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Sums Involving the Digamma Function Connected to the Incomplete Beta Function and the Bessel functions

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Abstract: We calculate some infinite sums containing the digamma function in closed form. These sums are related either to the incomplete beta function or to the Bessel functions. The calculations yield interesting new results as by-products, such as parameter differentiation formulas for the beta incomplete function, reduction formulas of ${}_3F_2$ hypergeometric functions, or a definite integral which does not seem to be tabulated in the most common literature. As an application of certain sums involving the digamma function, we calculated some reduction formulas for the parameter differentiation of the Mittag–Leffler function and the Wright function.

Keywords: digamma function; Bessel functions; incomplete beta function; Wright function; Mittag–Leffler function; differentiation with respect to parameters

MSC: 33B15; 33C20; 33E12; 33C10

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1. Introduction

In the existing literature [1,2], we found some compilations of series and finite sums involving the digamma function. Some authors contributed to these compilations, such as Doelder [3], Miller [4], and Cvijović [5]. More recently, the authors published some novel results in this regard [6].

Sums involving the digamma function occur in the expressions of the derivatives of the Mittag–Leffler function and the Wright function with respect to parameters [7,8]. In addition, they occur in the derivation of asymptotic expansions for Mellin–Barnes integrals [9,10]. Further, Doelder [3] calculate sums involving the digamma function in connection to the dilogarithm function [11]. As an application in physics, this type of sums arises in the evaluation of Feynman amplitudes in quantum field theory [12].

The aim of this paper is the derivation of some new sums involving the digamma function by using the derivative of the Pochhammer symbol and some reduction formulas of the generalized hypergeometric function. As a consistency test, for many particular values of the results obtained, we recover expressions given in the existing literature. In addition, we developed a MATHEMATICA program to numerically check all the new expressions derived in the paper. This program is available at https://bit.ly/3LG2gej (accessed on 19 April 2023).

This paper is organized as follows. In Section 2, we present some basic properties of the Pochhammer symbol, the beta and the digamma functions, as well as the definitions of the generalized hypergeometric function and the Meijer-G function. In Section 3, we derive some sums connected to the parameter differentiation of the incomplete beta function. In Section 4, we calculate, in a similar way, some other sums connected to the order derivatives of the Bessel and the modified Bessel functions. Section 5 is devoted to the application of some sums involving the digamma function to reduction formulas for the parameter differentiation of the Wright and Mittag–Leffler functions. Finally, we compile our conclusions in Section 6.

2. Preliminaries

The Pochhamer symbol is defined as [13], Equation 18:12:1

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}, \tag{1}$$

where $\Gamma(x)$ denotes the gamma function with the following basic properties [13] (Ch. 43):

$$\Gamma(z+1) = z\,\Gamma(z), \tag{2}$$
$$2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2z). \tag{3}$$

In addition, the beta function, defined as [14] (Equation 1.5.3)

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt,$$
$$\mathrm{Re}\,x,\ \mathrm{Re}\,y > 0,$$

satisfies the property [14] (Equation 1.5.5)

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{4}$$

Further, the incomplete beta function is defined as [15] (Equation 8.17.1):

$$\mathrm{B}_z(x,y) = \int_0^z t^{x-1}(1-t)^{y-1}dt, \tag{5}$$

which satisfies the property [13] (Equation 58:5:1),

$$\mathrm{B}_z(a,b) + \mathrm{B}_{1-z}(b,a) = \mathrm{B}(a,b). \tag{6}$$

A function related to the incomplete beta function is the Lerch function, defined as

$$\Phi(z,a,b) = \sum_{k=0}^{\infty} \frac{z^k}{(k+b)^a}. \tag{7}$$

According to (1), we have

$$\frac{d}{dx}[(x)_n] = (x)_n[\psi(x+n) - \psi(x)], \tag{8}$$

and

$$\frac{d}{dx}\left[\frac{1}{(x)_n}\right] = \frac{1}{(x)_n}[\psi(x) - \psi(x+n)], \tag{9}$$

where $\psi(x)$ denotes the digamma function [13] (Ch. 44)

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{10}$$

with the following properties [14] (Equations 1.3.3-4&8)

$$\psi\left(\frac{1}{2}\right) = -\gamma - 2\ln 2, \tag{11}$$
$$\psi(z+1) = \frac{1}{z} + \psi(z), \tag{12}$$
$$\psi(1-z) - \psi(z) = \pi\cot(\pi z), \tag{13}$$

with γ being the Euler–Mascheroni constant.

Finally, ${}_pF_q(z)$ denotes the generalized hypergeometric function, usually defined by means of the hypergeometric series [15] (Section 16.2):

$$ {}_pF_q\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots b_q \end{array} \middle| z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}, \tag{14} $$

whenever this series converges and elsewhere by analytic continuation.

In addition, the Meijer-G function is defined via the Mellin–Barnes integral representation [15] (Equation 16.17.1):

$$ \begin{aligned} & G_{p,q}^{m,n}\left(z \middle| \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots b_q \end{array} \right) \\ = & \frac{1}{2\pi i} \int_L \frac{\prod_{\ell=1}^{m} \Gamma(b_\ell - s) \prod_{\ell=1}^{m} \Gamma(1 - a_\ell + s)}{\prod_{\ell=m}^{q-1} \Gamma(1 - b_{\ell+1} + s) \prod_{\ell=n}^{p-1} \Gamma(a_{\ell+1} - s)} z^s ds, \end{aligned} $$

where the integration path L separates the poles of the factors $\Gamma(b_\ell - s)$ from those of the factors $\Gamma(1 - a_\ell + s)$.

3. Sums Connected to the Incomplete Beta Function

3.1. Derivatives of the Incomplete Beta Function with Respect to the Parameters

Theorem 1. *The following parameter derivative holds true:*

$$ \frac{\partial}{\partial a} \mathrm{B}_z(a,b) = \ln z \, \mathrm{B}_z(a,b) - \frac{z^a}{a^2} \, {}_3F_2\left(\begin{array}{c} 1-b, a, a \\ a+1, a+1 \end{array} \middle| z \right). \tag{15} $$

Proof. According to the definition of the incomplete beta function (5), we have

$$ \frac{\partial}{\partial a} \mathrm{B}_z(a,b) = \int_0^z t^{a-1} (1-t)^{b-1} \ln t \, dt. \tag{16} $$

Now, apply the formulas [13] (Equation 18:3:4),

$$ \frac{1}{(1-t)^{\nu}} = \sum_{k=0}^{\infty} (\nu)_k \frac{t^k}{k!}, \tag{17} $$

and [16] (Equation 1.6.1(18))

$$ \int x^p \ln x \, dx = x^{p+1} \left[\frac{\ln x}{p+1} - \frac{1}{(p+1)^2} \right], \tag{18} $$

in order to rewrite (16) as

$$ \frac{\partial}{\partial a} \mathrm{B}_z(a,b) = z^a \left\{ \ln z \sum_{k=0}^{\infty} \frac{(1-b)_k \, z^k}{k!(a+k)} - \sum_{k=0}^{\infty} \frac{(1-b)_k \, z^k}{k!(a+k)^2} \right\}. \tag{19} $$

