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A Note on Some Generalized Hypergeometric Reduction Formulas

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Abstract: Herein, we calculate reduction formulas for some generalized hypergeometric functions ${}_{m+1}F_m(z)$ in terms of elementary functions as well as incomplete beta functions. For this purpose, we calculate the n-th order derivative of the function $z^{\gamma}\,\mathrm{B}_z(\alpha,\beta)$ with respect to z. As corollaries, we obtain reduction formulas of these ${}_{m+1}F_m(z)$ functions for argument unity in terms of elementary functions, as well as beta functions.

Keywords: reduction formulas; generalized hypergeometric functions; incomplete beta functions

MSC: 33C20; 33B20

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1. Introduction

Mathematical and physical applications of the generalized hypergeometric functions ${}_pF_q(a_1,\ldots,a_p;b_1,\ldots b_q;z)$ are abundant in the existing literature (see [1] (Sections 16.23 and 16.24) and the references therein). For instance, a variety of problems in classical mechanics and mathematical physics lead to Picard–Fuchs equations, which are frequently solvable in terms of generalized hypergeometric functions [2]. As a consequence, the calculation of generalized hypergeometric functions for particular values of the parameters $a_1,\ldots,a_p$, $b_1,\ldots b_q$ and the argument z in terms of other special or elementary functions is of great interest. These reduction formulas are found in several compilations of the existing literature, such as those provided by Luke [3] (Sections 6.2 and 6.3) and Prudnikov et al. [4] (Chapter 7). A revision, as well as an extension of the tables presented by Prudnikov et al., was carried out by Krupnikov and Kölbig in [5]. More recently, Brychov [6] (Chapter 8) compiled new representations of ${}_pF_q(z)$ hypergeometric functions. However, the number of papers devoted to the calculation of reduction formulas of ${}_pF_q(z)$ for arbitrary argument z are relatively scarce (see, e.g., [7,8]), because there are many more papers devoted to the calculation of representations of ${}_pF_q(z)$ for particular values of z, such as [9,10] for $z=1$; [11,12] for $z=2$; and [13] for $z=-1$.

The goal of the present note is to fill this gap by calculating some reduction formulas of the ${}_{m+1}F_m(z)$ function for an arbitrary z in terms of elementary functions, as well as incomplete beta functions. As corollaries, we will obtain reduction formulas of these ${}_{m+1}F_m(z)$ functions for $z=1$ in terms of elementary functions, as well as beta functions.

This paper is organized as follows: In Section 2, we present some basic definitions and properties that we will use throughout the paper. In Section 3, we derive some n-th derivative formulas that we will apply in Section 4 for the calculation of the reduction formulas mentioned above. Finally, we collect our conclusions in Section 5.

2. Preliminaries

The gamma function is usually defined by the integral representation [1] (Equation 5.2.1)

$$\Gamma(z)=\int_0^{\infty}e^{-t}t^{z-1}dt,\quad \Re(z)>0, \tag{1}$$

and for $\Re(z) \leq 0$ by analytic continuation. An important property of the gamma function is [14] (Equation 1.2.1)

$$\Gamma(z+1) = z\,\Gamma(z). \tag{2}$$

The logarithmic derivative of the gamma function is [14] (Equation 1.3.1)

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}. \tag{3}$$

The Pochhammer polynomial can be defined in terms of the gamma function as [15] (Equation 18:12:1)

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}, \tag{4}$$

and it satisfies the property [15] (Equation 18:5:1)

$$(-x)_n = (-1)^n (x-n+1)_n. \tag{5}$$

The incomplete beta function is defined as [1] (Equation 8.17.1)

$$\mathrm{B}_z(a,b) = \int_0^z t^{a-1}(1-t)^{b-1} dt, \tag{6}$$

for $0 \leq z \leq 1$, and by analytical continuation for other real or complex values of z. For $z = 1$, the incomplete beta function is reduced to the beta function [1] (Equation 5.12.1)

$$\mathrm{B}(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}. \tag{7}$$

The generalized hypergeometric function ${}_pF_q(z)$ is usually defined by means of the hypergeometric series [1] (Section 16.2)

$${}_pF_q\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots b_q \end{array} \middle| z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}, \tag{8}$$

whenever this series converges, and elsewhere by analytic continuation. An important transformation formula was provided by Euler [16] (Equation 2.2.7):

$${}_2F_1\left(\begin{array}{c} a, b \\ c \end{array} \middle| z \right) = (1-z)^{-a}\, {}_2F_1\left(\begin{array}{c} a, c-b \\ c \end{array} \middle| \frac{z}{z-1} \right). \tag{9}$$

