mid-point of the corresponding interval of medians is adopted to guarantee that the 1-norm median is not empty even when $\mathrm{Me}(\inf X)$ or $\mathrm{Me}(\inf X)$ may not be unique.

Definition 4 The *d_θ-median* of a random interval X is the interval(s) $\mathrm{M}_\theta[X] \in \mathcal{K}_c(\mathbb{R})$ such that minimizes $E[d_\theta(X, K)]$ over $K \in \mathcal{K}_c(\mathbb{R})$ (whenever this expectation exists).

In contrast to what happens with the Hausdorff-type median, the d_θ-median depends on the value θ.

Finally, M-estimators could be understood as "intermediaries" between the Aumann mean and interval-valued medians because they weigh distances by means of a loss function that is generally less rapidly increasing than the square function.

Definition 5 Given a continuous and non-decreasing loss function $\rho : \mathbb{R}^+ \to \mathbb{R}$ that vanishes at 0, the *M-location measure* of a random interval X is the interval(s) $\mathrm{K}^M_\rho[X] \in \mathcal{K}_c(\mathbb{R})$ which minimizes $E[\rho(d_\theta(X, K))]$ over $K \in \mathcal{K}_c(\mathbb{R})$ (whenever this expectation exists).

A simple random sample $(X_1, \ldots, X_n)$ from a random interval X consists of n independent random intervals X_i, $i = 1, ..., n$, that are identically distributed as X. In this context,

- the sample Hausdorff-type median is given by
  $$\widehat{\mathrm{Med}[X]}_n = [\widehat{\mathrm{Me}(\mathrm{mid}\, X)}_n - \widehat{\mathrm{Me}(\mathrm{spr}\, X)}_n, \widehat{\mathrm{Me}(\mathrm{mid}\, X)}_n + \widehat{\mathrm{Me}(\mathrm{spr}\, X)}_n];$$
- the sample 1-norm median is $\widehat{\mathrm{Me}[X]}_n = [\widehat{\mathrm{Me}(\inf X)}_n, \widehat{\mathrm{Me}(\sup X)}_n]$;
- the sample d_θ-median is the random interval $\widehat{\mathrm{M}_\theta[X]}_n$ that takes, for each realization $\mathbf{x}_n = (x_1, \ldots x_n)$, the value $\widehat{\mathrm{M}_\theta[\mathbf{x}_n]}$ that minimizes $\frac{1}{n}\sum_{i=1}^n d_\theta(x_i, K)$ over $K \in \mathcal{K}_c(\mathbb{R})$;
- the M-estimator of location is the random interval $\widehat{\mathrm{K}^M_\rho[X]}_n$ that takes, for each realization $\mathbf{x}_n$, the value $\widehat{\mathrm{K}^M_\rho[\mathbf{x}_n]}$ that minimizes $\frac{1}{n}\sum_{i=1}^n \rho(d_\theta(x_i, K))$ over $K \in \mathcal{K}_c(\mathbb{R})$.

The sample d_θ-median always exists and is unique whenever the points $\{(\mathrm{mid}\, X_i, \mathrm{spr}\, X_i)\}_{i=1}^n$ are not all collinear. On the other hand, the Representer

Theorem (see Sinova, 2016) provides necessary and sufficient conditions to express M-estimators of location as a weighted mean of the interval-valued observations, which hold for some well-known loss functions, such as the function by Hampel (1974) under some additional conditions. Unfortunately, neither of these two measures admits an explicit expression. For details about their practical computation, see Sinova (2016); Sinova and Van Aelst (2018).

4 The d_θ-depth-based interval trimmed mean

Sinova (2022) has followed a different approach for summarizing the central tendency by studying trimmed means based on a depth function in the fuzzy number-valued setting (one-dimensional fuzzy sets). The main novelty of the work by Sinova (2022) consists in introducing a depth function for fuzzy-number valued data for the first time and applying it to adapt depth-based trimmed means to the fuzzy framework. Indeed, such depth-based trimmed means have shown very promising empirical results and, for this reason, it would be interesting to analyze whether their advantages remain when data are interval-valued.

The particularization of the depth-based trimmed means by Sinova (2022) to the interval-valued setting would be as follows.

Definition 6 Given a random interval X and a fixed value $\theta \in (0, \infty)$, the *d_θ-depth* of $K \in \mathcal{K}_c(\mathbb{R})$ with respect to the distribution of X is given by

$$DD_\theta(K; X) = \frac{1}{1 + E[d_\theta(X, K)]}.$$

Given a simple random sample from X, $(X_1, \ldots, X_n)$, the *empirical d_θ-depth* of K is given by

$$DD_{\theta,n}(K; (X_1, \ldots, X_n)) = \frac{1}{1 + \frac{1}{n}\sum_{i=1}^n d_\theta(X_i, K)}.$$

Notice that the centremost element with respect to the d_θ-depth is the d_θ-median.