Taking into account the property

$$ \frac{1}{\alpha + k} = \frac{(\alpha)_k}{\alpha (\alpha+1)_k}, $$

and the definition of the generalized hypergeometric function (14), we may recast the the sums given in (19) as

$$ \frac{\partial}{\partial a} \mathrm{B}_z(a,b) = z^a \left\{ \frac{\ln z}{a} \, {}_2F_1\left(\begin{array}{c} 1-b, a \\ a+1 \end{array} \middle| z \right) - \frac{1}{a^2} \, {}_3F_2\left(\begin{array}{c} 1-b, a, a \\ a+1, a+1 \end{array} \middle| z \right) \right\}. \tag{20} $$

Finally, apply to (20) the reduction formula [17] (Equation 7.3.1(28))

$$ {}_2F_1\left(\begin{array}{c} \alpha, \beta \\ \beta+1 \end{array} \middle| z \right) = \beta\, z^{-\beta}\, \mathrm{B}_z(\beta, 1-\alpha), \tag{21} $$

in order to arrive at (15), as we wanted to prove. □

As a consequence of the last theorem, we calculate the next integral, which does not seem to be tabulated in the most common literature.

Theorem 2. *For* $\mathrm{Re}\,\alpha > -1$ *and* $\mathrm{Re}\,z > 0$, *the following integral holds true:*

$$ \int_0^z \frac{u^\alpha}{1-u^2} \ln u \, du = \frac{1}{2} \ln z \, \mathrm{B}_{z^2}\left(\frac{1+\alpha}{2}, 0 \right) - \frac{z^{\alpha+1}}{4}\, \Phi\left(z^2, 2, \frac{1+\alpha}{2} \right). \tag{22} $$

Proof. According to [13] (Equation 58:14:7), we have

$$ \mathrm{B}_{\tanh^2(x)}(\lambda, 0) = 2 \int_0^x \tanh^{2\lambda-1} t \, dt. $$

Performing the substitutions $z = \tanh x$ and $u = \tanh t$, we obtain

$$ \mathrm{B}_{z^2}(\lambda, 0) = 2 \int_0^z \frac{u^{2\lambda-1}}{1-u^2} du. \tag{23} $$

On the one hand, calculate the derivative of the LHS of (23) with respect to the parameter λ, taking into account (15),

$$ \frac{\partial}{\partial \lambda} \mathrm{B}_{z^2}(\lambda, 0) = 2 \ln z \, \mathrm{B}_{z^2}(\lambda, 0) - \frac{z^{2\lambda}}{\lambda^2}\, {}_3F_2\left(\begin{array}{c} 1, \lambda, \lambda \\ \lambda+1, \lambda+1 \end{array} \middle| z^2 \right). \tag{24} $$

In order to calculate the ${}_3F_2$ function given in (24), we apply the reduction formula [17] (Equation 7.4.1(5))

$$ {}_3F_2\left(\begin{array}{c} a, b, c \\ a+1, b+1 \end{array} \middle| x \right) = \frac{1}{b-a}\left[b\, {}_2F_1\left(\begin{array}{c} a, c \\ a+1 \end{array} \middle| x \right) - a\, {}_2F_1\left(\begin{array}{c} b, c \\ b+1 \end{array} \middle| x \right) \right]. $$

Thus, taking $c = 1$ and applying the reduction formula [17] (Equation 7.3.1(122))

$$ {}_2F_1\left(\begin{array}{c} 1, b \\ b+1 \end{array} \middle| x \right) = b\, \Phi(x, 1, b), $$

as well as the definition of the Lerch function (7), we have

$$ \begin{aligned} {}_3F_2\left(\begin{array}{c} 1, a, b \\ a+1, b+1 \end{array} \middle| x \right) &= \frac{ab}{b-a}[\Phi(x,1,a) - \Phi(x,1,b)] \\ &= \frac{ab}{b-a} \sum_{k=0}^\infty x^k \left(\frac{1}{k+a} - \frac{1}{k+b} \right) \\ &= ab \sum_{k=0}^\infty \frac{x^k}{(k+a)(k+b)}. \end{aligned} $$

Therefore, taking $a = \lambda$, $b = \lambda + \epsilon$, we have

$$ \begin{aligned} {}_3F_2\left(\begin{array}{c} 1, \lambda, \lambda \\ \lambda+1, \lambda+1 \end{array} \middle| z^2 \right) &= \lim_{\epsilon \to 0} \lambda(\lambda+\epsilon) \sum_{k=0}^\infty \frac{z^{2k}}{(k+\lambda)(k+\lambda+\epsilon)} \\ &= \lambda^2\, \Phi\left(z^2, 2, \lambda \right). \end{aligned} \tag{25} $$

Insert (25) in (24) to obtain

$$\frac{\partial}{\partial \lambda} \mathrm{B}_{z^2}(\lambda, 0) = 2 \ln z \, \mathrm{B}_{z^2}(\lambda, 0) - z^{2\lambda} \, \Phi\left(z^2, 2, \lambda\right). \tag{26}$$

On the other hand, calculate the derivative of the RHS of (23) as

$$2 \frac{\partial}{\partial \lambda} \int_0^z \frac{u^{2\lambda - 1}}{1 - u^2} \, du = 4 \int_0^z \frac{u^{2\lambda - 1}}{1 - u^2} \ln u \, du. \tag{27}$$

Finally, equate (26) to (27) and perform the substitution $\alpha = 2\lambda - 1$ to complete the proof. □

Lemma 1. *For* $\alpha \neq 1$, $\operatorname{Re} \alpha < 2$, *the following reduction formula holds true:*

$${}_3F_2\left(\left. \begin{array}{c} \alpha, \beta, \beta \\ \beta + 1, \beta + 1 \end{array} \right| 1 \right) = \beta^2 \, \mathrm{B}(1 - \alpha, b) [\psi(1 + \beta - \alpha) - \psi(\beta)]. \tag{28}$$