Finally, Gauss's summation formula reads as follows [16] (Theorem 2.2.2):

$${}_2F_1\left(\begin{array}{c} a, b \\ c \end{array} \middle| 1 \right) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}, \tag{10}$$

for $\Re(c-a-b) > 0$.

3. Formulas for n-th Order Derivatives

Following the notation given in [6] (Chapter 1), we denote the n-th order derivative of a function $f(z)$ with respect to its argument z as

$$D^n[f(z)] = \frac{d^n f(z)}{dz^n}.$$

According to this notation, Leibniz's differentiation formula [1] (Equation 1.4.2) is written as

$$D^n[f(z)g(z)] = \sum_{k=0}^{n} \binom{n}{k} D^k[f(z)]\, D^{n-k}[g(z)]. \tag{11}$$

Also, it is easy to prove that (see [6] (Equation 1.1.2 [1]))

$$D^n[z^\alpha] = (\alpha-n+1)_n z^{\alpha-n} = (-1)^n(-\alpha)_n z^{\alpha-n}, \tag{12}$$

$$D^n\left[(1-z)^\alpha\right] = (-1)^m(\alpha-n+1)_n(1-z)^{\alpha-n} = (-\alpha)_n(1-z)^{\alpha-n}. \tag{13}$$

Further, according to [1] (Equation 16.3.2), for $n=0,1,2,\ldots$, we have

$$\begin{aligned} &{}_{p+1}F_{q+1}\left(\begin{array}{c} a+n, a_1,\ldots,a_p \\ a, b_1,\ldots b_q \end{array} \middle| z \right) \\ &= \frac{z^{1-a}}{(a)_n} D^n\left[z^{n+a-1}\, {}_pF_q\left(\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots b_q \end{array} \middle| z \right)\right]. \end{aligned} \tag{14}$$

Theorem 1. *For $n=0,1,2,\ldots$, the following n-th derivative formulas hold true:*

$$\mathcal{H}^n(\alpha,\beta,\gamma;z) := D^n[z^\gamma\,\mathrm{B}_z(\alpha,\beta)] \tag{15}$$

$$\begin{aligned} = {}& (-1)^n z^{\gamma-n}\left\{ (-\gamma)_n\,\mathrm{B}_z(\alpha,\beta) - \frac{z^\alpha}{(1-z)^{1-\beta}} \right. \\ & \left. \sum_{k=0}^{n-1}\binom{n}{k+1}(-\gamma)_{n-k-1}(1-\alpha)_k\,{}_2F_1\left(\begin{array}{c} -k, 1-\beta \\ \alpha-k \end{array} \middle| \frac{z}{z-1} \right)\right\} \end{aligned} \tag{16}$$

$$\begin{aligned} = {}& (-1)^n z^{\gamma-n}\left\{ (-\gamma)_n\,\mathrm{B}_z(\alpha,\beta) - z^\alpha \right. \\ & \left. \sum_{k=0}^{n-1}\binom{n}{k+1}(-\gamma)_{n-k-1}(1-\alpha)_k\,{}_2F_1\left(\begin{array}{c} \alpha, 1-\beta \\ \alpha-k \end{array} \middle| z \right)\right\}. \end{aligned} \tag{17}$$

Proof. Applying Leibniz's differentiation formula (11) and the property (12), we obtain

$$\begin{aligned} & D^n[z^\gamma\,\mathrm{B}_z(\alpha,\beta)] \\ = {}& \sum_{k=0}^{n}\binom{n}{k} D^{n-k}[z^\gamma]\, D^k[\mathrm{B}_z(\alpha,\beta)] \\ = {}& D^n[z^\gamma]\,\mathrm{B}_z(\alpha,\beta) + \sum_{k=1}^{n}\binom{n}{k} D^{n-k}[z^\gamma]\, D^k[\mathrm{B}_z(\alpha,\beta)] \\ = {}& (-1)^n(-\gamma)_n\, z^{\gamma-n}\,\mathrm{B}_z(\alpha,\beta) \\ & + \sum_{k=0}^{n-1}\binom{n}{k+1}(-1)^{n-k-1}(-\gamma)_{n-k-1} z^{\gamma-n+k+1}\, D^{k+1}[\mathrm{B}_z(\alpha,\beta)]. \end{aligned} \tag{18}$$