Definition 7 Given a simple random sample $(X_1, \ldots, X_n)$ from a random interval X and a trimming proportion $\beta \in (0, 1)$, the *d_θ-depth-based interval trimmed mean* estimator is defined as

$$DD_\theta\text{-}\overline{X}_{n,\beta} = \frac{\sum_{i=1}^n I_{[\gamma,\infty)}(DD_{\theta,n}(X_i)) \cdot X_i}{\sum_{i=1}^n I_{[\gamma,\infty)}(DD_{\theta,n}(X_i))},$$

with $\frac{1}{n}\sum_{i=1}^n I_{[\gamma,\infty)}(DD_{\theta,n}(X_i)) \simeq 1 - \beta$.

As mentioned in Sinova (2022), the d_θ-depth is not affine invariant, but it is enough to replace the distances $d_\theta(X,K)$ by the standardized distances $d_\theta(X,K)/\sigma_X$ (where $\sigma_X = \sqrt{E((d_\theta(X,E[X]))^2)}$ denotes the standard deviation of the random interval X) to get an affine invariant version.

4.1 Comparative simulation study

The empirical behaviour of the d_θ-depth-based trimmed mean has been compared to that of the central tendency measures recalled in Sect. 3. The common choice $\theta = 1/3$ has been considered for the computation of the location measures (whenever the d_θ distance is involved), and the estimates of their bias, variance and mean square error (MSE): Bias $= d_{1/3}(1/N\sum_{i=1}^N \widehat{T}_i, T)$, Var $= 1/N\sum_{i=1}^N (d^2_{1/3}(\widehat{T}_i, \sum_{i=1}^N \widehat{T}_i/N))$ and MSE $= 1/N\sum_{i=1}^N (d^2_{1/3}(\widehat{T}_i, T))$, with T the population value of a location measure, N the number of samples and $\widehat{T}_i$ the estimate of T for the ith generated sample.

Regarding the M-estimator of location, the Hampel loss function with tuning parameters the median, the 75th and the 85th percentiles of the distribution of sample distances (see Sinova, 2016, for details) has been considered due to its flexibility and good empirical performance in previous studies. Simulations have been designed as follows, inspired by the simulation strategy proposed in De la Rosa de Sáa et al. (2015).

1. A random sample of 100 interval-valued observations is generated from a random interval $X = [X_1 - X_2, X_1 + X_2]$, where
$$X_1 \rightsquigarrow \beta(6,1), \quad X_2 \rightsquigarrow \begin{cases} \exp\,(100 + 4X_1) & \text{if } X_1 < .25, \\ \exp\,(200) & \text{if } .25 \le X_1 \le .75, \\ \exp\,(500 - 4X_1) & \text{if } X_1 > .75. \end{cases}$$
2. A proportion $c_p = 0, .1, .2, .3$ or $.4$ of the observations is then contaminated in both location and spread with
$$X_1 \rightsquigarrow \beta(1,6), \quad X_2 \rightsquigarrow \begin{cases} \exp\,(100 + 4X_1)\,/(C_D^2 + 1) & \text{if } X_1 < .25, \\ \exp\,(200)\,/(C_D^2 + 1) & \text{if } .25 \le X_1 \le .75, \\ \exp\,(500 - 4X_1)\,/(C_D^2 + 1) & \text{if } X_1 > .75, \end{cases}$$
where $C_D = 0, 1, 5, 10$ or 100 measures the relative distance between the distributions of the regular and contaminated observations.
3. The population parameters T are approximated by Monte Carlo simulation using $N = 10000$ replications of *Step 1*.
4. For each (c_p, C_D), *Steps 1-2* are repeated $N = 10000$ times and, for each of these contaminated samples, the location estimates $\widehat{T}_i$ are calculated.

Table 1 shows the outputs and the smallest value of bias, variance and MSE for each choice of c_p and C_D has been highlighted in bold. Bias, variance and MSE have also been computed in terms of the Hausdorff and 1-norm distances, but conclusions do not differ.

Table 1 Bias, variance and MSE (all multiplied by a factor 1000) of the Aumann mean, the 1-norm, Hausdorff-type and $d_{1/3}$- medians, the Hampel M-estimator of location and the $d_{1/3}$-depth-based trimmed means (DTM) with $\beta = .2$ and $\beta = .45$.