Proof. Take $a = \beta + \epsilon$, $b = \beta$, $c = \alpha$ and calculate the limit $\epsilon \to 0$ in the following reduction formula [17] (Equation 7.4.4(16))

$$\begin{aligned} &{}_3F_2\left(\left. \begin{array}{c} a, b, c \\ a + 1, b + 1 \end{array} \right| 1 \right) = \frac{a\, b}{a - b} \Gamma(1 - c) \left\{ \frac{\Gamma(b)}{\Gamma(1 + b - c)} - \frac{\Gamma(a)}{\Gamma(1 + a - c)} \right\}, \\ &a \neq b, \; c \neq 1, \; \operatorname{Re} c < 2, \end{aligned}$$

to obtain:

$$\begin{aligned} & {}_3F_2\left(\left. \begin{array}{c} \alpha, \beta, \beta \\ \beta + 1, \beta + 1 \end{array} \right| 1 \right) \\ = & \lim_{\epsilon \to 0} \beta(\beta + \epsilon) \, \Gamma(1 - \alpha) \, \frac{\Gamma(\beta) \, \Gamma(1 + \beta - \alpha + \epsilon) - \Gamma(1 + \beta - \alpha) \, \Gamma(\beta + \epsilon)}{\epsilon \, \Gamma(1 + \beta - \alpha) \, \Gamma(1 + \beta - \alpha + \epsilon)}. \end{aligned} \tag{29}$$

Apply the Taylor series expansion

$$\Gamma(x + \epsilon) = \Gamma(x) + \Gamma(x) \, \psi(x) \, \epsilon + O\left(\epsilon^2\right),$$

to calculate (29). After simplification, we arrive at (28), as we wanted to prove. □

Theorem 3. *The following parameter derivative holds true:*

$$\begin{aligned} \frac{\partial}{\partial b} \mathrm{B}_z(a, b) \;\; = \;\; & \frac{(1 - z)^b}{b^2} \, {}_3F_2\left(\left. \begin{array}{c} 1 - a, b, b \\ b + 1, b + 1 \end{array} \right| 1 - z \right) \\ & - \ln(1 - z) \, \mathrm{B}_{1 - z}(b, a) - \mathrm{B}(a, b) [\psi(a + b) - \psi(b)]. \end{aligned} \tag{30}$$

Proof. According to the definition of the incomplete beta function (5), we have

$$\frac{\partial}{\partial b} \mathrm{B}_z(a, b) = \int_0^z t^{a - 1} (1 - t)^{b - 1} \ln(1 - t) dt.$$

Perform the substitution $\tau = 1 - t$ and apply the Formulas (17) and (18) to obtain

$$\begin{aligned}
&\frac{\partial}{\partial b}\mathrm{B}_z(a,b) \qquad (31)\\
= \quad & (1-z)^b \left\{ \sum_{k=0}^{\infty} \frac{(1-a)_k (1-z)^k}{k!(b+k)^2} - \ln(1-z) \sum_{k=0}^{\infty} \frac{(1-a)_k (1-z)^k}{k!(b+k)} \right\} \\
& - \sum_{k=0}^{\infty} \frac{(1-a)_k}{k!(b+k)^2}.
\end{aligned}$$

Finally, write the sums given in (31) as hypergeometric functions and apply the results given in (21) and (28) to arrive at (30), as we wanted to prove. □

Proof. (Alternative). Consider (15) perform the substitutions $a \leftrightarrow b$ and $z \to 1-z$ to obtain

$$\frac{\partial}{\partial b}\mathrm{B}_{1-z}(b,a) = \ln(1-z)\,\mathrm{B}_{1-z}(b,a) - \frac{(1-z)^b}{b^2}\,{}_3F_2\left(\left. \begin{array}{c} 1-a, b, b \\ b+1, b+1 \end{array} \right| 1-z \right). \qquad (32)$$

Take into account (6) in order to rewrite (32) as

$$\begin{aligned}
&\frac{\partial}{\partial b}\mathrm{B}_z(a,b) \\
= \quad & \frac{(1-z)^b}{b^2}\,{}_3F_2\left(\left. \begin{array}{c} 1-a, b, b \\ b+1, b+1 \end{array} \right| 1-z \right) - \ln(1-z)\,\mathrm{B}_{1-z}(b,a) - \frac{\partial}{\partial b}\mathrm{B}(a,b). \qquad (33)
\end{aligned}$$

According to (4) and (10), note that

$$\begin{aligned}
\frac{\partial}{\partial b}\mathrm{B}(a,b) &= \frac{\partial}{\partial b}\left(\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} \right) \\
&= \mathrm{B}(a,b)[\psi(b) - \psi(a+b)]. \qquad (34)
\end{aligned}$$

Insert (34) in (33) to complete the proof. □

3.2. Calculation of Sums Involving the Digamma Function

Theorem 4. *For $b \neq c+1$ and $z \in \mathbb{C}$, $|z| < 1$ the following sum holds true:*

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k} \psi(b+k)\, z^k \qquad (35)\\
= \quad & (c-1) z^{1-c} \left\{ \frac{1}{(b-c+1)^2}\,{}_3F_2\left(\left. \begin{array}{c} 2-c, b-c+1, b-c+1 \\ b-c+2, b-c+2 \end{array} \right| 1-z \right) \right. \\
& \left. + (1-z)^{c-b-1} [(\psi(b-c+1) - \ln(1-z))\,\mathrm{B}(b-c+1, c-1) + \psi(b)\,\mathrm{B}_{1-z}(b-c+1, c-1)] \right\}
\end{aligned}$$

Proof. On the one hand, applying the ratio test, we see that the sum given in (35) converges for $|z| < 1$ and diverges for $|z| > 1$. Indeed, taking

$$a_k = \frac{(b)_k}{(c)_k} \psi(b+k) z^k,$$

and taking into account (12), we have

$$\begin{aligned}
\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| &= \lim_{k\to\infty} \left| \frac{(b+k)\psi(b+k+1)}{(c+k)\psi(b+k)} z \right| \\
&= \lim_{k\to\infty} \left| \frac{b+k}{c+k} \left(\frac{1}{(b+k)\psi(b+k)} + 1 \right) z \right| = |z|.
\end{aligned}$$

On the other hand, let us differentiate both sides of the reduction formula [17] (Equation 7.3.1(119)) with respect to parameter b:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k} z^k &= {}_2F_1\left(\begin{array}{c} 1, b \\ c \end{array} \middle| z \right) \\ &= z^{1-c}(1-z)^{c-b-1}(c-1)\,\mathrm{B}_z(c-1, b-c+1). \end{aligned} \tag{36}$$

Apply (8) and (36) to the LHS of (36), to obtain:

$$\begin{aligned} &\frac{\partial}{\partial b} \sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k} z^k = \sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k} [\psi(b+k) - \psi(b)] z^k \\ = &\sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k} [\psi(b+k)] z^k - \psi(b)\, z^{1-c}(1-z)^{c-b-1}(c-1)\,\mathrm{B}_z(c-1, b-c+1). \end{aligned} \tag{37}$$

On the RHS of (36), we obtain

$$\begin{aligned} &(c-1)z^{1-c} \frac{\partial}{\partial b}\left[(1-z)^{c-b-1}\,\mathrm{B}_z(c-1, b-c+1)\right] \\ = &(c-1)z^{1-c}(1-z)^{c-b-1}\left[-\ln(1-z)\mathrm{B}_z(c-1, b-c+1) + \frac{\partial}{\partial b}\mathrm{B}_z(c-1, b-c+1)\right]. \end{aligned} \tag{38}$$