According to the definition of the incomplete beta function (6), and applying again Leibniz's differentiation formula (11), as well as the formulas given in (12) and (13), we arrive at

$$\begin{aligned} & D^{k+1}[\mathrm{B}_z(\alpha,\beta)] = D^k\left[z^{\alpha-1}(1-z)^{\beta-1}\right] \\ = {}& \sum_{s=0}^{k}\binom{k}{s} D^s\left[(1-z)^{\beta-1}\right] D^{k-s}\left[z^{\alpha-1}\right] \\ = {}& \frac{(-1)^k z^{\alpha-1-k}}{(1-z)^{1-\beta}}\sum_{s=0}^{k}\binom{k}{s}(1-\beta)_s(1-\alpha)_{k-s}\left(\frac{z}{z-1}\right)^s. \end{aligned} \tag{19}$$

Note that we can recast the finite sum given in (19) in terms of a hypergeometric function. Indeed, applying to the definition (8) the property (5) and the definition of the Pochhammer symbol (4), we calculate

$$\begin{aligned} & (1-\alpha)_k \, {}_2F_1\left(\begin{array}{c} -k, 1-\beta \\ \alpha-k \end{array} \middle| x \right) \\ = \; & (1-\alpha)_k \sum_{s=0}^{\infty} \frac{(-k)_s (1-\beta)_s}{s! \, (\alpha-k)_s} x^s \\ = \; & \sum_{s=0}^{k} \binom{k}{s} (1-\beta)_s (1-\alpha)_{k-s} \, x^s. \end{aligned} \tag{20}$$

Therefore, applying (20) and taking into account Euler's transformation formula (9), we rewrite (19) as

$$\begin{aligned} & D^{k+1}[\mathrm{B}_z(\alpha,\beta)] \\ = \; & \frac{(-1)^k z^{\alpha-1-k}}{(1-z)^{1-\beta}} (1-\alpha)_k \, {}_2F_1\left(\begin{array}{c} -k, 1-\beta \\ \alpha-k \end{array} \middle| \frac{z}{z-1} \right) \qquad (21) \\ = \; & (-1)^k z^{\alpha-1-k} (1-\alpha)_k \, {}_2F_1\left(\begin{array}{c} \alpha, 1-\beta \\ \alpha-k \end{array} \middle| z \right). \qquad (22) \end{aligned}$$

Now, we insert (21) and (22) into (18) and simplify the result to complete the proof. □

Remark 1. *It is worth noting that, according to (21), the $\mathcal{H}^n(\alpha,\beta,\gamma;z)$ function is given in terms of elementary functions and incomplete beta functions.*

Corollary 1. *For $\Re(\beta) > n$, we have*

$$\mathcal{H}^n(\alpha,\beta,\gamma;1) = (-1)^n (-\gamma)_n \, \mathrm{B}(\alpha,\beta). \tag{23}$$

Proof. Considering (17) for $z = 1$ and applying Gauss's summation formula (10), we obtain

$$\begin{aligned} \mathcal{H}^n(\alpha,\beta,\gamma;1) = (-1)^n \Bigg\{ & (-\gamma)_n \, \mathrm{B}(\alpha,\beta) \\ & - \sum_{k=0}^{n-1} \binom{n}{k+1} (-\gamma)_{n-k-1} (1-\alpha)_k \frac{\Gamma(\alpha-k)\Gamma(\beta-k-1)}{\Gamma(-k)\Gamma(\alpha+\beta-k-1)} \Bigg\}. \end{aligned} \tag{24}$$

However, the finite sum given in (24) vanishes; thus, we obtain (23), as we intended to prove. □