c_p	C_D		$E[X]$	Me$[X]$	Med$[X]$	M$_{1/3}[X]$	K$^M_{Hampel}$	DTM$_{\beta=.2}$	DTM$_{\beta=.45}$
.0	0	Bias	.0	**.0**	.1	.1	.0	.1	.2
		Var	.2	.2	.2	.2	**.1**	.2	.2
		MSE	.2	.2	.2	.2	**.1**	.2	.2
.1	0	Bias	71.4	17.3	17.3	17.3	**2.5**	14.3	14.2
		Var	**.2**	.3	.3	.3	.2	.2	.3
		MSE	5.3	.6	.6	.6	**.2**	.4	.5
.1	1	Bias	71.3	17.3	17.3	17.3	**2.5**	14.2	14.2
		Var	**.2**	.3	.3	.3	.2	.2	.3
		MSE	5.2	.6	.6	.6	**.2**	.4	.5
.1	5	Bias	71.3	17.3	17.3	17.3	**2.7**	14.3	14.2
		Var	**.2**	.3	.3	.3	.2	.2	.3
		MSE	5.2	.6	.6	.6	**.2**	.4	.5
.1	10	Bias	71.6	17.5	17.6	17.6	**2.9**	14.6	14.3
		Var	**.2**	.3	.3	.3	.2	.2	.3
		MSE	5.3	.6	.6	.6	**.2**	.4	.5
.1	100	Bias	71.6	17.4	17.4	17.4	**2.8**	14.4	14.2
		Var	**.2**	.3	.3	.3	.2	.2	.3
		MSE	5.3	.6	.6	.6	**.2**	.4	.5
.2	0	Bias	142.9	41.7	41.7	41.6	**20.6**	44.3	32.1
		Var	**.2**	.4	.4	.4	.2	.2	.4
		MSE	20.6	2.1	2.2	2.1	**.7**	2.2	1.4
.2	1	Bias	142.6	41.4	41.4	41.4	**21.0**	44.1	31.5
		Var	**.2**	.4	.4	.4	.2	.2	.4
		MSE	20.5	2.1	2.1	2.1	**.7**	2.1	1.4
.2	5	Bias	142.8	41.5	41.5	41.5	**21.7**	44.2	31.4
		Var	**.2**	.4	.4	.4	.2	.2	.4
		MSE	20.5	2.1	2.1	2.1	**.7**	2.1	1.4
.2	10	Bias	142.8	41.8	41.9	41.9	**21.9**	44.4	32.0
		Var	**.2**	.4	.4	.4	.2	.2	.4
		MSE	20.5	2.2	2.2	2.2	**.7**	2.2	1.4
.2	100	Bias	142.9	41.8	41.9	41.9	**21.8**	44.4	32.0
		Var	**.2**	.4	.4	.4	.2	.2	.4
		MSE	20.6	2.2	2.2	2.2	**.7**	2.2	1.4
.3	0	Bias	214.3	79.2	79.2	78.9	127.3	116.9	**52.0**
		Var	**.2**	.6	.6	.6	1.5	.2	.5
		MSE	46.1	6.9	6.9	6.9	17.7	13.9	**3.2**
.3	1	Bias	214.0	78.6	78.7	78.5	133.7	116.6	**51.6**
		Var	**.1**	.6	.6	.6	1.4	.2	.5
		MSE	45.9	6.8	6.8	6.8	19.2	13.8	**3.1**
.3	5	Bias	214.3	79.5	79.5	79.3	138.3	117.0	**52.2**
		Var	**.2**	.6	.6	.6	1.3	.2	.5
		MSE	46.1	7.0	7.0	6.9	20.4	13.9	**3.2**
.3	10	Bias	214.4	79.5	79.6	79.3	138.4	117.0	**51.9**
		Var	**.2**	.6	.6	.6	1.3	.2	.5
		MSE	46.1	7.0	7.0	6.9	20.5	13.9	**3.2**
.3	100	Bias	214.3	79.5	79.5	79.3	138.1	117.0	**52.2**
		Var	**.2**	.6	.6	.6	1.3	.2	.5
		MSE	46.1	7.0	7.0	6.9	20.4	13.9	**3.2**
.4	0	Bias	285.6	147.9	148.0	147.1	241.6	200.5	**96.3**
		Var	**.2**	1.2	1.2	1.2	2.0	.2	1.5
		MSE	81.7	23.1	23.1	22.9	60.4	40.4	**10.8**
.4	1	Bias	285.5	148.0	148.1	147.3	246.5	200.5	**96.1**
		Var	**.2**	1.2	1.2	1.2	1.6	.2	1.5
		MSE	81.7	23.1	23.1	22.9	62.3	40.4	**10.7**
.4	5	Bias	285.6	148.4	148.5	147.5	250.3	200.6	**96.8**
		Var	**.2**	1.2	1.2	1.2	1.2	.2	1.5
		MSE	81.7	23.2	23.3	23.0	63.9	40.5	**10.9**
.4	10	Bias	285.8	148.5	148.6	147.8	249.7	200.8	**96.9**
		Var	**.2**	1.2	1.2	1.2	1.3	.2	1.5
		MSE	81.8	23.3	23.3	23.1	63.6	40.5	**10.9**
.4	100	Bias	285.8	148.4	148.5	147.6	250.3	200.8	**96.6**
		Var	**.2**	1.2	1.2	1.2	1.2	.2	1.4
		MSE	81.8	23.2	23.2	23.0	63.9	40.5	**10.8**

5 Concluding remarks

Table 1 shows that there is no uniformly most appropriate location estimate. However, the performance of the d_θ-depth-based trimmed mean (with an appropriate choice of the trimming parameter) is among the best: it can outperform the behaviour of the other measures or, at least, become the second best option in terms of bias and MSE.

Acknowledgements This research has been partially supported by the Spanish Ministry of Science, Innovation and Universities (Grant MTM-PID2019-104486GB-I00) and by Principality of Asturias/FEDER Grants (SV-PA-21-AYUD/2021/50897).

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