According to (30), we have

$$\begin{aligned} &\frac{\partial}{\partial b}\mathrm{B}_z(c-1, b-c+1) \\ = &\frac{(1-z)^{b-c+1}}{(b-c+1)^2}\, {}_3F_2\left(\begin{array}{c} 2-c, b-c+1, b-c+1 \\ b-c+2, b-c+2 \end{array} \middle| 1-z \right) \\ &-\ln(1-z)\,\mathrm{B}_{1-z}(b-c+1, c-1) + \mathrm{B}(c-1, b-c+1)[\psi(1+b-c) - \psi(b)]. \end{aligned} \tag{39}$$

Now, insert (39) in (38) and apply the Formula (6) to arrive at

$$\begin{aligned} &(c-1)z^{1-c} \frac{\partial}{\partial b}\left[(1-z)^{c-b-1}\,\mathrm{B}_z(c-1, b-c+1)\right] \\ = &(c-1)z^{1-c}\Big\{(1-z)^{c-b+1}\mathrm{B}(c-1, b-c+1)[\psi(1+b-c) - \psi(b) - \ln(1-z)] \\ &+ \frac{1}{(b-c+1)^2}\, {}_3F_2\left(\begin{array}{c} 2-c, b-c+1, b-c+1 \\ b-c+2, b-c+2 \end{array} \middle| 1-z \right)\Big\}. \end{aligned} \tag{40}$$

Finally, equate the results given in (37) and (40) and apply (6) again to complete the proof. □

Remark 1. *It is worth noting that for $z = 1$, the sum given in (35) can be calculated taking $a = 1$ in [6] (Equation (23)):*

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(c)_k} \psi(b+k) \\ = &\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)}[\psi(c-b) - \psi(c-a-b) + \psi(b)], \\ &\mathrm{Re}(c-a-b) > 0, \end{aligned} \tag{41}$$

thus

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k} \psi(b+k) = \frac{c-1}{c-b-1}\left[\frac{1}{c-b-1} + \psi(b)\right], \\ &\mathrm{Re}(c-b) > 1. \end{aligned}$$

Corollary 1. *For $z \in \mathbb{C}$ and $|z| < 1$ the following formula holds true:*

$$\begin{aligned}
&\sum_{k=1}^{\infty} \psi(b+k) z^k \qquad (42)\\
&= (b-1) z^{1-b} {}_3F_2\left(\begin{array}{c} 1,1,2-b \\ 2,2 \end{array} \middle| 1-z \right) + \frac{z^{1-b}}{z-1}[\gamma + \ln(1-z)] + \frac{z^{1-b}}{z-1}\psi(b).
\end{aligned}$$

Proof. Put apart the term for $k = 0$ in (35) and take $b = c$. □

Corollary 2. *For $z \in \mathbb{C}$, and $a \neq 1$, the following reduction formula holds true:*

$$ {}_3F_2\left(\begin{array}{c} 1,1,a \\ 2,2 \end{array} \middle| z \right) = \frac{\psi(2-a) + \gamma + \ln z + \mathrm{B}_{1-z}(2-a,0)}{(1-a)z}. \qquad (43)$$

Proof. Taking into account (36) for $c = b+1$, compare (42) to the result found in the existing literature [4] (Equation (1.2)):

$$\begin{aligned}
\sum_{k=1}^{\infty} \psi(b+k) z^k &= \frac{z}{1-z}\left[\psi(b) + \frac{1}{b}\, {}_2F_1\left(\begin{array}{c} 1,b \\ b+1 \end{array} \middle| z \right) \right] \\
&= \frac{z}{1-z}\left[\psi(b) + \frac{\mathrm{B}_z(b,0)}{z^b} \right],
\end{aligned}$$

and solve for the ${}_3F_2$ function with $a = 2-b$ to obtain the desired result. □

Remark 2. *It is worth noting that for $(2-a) \in \mathbb{Q} - \{-1,-2,\ldots\}$, the incomplete beta function $\mathrm{B}_{1-z}(2-a,0)$ given in (43) can be expressed in terms of elementary functions [18]. For instance, taking $a = 3/2$ in (43) and considering (11) and the formula for $n = 0,1,\ldots$ [18]*

$$\mathrm{B}_z\left(n+\frac{1}{2},0\right) = 2\left(\tanh^{-1}\sqrt{z} - \sum_{k=0}^{n-1}\frac{z^{k+1/2}}{2k+1}\right),$$

we arrive at

$$ {}_3F_2\left(\begin{array}{c} 1,1,\frac{3}{2} \\ 2,2 \end{array} \middle| z \right) = \frac{4}{z}\ln\left(\frac{2\left(1-\sqrt{1-z}\right)}{z}\right),$$

which is given in the existing literature [17] (Equation 7.4.2(365)).

Remark 3. *As a consistency test, we can recover a known formula by taking the limit $a \to 1$ in (43). Indeed,*

$$ {}_3F_2\left(\begin{array}{c} 1,1,1 \\ 2,2 \end{array} \middle| z \right) = \lim_{a\to 1}\frac{\psi(2-a) + \gamma + \ln z + \mathrm{B}_{1-z}(2-a,0)}{(1-a)z}.$$

Perform the substitution $b = 1-a$, take into account (5) and the Formula [15] (Equation 5.9.16):

$$\psi(z) + \gamma = \int_0^1 \frac{1-t^{z-1}}{1-t}dt,$$

to obtain

$$\begin{aligned}
{}_3F_2\left(\begin{array}{c} 1,1,1 \\ 2,2 \end{array} \middle| z \right) &= \lim_{b\to 0}\frac{\psi(1+b) + \gamma + \ln z + \mathrm{B}_{1-z}(1+b,0)}{b\,z} \\
&= \lim_{b\to 0}\frac{1}{b\,z}\left[\ln z + \int_0^1 \frac{1-t^b}{1-t}dt + \int_0^{1-z}\frac{t^b}{1-t}dt \right].
\end{aligned}$$

Now, perform the susbstitution $\tau = 1 - t$, and apply the Taylor series:

$$(1-\tau)^b = \sum_{n=0}^{\infty} \frac{\ln^n(1-\tau)}{n!} b^n,$$

to arrive at

$$\begin{aligned}
&{}_3F_2\left(\begin{array}{c} 1,1,1 \\ 2,2 \end{array} \middle| z \right) \\
=\ & \lim_{b\to 0} \frac{1}{b\,z} \left[\ln z - \int_0^1 \frac{1-(1-\tau)^b}{\tau} d\tau + \int_1^z \frac{(1-\tau)^b}{\tau} dt \right] \\
=\ & \lim_{b\to 0} \frac{1}{b\,z} \left[\ln z - \int_0^1 \frac{\ln(1-\tau)}{\tau} b\, d\tau - \int_1^z \left(\frac{1}{\tau} + \frac{\ln(1-\tau)}{\tau} b \right) dt \right] \\
=\ & -\frac{1}{z} \int_0^z \frac{\ln(1-\tau)}{\tau} d\tau.
\end{aligned}$$