4. Reduction Formulas

Theorem 2. *For $m = 1, 2, \ldots$, and $b_i \neq b_j$ $(i \neq j)$, the following reduction formula holds true:*

$${}_{m+1}F_m\left(\begin{array}{c} a, b_1, \ldots, b_m \\ b_1+1, \ldots, b_m+1 \end{array} \middle| z \right) = \prod_{s=1}^{m} b_s \sum_{j=1}^{m} \frac{z^{-b_j} \, \mathrm{B}_z(b_j, 1-a)}{\prod_{\ell \neq j}^{m} (b_\ell - b_j)}. \tag{25}$$

Proof. According to (4) and (2), we have

$$\frac{(x)_k}{(x+1)_k} = \frac{x}{x+k}. \tag{26}$$

Thus,

$$
\begin{aligned}
{}_{m+1}F_m\left(\begin{array}{c} a, b_1, \ldots, b_m \\ b_1+1, \ldots, b_m+1 \end{array}\middle| z\right) &= \sum_{k=0}^{\infty} \frac{(a)_k (b_1)_k \cdots (b_m)_k\, z^k}{k!(b_1+1)_k \cdots (b_m+1)_k} \\
&= \prod_{s=1}^{m} b_s \sum_{k=0}^{\infty} \frac{(a)_k\, z^k}{k!} \prod_{j=1}^{m} \frac{1}{b_j+k}.
\end{aligned} \tag{27}
$$

Now, if $p_m(x)$ is a polynomial of degree m, where r_j $(j = 1, \ldots, m)$ are its corresponding roots with $r_i \neq r_j$ $(i \neq j)$, then the following formula is satisfied [15] (Equation 17:13:10):

$$
\frac{1}{p_m(x)} = \sum_{j=1}^{m} \frac{1}{p'(r_j)\,(x - r_j)}. \tag{28}
$$

If we consider the polynomial

$$
p_m(x) = \prod_{j=1}^{m} (x - r_j) \to p'_m(r_j) = \prod_{\ell \neq j}^{m} (r_j - r_\ell),
$$

and we take $r_j = -b_j$ and $x = k$, Equation (28) becomes

$$
\prod_{j=1}^{m} \frac{1}{b_j + k} = \sum_{j=1}^{m} \frac{1}{(b_j + k) \prod_{\ell \neq j}^{m} (b_\ell - b_j)}. \tag{29}
$$

Inserting (26) into (27) and exchanging the summation order, we obtain

$$
\begin{aligned}
&{}_{m+1}F_m\left(\begin{array}{c} a, b_1, \ldots, b_m \\ b_1+1, \ldots, b_m+1 \end{array}\middle| z\right) \\
&= \prod_{s=1}^{m} b_s \sum_{j=1}^{m} \frac{1}{\prod_{\ell \neq j}^{m} (b_\ell - b_j)} \sum_{k=0}^{\infty} \frac{(a)_k\, z^k}{k!\,(b_j + k)}.
\end{aligned}
$$

Applying again (26) and recasting the result as an hypergeometric function, we obtain

$$
\begin{aligned}
&{}_{m+1}F_m\left(\begin{array}{c} a, b_1, \ldots, b_m \\ b_1+1, \ldots, b_m+1 \end{array}\middle| z\right) \\
&= \prod_{s=1}^{m} b_s \sum_{j=1}^{m} \frac{1}{b_j \prod_{\ell \neq j}^{m} (b_\ell - b_j)}\, {}_2F_1\left(\begin{array}{c} a, b_j \\ b_j+1 \end{array}\middle| z\right).
\end{aligned}
$$

Taking into account the reduction formula [4] (Equation 7.3.1 (28))

$$
{}_2F_1\left(\begin{array}{c} a, b \\ b+1 \end{array}\middle| z\right) = b\, z^{-b}\, \mathrm{B}_z(b, 1-a),
$$

we arrive at (25), as we intended to prove. □

Remark 2. *Particular cases of (25) are found in [4] (Equation 7.4.1 (5)) for $m = 2$ and [4] (Equation 7.5.1 (2)) for $m = 3$ in terms of ${}_2F_1(z)$ hypergeometric functions. However, when we have a parameter $b_i = b_j$ for $i \neq j$, we obtain in (25) an indeterminate expression. Nonetheless, the particular case $m = 2$ with $b_1 = b_2$ is given by the authors in [17]:*

$$
{}_3F_2\left(\begin{array}{c} a, b, b \\ b+1, b+1 \end{array}\middle| z\right) = b^2 z^{-b} \left\{ \log z\, \mathrm{B}_z(b, 1-a) - \frac{\partial}{\partial b} \mathrm{B}_z(b, 1-a) \right\}.
$$