From the following formula of the dilogarithm function [15] (Equation 25.12.2)

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-\tau)}{\tau} d\tau,$$

we recover the following result found in the existing literature [17] (Equation 7.4.2(355)):

$${}_3F_2\left(\begin{array}{c} 1,1,1 \\ 2,2 \end{array} \middle| z \right) = \frac{\mathrm{Li}_2(z)}{z}.$$

Theorem 5. *For $a \neq 1$ and $z \in \mathbb{C}$, $|z| < 1$, the following sum holds true:*

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(b+1)_k} \psi(a+k) z^k \qquad (44) \\
=\ & b\,z^{-b} \{ [\ln(1-z) - \psi(a)]\, \mathrm{B}_{1-z}(1-a,b) + [\psi(1+b-a) - \pi \cot(\pi a)]\, \mathrm{B}(b,1-a) \\
& - \frac{(1-z)^{1-a}}{(1-a)^2}\, {}_3F_2\left(\begin{array}{c} 1-b, 1-a, 1-a \\ 2-a, 2-a \end{array} \middle| 1-z \right) \bigg\}.
\end{aligned}$$

Proof. On the one hand, applying the ratio test, we see that the sum given in (44) converges for $|z| < 1$ and diverges for $|z| > 1$. Indeed, taking

$$c_k = \frac{(a)_k (b)_k}{k!(b+1)_k} \psi(a+k) z^k$$

and taking into account (12), we have

$$\begin{aligned}
\lim_{k\to\infty} \left| \frac{c_{k+1}}{c_k} \right| &= \lim_{k\to\infty} \left| \frac{(a+k)(b+k)\psi(a+k+1)}{(k+1)(b+1+k)\psi(a+k)} z \right| \\
&= \lim_{k\to\infty} \left| \frac{(a+k)(b+k)}{(k+1)(b+1+k)} \left(\frac{1}{(a+k)\psi(a+k)} + 1 \right) z \right| = |z|.
\end{aligned}$$

On the other hand, taking into account (8), differentiate the reduction Formula (21), i.e.,

$${}_2F_1\left(\begin{array}{c} a,b \\ b+1 \end{array} \middle| z \right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(b+1)_k} z^k = b\,z^{-b}\, \mathrm{B}_z(b,1-a),$$

with respect to the parameter a to obtain

$$\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!(b+1)_k}[\psi(a+k)-\psi(a)]z^k = b\,z^{-b}\frac{\partial}{\partial a}\mathrm{B}_z(b,1-a). \tag{45}$$

Apply (21) on the LHS of (45) and (30) on the RHS of (45) to arrive at

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!(b+1)_k}\psi(a+k)z^k \qquad (46)\\ =\;& b\,z^{-b}\{\ln(1-z)\,\mathrm{B}_{1-z}(1-a,b)+\mathrm{B}(b,1-a)[\psi(1+b-a)-\psi(1-a)]\\ &+\psi(a)\,\mathrm{B}_z(b,1-a)-\frac{(1-z)^{1-a}}{(1-a)^2}\,{}_3F_2\left(\begin{array}{c}1-b,1-a,1-a\\ 2-a,2-a\end{array}\middle|1-z\right)\right\}.\end{aligned}$$

Finally, apply (6) and (13) in order to reduce (46) to (44), as we wanted to prove. □

Remark 4. *It is worth noting that for $z=1$, the sum given in (44) can be calculated taking $c=b+1$ in (41) and applying (2), (4) and (13), to obtain:*

$$\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!(b+1)_k}\psi(b+k) = b\,\mathrm{B}(b,1-a)[\psi(1+b-a)-\pi\cot(\pi a)]\,,$$
$$\mathrm{Re}\,a<1.$$

Corollary 3. *For $a\neq 1$, and $z\in\mathbb{C}$, $|z|<1$ the following formula holds true:*

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(a)_k}{(k+1)!}\psi(a+k)z^k = \frac{1}{(1-a)z} \qquad (47)\\ &\left\{(1-z)^{1-a}\left[\ln(1-z)-\psi(a)+\frac{1}{a-1}\right]+\psi(2-a)-\pi\cot(\pi a)\right\}.\end{aligned}$$

Proof. Take $b=1$ in (44) and consider that for $a\neq 1$ we have

$$\begin{aligned}\mathrm{B}_{1-z}(1-a,1) &= \frac{(1-z)^{1-a}}{1-a},\\ \mathrm{B}(1,1-a) &= \frac{1}{1-a}.\end{aligned}$$

□

4. Sums Connected to Bessel Functions

If we differentiate the following sum formulas [17] (Equation 7.13.1(1)):

$${}_0F_1\left(\begin{array}{c}-\\ b\end{array}\middle|-z\right) = \sum_{k=0}^{\infty}\frac{(-z)^k}{k!(b)_k} = \Gamma(b)\,z^{(1-b)/2}\,J_{b-1}(2\sqrt{z}),$$

and

$${}_0F_1\left(\begin{array}{c}-\\ b\end{array}\middle|z\right) = \sum_{k=0}^{\infty}\frac{z^k}{k!(b)_k} = \Gamma(b)\,z^{(1-b)/2}\,I_{b-1}(2\sqrt{z}),$$

with respect to parameter b, taking into account (9), we obtain:

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(-z)^k\psi(k+b)}{k!(b)_k} \qquad (48)\\ =\;& z^{(1-b)/2}\Gamma(b)\left[J_{b-1}(2\sqrt{z})\ln\sqrt{z}-\frac{\partial J_{b-1}(2\sqrt{z})}{\partial b}\right],\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{z^k \psi(k+b)}{k!(b)_k} \\
&= z^{(1-b)/2}\Gamma(b)\left[I_{b-1}(2\sqrt{z})\ln\sqrt{z} - \frac{\partial I_{b-1}(2\sqrt{z})}{\partial b}\right],
\end{aligned} \tag{49}$$

which are found in an equivalent form in [1] (Equations 55.7.11-12). For $b = n \in \mathbb{N}$, we found a closed-form expression for (49) in [4] (Equation (5.11)). We can obtain closed-form expressions for other values of b using the following formulas [19] (Equations (93) and (99)) for $\nu \geq 0$, $\operatorname{Re} z > 0$:

$$\begin{aligned}
\frac{\partial J_\nu(z)}{\partial \nu} &= \frac{\pi}{2}\left[\frac{Y_\nu(z)(z/2)^{2\nu}}{\Gamma^2(\nu+1)}\,{}_2F_3\left(\left.\begin{array}{c}\nu, \nu+\frac{1}{2} \\ 2\nu+1, \nu+1, \nu+1\end{array}\right| -z^2\right)\right. \\
&\quad \left. - \frac{\nu J_\nu(z)}{\sqrt{\pi}} G^{3,0}_{2,4}\left(z^2\left|\begin{array}{c}\frac{1}{2}, 1 \\ 0, 0, \nu, -\nu\end{array}\right.\right)\right],
\end{aligned} \tag{50}$$

and

$$\begin{aligned}
\frac{\partial I_\nu(z)}{\partial \nu} &= \frac{-\nu I_\nu(z)}{2\sqrt{\pi}} G^{3,1}_{2,4}\left(z^2\left|\begin{array}{c}\frac{1}{2}, 1 \\ 0, 0, \nu, -\nu\end{array}\right.\right) \\
&\quad - \frac{K_\nu(z)(z/2)^{2\nu}}{\Gamma^2(\nu+1)}\,{}_2F_3\left(\left.\begin{array}{c}\nu, \nu+\frac{1}{2} \\ 2\nu+1, \nu+1, \nu+1\end{array}\right| z^2\right).
\end{aligned} \tag{51}$$

Theorem 6. *For $b \geq 1$ and $\operatorname{Re} z > 0$, the following sum holds true:*

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(-z)^k \psi(k+b)}{k!(b)_k} \\
&= \frac{z^{-(1+b)/2}}{8\,\Gamma(b)} \\
&\left\{\Gamma^2(b) J_{b-1}(2\sqrt{z})\left[\sqrt{\pi}(b-1) G^{3,0}_{2,4}\left(4z\left|\begin{array}{c}\frac{3}{2}, 2 \\ 1, 1, b, 2-b\end{array}\right.\right) + 4z\ln z\right]\right. \\
&\left. - 4\pi\, z^b Y_{b-1}(2\sqrt{z})\,{}_2F_3\left(\left.\begin{array}{c}b-1, b-\frac{1}{2} \\ b, b, 2b-1\end{array}\right| -4z\right)\right\}.
\end{aligned} \tag{52}$$

Proof. Calculate (48) taking into account (50) to arrive at the desired result. □

Theorem 7. *For $b \geq 1$ and $\operatorname{Re} z > 0$, the following sum holds true:*

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{z^k \psi(k+b)}{k!(b)_k} \\
&= \frac{z^{-(1+b)/2}}{8\sqrt{\pi}\Gamma(b)} \\
&\left\{\Gamma^2(b) I_{b-1}(2\sqrt{z})\left[(b-1) G^{3,1}_{2,4}\left(4z\left|\begin{array}{c}\frac{3}{2}, 2 \\ 1, 1, b, 2-b\end{array}\right.\right) + 4\sqrt{\pi} z\ln z\right]\right. \\
&\left. + 8\sqrt{\pi}\, z^b K_{b-1}(2\sqrt{z})\,{}_2F_3\left(\left.\begin{array}{c}b-1, b-\frac{1}{2} \\ b, b, 2b-1\end{array}\right| 4z\right)\right\}.
\end{aligned} \tag{53}$$

Proof. Calculate (49) taking into account (51) to arrive at the desired result. □

Theorem 8. *For $b \geq 1$ and $\operatorname{Re} z > 0$, the following sum holds true:*

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{z^k \psi(2k+b)}{k! \left(\frac{1}{2}\right)_k \left(\frac{b}{2}\right)_k \left(\frac{b+1}{2}\right)_k} \\
= &\frac{\Gamma(b)}{2^b z^{(b-1)/4}} \\
&\left\{ \ln\left(2\, z^{1/4}\right) \left[J_{b-1}\left(4\, z^{1/4}\right) + I_{b-1}\left(4\, z^{1/4}\right) \right] - \frac{\partial J_{b-1}\left(4\, z^{1/4}\right)}{\partial b} - \frac{\partial I_{b-1}\left(4\, z^{1/4}\right)}{\partial b} \right\},
\end{aligned} \tag{54}$$

where the order derivatives of the Bessel functions are calculated using (50) and (51).

Proof. Sum up (48) and (49) using the duplication formula of the gamma function (3) to arrive at the desired result. □

Theorem 9. *For $b \geq 1$ and $\mathrm{Re}\, z > 0$, the following sum holds true:*

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{z^k \psi(2k+b)}{k! \left(\frac{3}{2}\right)_k \left(\frac{b}{2}\right)_k \left(\frac{b+1}{2}\right)_k} \\
= &\frac{\Gamma(b)}{2^{b+1} z^{b/4}} \\
&\left\{ \ln\left(2\, z^{1/4}\right) \left[I_{b-2}\left(4\, z^{1/4}\right) - J_{b-2}\left(4\, z^{1/4}\right) \right] - \frac{\partial I_{b-2}\left(4\, z^{1/4}\right)}{\partial b} + \frac{\partial J_{b-2}\left(4\, z^{1/4}\right)}{\partial b} \right\},
\end{aligned} \tag{55}$$

where the order derivatives of the Bessel functions are calculated using (50) and (51).

Proof. Substract (49) from (48) and apply (3) again to arrrive at the desired result. □

5. Application to the Parameter Derivative of Some Special Functions

5.1. Application to the Derivative of the Wright Function with Respect to the Parameters

The Wright function is defined as [15] (Equation 10.46.1):

$$\mathrm{W}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\, \Gamma(\alpha k + \beta)}, \quad \alpha > -1,$$

thus,

$$\frac{\partial \mathrm{W}_{\alpha,\beta}(z)}{\partial \alpha} = -\sum_{k=1}^{\infty} \frac{k\, z^k \psi(\alpha k + \beta)}{k!\, \Gamma(\alpha k + \beta)}, \tag{56}$$

$$\frac{\partial \mathrm{W}_{\alpha,\beta}(z)}{\partial \beta} = -\sum_{k=0}^{\infty} \frac{z^k \psi(\alpha k + \beta)}{k!\, \Gamma(\alpha k + \beta)}, \tag{57}$$

and the following equation is satisfied:

$$\frac{\partial \mathrm{W}_{\alpha,\beta}(z)}{\partial \alpha} = z \frac{\partial}{\partial z} \left(\frac{\partial \mathrm{W}_{\alpha,\beta}(z)}{\partial \beta} \right). \tag{58}$$

In reference [20], we found some reduction formulas for the first derivative of the Wright function with respect to the parameters for particular values of α and β. Next, we extend these reduction formulas. For this purpose, apply (53) to arrive at the following result:

Theorem 10. *For $\beta \geq 1$ and $\operatorname{Re} z > 0$, we have*

$$\begin{aligned} & \left.\frac{\partial \mathrm{W}_{\alpha,\beta}(z)}{\partial \beta}\right|_{\alpha=1} \qquad (59) \\ = \; & z^{-(1+\beta)/2}\left\{ I_{\beta-1}\left(2\sqrt{z}\right)\left[\frac{1-\beta}{8\sqrt{\pi}} G_{2,4}^{3,1}\left(4z\left|\begin{array}{c} 3/2, 2 \\ 1, 1, \beta, 2-\beta \end{array}\right.\right) - \frac{z}{2}\ln z\right] \right. \\ & \left. - \frac{z^{\beta}}{\Gamma^2(\beta)} K_{\beta-1}\left(2\sqrt{z}\right) {}_2F_3\left(\begin{array}{c} \beta-1, \beta-\frac{1}{2} \\ \beta, \beta, 2\beta-1 \end{array}\left| 4z\right.\right)\right\}. \end{aligned}$$

Remark 5. *It is worth noting that for $\beta = 1$, Equation (59) is reduced to*

$$\left.\frac{\partial \mathrm{W}_{\alpha,\beta}(z)}{\partial \beta}\right|_{\alpha=\beta=1} = -\frac{1}{2}\ln z\, I_0\left(2\sqrt{z}\right) - K_0\left(2\sqrt{z}\right),$$

which is found in [20] (Equation (6.8)).

Further, from (58) and (59) and with the aid of the MATHEMATICA program, we arrive at the following result:

Theorem 11. *For $\beta \geq 1$ and $\operatorname{Re} z > 0$, we have*

$$\begin{aligned} & \left.\frac{\partial \mathrm{W}_{\alpha,\beta}(z)}{\partial \alpha}\right|_{\alpha=1} = \frac{z^{-(\beta+1)/2}}{2} \qquad (60) \\ & \left\{\frac{\beta-1}{8\sqrt{\pi}}\left\{(\beta-1)I_{\beta-1}\left(2\sqrt{z}\right) - \sqrt{z}\left[I_{\beta-2}\left(2\sqrt{z}\right) + I_{\beta}\left(2\sqrt{z}\right)\right]\right\} G_{2,4}^{3,1}\left(4z\left|\begin{array}{c} 3/2, 2 \\ 1, 1, \beta, 2-\beta \end{array}\right.\right)\right. \\ & + \frac{z^{\beta}}{\Gamma^2(\beta)}\left\{(\beta-1)K_{\beta-1}\left(2\sqrt{z}\right) + \sqrt{z}\left[K_{\beta-2}\left(2\sqrt{z}\right) + K_{\beta}\left(2\sqrt{z}\right)\right]\right\} {}_2F_3\left(\begin{array}{c} \beta-1, \beta-\frac{1}{2} \\ \beta, \beta, 2\beta-1 \end{array}\left| 4z\right.\right) \\ & + \frac{I_{\beta-1}\left(2\sqrt{z}\right)}{4\sqrt{\pi}}\left[2\sqrt{\pi}z\left[(\beta-1)\ln z - 2\right] + (\beta-1) G_{1,3}^{2,1}\left(4z\left|\begin{array}{c} 3/2 \\ 1, \beta, 2-\beta \end{array}\right.\right)\right] \\ & \left. - \frac{z^{3/2}\ln z}{2}\left[I_{\beta-2}\left(2\sqrt{z}\right) + I_{\beta}\left(2\sqrt{z}\right)\right] + \frac{2(1-\beta)z^{\beta}}{\Gamma^2(\beta)} K_{\beta-1}\left(2\sqrt{z}\right) {}_1F_2\left(\begin{array}{c} \beta-\frac{1}{2} \\ \beta, 2\beta-1 \end{array}\left| 4z\right.\right)\right\}. \end{aligned}$$

Remark 6. *It is worth noting that for $\beta = 1$, Equation (60) is reduced to*

$$\left.\frac{\partial \mathrm{W}_{\alpha,\beta}(z)}{\partial \alpha}\right|_{\alpha=\beta=1} = \frac{\sqrt{z}\left[K_1\left(2\sqrt{z}\right) - \ln z\, I_1\left(2\sqrt{z}\right)\right] - I_0\left(2\sqrt{z}\right)}{2},$$

which is also found in [20] (Equation (6.14)).

5.2. Application to the Derivative of the Mittag–Leffler Function with Respect to the Parameters

The two-parameter Mittag–Leffler function is defined as [15] (Equation 10.46.3):

$$\mathrm{E}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0, \qquad (61)$$

thus,

$$\frac{\partial \mathrm{E}_{\alpha,\beta}(z)}{\partial \alpha} = -\sum_{k=0}^{\infty}\frac{k\, z^k \psi(\alpha k+\beta)}{\Gamma(\alpha k+\beta)}, \qquad (62)$$

$$\frac{\partial \mathrm{E}_{\alpha,\beta}(z)}{\partial \beta} = -\sum_{k=0}^{\infty}\frac{z^k \psi(\alpha k+\beta)}{\Gamma(\alpha k+\beta)}, \qquad (63)$$

and

$$\frac{\partial \mathrm{E}_{\alpha,\beta}(z)}{\partial \alpha} = z\frac{\partial}{\partial z}\left(\frac{\partial \mathrm{E}_{\alpha,\beta}(z)}{\partial \beta}\right). \tag{64}$$

For this purpose, consider the following functions:

Definition 1. *According to [2] (Equation 6.2.1(63)), define*

$$\begin{aligned} Q(a,t) &= \sum_{k=0}^{\infty} \frac{t^k}{(a)_k}\psi(k+a) \\ &= \psi(a) + e^t\left[t^{1-a}\psi(a)\gamma(a,t) + \frac{t}{a^2}\,{}_2F_2\left(\begin{array}{c} a,a \\ a+1,a+1 \end{array}\middle| -t\right)\right], \end{aligned} \tag{65}$$

thus

$$\begin{aligned} P(a,t) &= \frac{\partial Q(a,t)}{\partial t} \\ &= \psi(a) + e^t \\ &\left\{\frac{t-a+1}{a^2}\,{}_2F_2\left(\begin{array}{c} a,a \\ a+1,a+1 \end{array}\middle| -t\right) + t^{-a}\gamma(a,t)[1+(t-a+1)\psi(a)]\right\}. \end{aligned} \tag{66}$$

In reference [20], we found some reduction formulas of the first derivative of the Mittag–Leffler function with respect to the parameters for particular values of α and β. In particular, we found for $q = 1,2,\ldots$ that

$$\begin{aligned} &\left.\frac{\partial \mathrm{E}_{\alpha,\beta}(z)}{\partial \alpha}\right|_{\alpha=1/q} \\ &= -\sum_{h=0}^{q-1} \frac{h\left[\psi\left(\frac{h}{q}+\beta\right) + \tilde{Q}\left(\frac{h}{q}+\beta, z^q\right) + q\,z^q\,P\left(\frac{h}{q}+\beta, z^q\right)\right]}{\Gamma\left(\frac{h}{q}+\beta\right)} z^h, \end{aligned} \tag{67}$$

and

$$\left.\frac{\partial \mathrm{E}_{\alpha,\beta}(z)}{\partial \beta}\right|_{\alpha=1/q} = -\sum_{h=0}^{q-1} \frac{\psi\left(\frac{h}{q}+\beta\right) + \tilde{Q}\left(\frac{h}{q}+\beta, z^q\right)}{\Gamma\left(\frac{h}{q}+\beta\right)} z^h, \tag{68}$$

where

$$\tilde{Q}(a,t) = Q(a,t) - \psi(a).$$

Next, we extend these reduction formulas to other values of the parameters. For this purpose, consider the following lemma.

Lemma 2. *For $n = 1,2,\ldots$, the following sum identity holds true:*

$$\sum_{k=0}^{\infty} a_{nk} = \sum_{k=0}^{\infty} \theta_{n,k}\, a_k, \tag{69}$$

where

$$\theta_{n,k} = \frac{1}{n}\sum_{m=1}^{n} \exp\left(\frac{2\pi\, i\, m\, k}{n}\right). \tag{70}$$

Theorem 12. *For $n = 1,2,\ldots$, the following reduction formula holds true:*

$$\left.\frac{\partial \mathrm{E}_{\alpha,\beta}(z)}{\partial \beta}\right|_{\alpha=n} = -\frac{1}{n\,\Gamma(\beta)}\sum_{m=1}^{n} Q\left(\beta, z^{1/n} e^{i2\pi m/n}\right). \tag{71}$$

Proof. According to (63) and Lemma 2, we have

$$\begin{aligned}
\left.\frac{\partial \mathrm{E}_{\alpha,\beta}(z)}{\partial \beta}\right|_{\alpha=n} &= -\sum_{k=0}^{\infty} \frac{z^k \psi(nk+\beta)}{\Gamma(nk+\beta)} \\
&= -\sum_{k=0}^{\infty} \theta_{n,k} \frac{z^{k/n} \psi(k+\beta)}{\Gamma(k+\beta)} \\
&= -\frac{1}{n\,\Gamma(\beta)} \sum_{m=1}^{n} \exp\left(\frac{2\pi\, i\, m\, k}{n}\right) \sum_{k=0}^{\infty} \frac{z^{k/n} \psi(k+\beta)}{(\beta)_k}.
\end{aligned}$$

Finally, take into account (65) to arrive at the desired result. □

Theorem 13. *For $n = 1, 2, \ldots$, the following reduction formula holds true:*

$$\left.\frac{\partial \mathrm{E}_{\alpha,\beta}(z)}{\partial \alpha}\right|_{\alpha=n} = -\frac{z^{1/n}}{n^2\,\Gamma(\beta)} \sum_{m=1}^{n} e^{i2\pi m/n} P\left(\beta, z^{1/n} e^{i2\pi m/n}\right). \tag{72}$$

Proof. Apply (64) to (71) and take into account the definition given in (66). □

Remark 7. *It is worth noting that for $\alpha = 1$, (72) is equivalent to (67) and (71) is equivalent to (68).*

For particular values of α and β, the first derivative of the Mittag–Leffler function with respect to the parameters are shown in Tables 1 and 2, using the results given in (71) and (72) with the aid of the MATHEMATICA program.

Table 1. First derivative of the Mittag–Leffler function with respect to α.

α	β	$\frac{\partial \mathrm{E}_{\alpha,\beta}(z)}{\partial \alpha}$
1	1	$1 - e^z\{z[\ln z + \Gamma(0,z)] + 1\}$
1	2	$\frac{1}{z}\{1 + \gamma - e^z[1 + (z-1)(\ln z + \Gamma(0,z))]\}$
2	1	$\frac{1}{8} e^{-\sqrt{z}} \left\{ \sqrt{z}\left[\ln z - 2\,\mathrm{Ei}(\sqrt{z}) - e^{2\sqrt{z}}\left(2\,\mathrm{E}_1(\sqrt{z}) + \ln z\right)\right] - 2\left(e^{\sqrt{z}} - 1\right)^2 \right\}$
2	2	$\frac{1}{8\sqrt{z}} e^{-\sqrt{z}} \left\{ e^{2\sqrt{z}}\left[(1-\sqrt{z})\left(2\,\mathrm{E}_1(\sqrt{z}) + \ln z\right) - 2\right] + (1+\sqrt{z})\left[2\,\mathrm{Ei}(\sqrt{z}) - \ln z\right] + 2 \right\}$

Table 2. First derivative of the Mittag–Leffler function with respect to β.

α	β	$\frac{\partial \mathrm{E}_{\alpha,\beta}(z)}{\partial \beta}$
1	1	$-e^z[\ln z + \Gamma(0,z)]$
1	2	$-\frac{1}{z}\{e^z[\ln z + \Gamma(0,z)] + \gamma\}$
2	1	$\frac{1}{4} e^{-\sqrt{z}} \left\{ 2\,\mathrm{Ei}(\sqrt{z}) - \ln z - e^{2\sqrt{z}}\left[\ln z + 2\,\Gamma(0,\sqrt{z})\right] \right\}$
2	2	$\frac{1}{4\sqrt{z}} e^{-\sqrt{z}} \left\{ \ln z - 2\,\mathrm{Ei}(\sqrt{z}) - e^{2\sqrt{z}}\left[\ln z + 2\,\Gamma(0,\sqrt{z})\right] \right\}$

6. Conclusions

We calculated some new infinite sums involving the digamma function. On the one hand, some of these new sums are connected to the incomplete beta function, i.e., Equations (35) and (44). For this purpose, we derived a new ${}_3F_2$ hypergeometric sum at argument unity in (28). We also calculated new expressions for the derivatives of the incomplete beta function $\mathrm{B}_z(a,b)$ with respect to the parameters a and b in (15) and (30). As a consequence of the latter, we obtained a definite integral in (22) that does not seem to be tabulated in the most common literature. In addition, in (43) we derived a new reduction formula for a ${}_3F_2$ hypergeometric function.

On the other hand, we calculated sums involving the digamma function which are connected to the Bessel functions, i.e., Equations (52)–(55). For this purpose, we used the derivative of the Pochhammer symbol given in (9), as well as some expressions found in the existing literature for the order derivatives of $J_\nu(z)$ and $I_\nu(z)$, given in (50) and (51) respectively.

Finally, we calculated some reduction formulas for the derivatives of some special functions with respect to the parameters as an application of the sums involving the digamma function. In particular, we applied the sum presented in (53) to the calculation of the reduction Formulas (59) and (60) for the derivatives of the Wright function with respect to the parameters. Similarly, applying the sum given in (65), we calculated the reduction Formulas (71) and (72) for the derivatives of the Mittag–Leffler function with respect to the parameters.

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