Corollary 2. *For $m = 1, 2, \ldots$ and $b_i \neq b_j$ $(i \neq j)$, the following reduction formula holds true:*

$$ {}_{m+1}F_m\left(\begin{array}{c} a, b_1, \ldots, b_m \\ b_1 + 1, \ldots, b_m + 1 \end{array} \middle| 1 \right) = \prod_{s=1}^{m} b_s \sum_{j=1}^{m} \frac{\mathrm{B}\left(b_j, 1-a\right)}{\prod_{\ell \neq j}^{m} \left(b_\ell - b_j\right)}. \tag{30} $$

Remark 3. *The particular case $m = 2$ with $b_1 = b_2$ is given by the authors in [17]. For $a \neq 1$ and $\Re\, a < 2$,*

$$ {}_3F_2\left(\begin{array}{c} a, b, b \\ b+1, b+1 \end{array} \middle| 1 \right) = b^2\, \mathrm{B}(1-a, b)\{\psi(1+b-a) - \psi(b)\}. $$

Theorems 1 and 2 allow us to obtain the reduction formula presented below.

Theorem 3. *For $n = 0, 1, 2, \ldots$; $m = 1, 2, \ldots$; and $c_i \neq c_j$ $(i \neq j)$, the following reduction formula holds true:*

$$ \begin{aligned} & {}_{m+2}F_{m+1}\left(\begin{array}{c} a+n, b, c_1, \ldots, c_m \\ a, c_1 + 1, \ldots, c_m + 1 \end{array} \middle| z \right) \\ &= \frac{z^{1-a}}{(a)_n} \prod_{s=1}^{m} c_s \sum_{j=1}^{m} \frac{\mathcal{H}^n\left(c_j, 1-b, n+a-c_j-1; z\right)}{\prod_{\ell \neq j}^{m} \left(c_\ell - c_j\right)}, \end{aligned} \tag{31} $$

where the $\mathcal{H}^n(\alpha, \beta, \gamma; z)$ function is given in Theorem 1.

Proof. Taking into account (15), consider (14) for $p = m+1$ and $q = m$, as well as the reduction formula (25), to obtain the desired result. □

Remark 4. *Since $\mathcal{H}^n(\alpha, \beta, \gamma; z)$ can be expressed in terms of elementary functions and the beta incomplete function (see Remark 1), it turns out that the generalized hypergeometric function given in (31) can also be expressed in terms of elementary functions and incomplete beta functions.*

Corollary 3. *For $n = 0, 1, 2, \ldots$; $m = 1, 2, \ldots$; $c_i \neq c_j$ $(i \neq j)$; and $\Re(b) < 1 - n$, the following reduction formula holds true:*

$$ \begin{aligned} & {}_{m+2}F_{m+1}\left(\begin{array}{c} a+n, b, c_1, \ldots, c_m \\ a, c_1 + 1, \ldots, c_m + 1 \end{array} \middle| 1 \right) \\ &= \frac{1}{(a)_n} \prod_{s=1}^{m} c_s \sum_{j=1}^{m} \frac{\left(a-c_j\right)_n \mathrm{B}\left(c_j, 1-b\right)}{\prod_{\ell \neq j}^{m} \left(c_\ell - c_j\right)}. \end{aligned} \tag{32} $$

Proof. Consider (31) for $z = 1$, taking into account (23) and (5). □

It is worth noting that the particular case $m = 0$ of (31) is given in the literature [4] (Equations 7.3.1 (21 and 140)). Next, we provide a simple derivation.

Theorem 4. *For $n = 0, 1, 2, \ldots$ and $z \neq 1$, the following reduction formula holds true:*

$$ {}_2F_1\left(\begin{array}{c} a+n, b \\ a \end{array} \middle| z \right) = (1-z)^{-b} \sum_{k=0}^{n} \binom{n}{k} \frac{(b)_k}{(a)_k} \left(\frac{z}{1-z} \right)^k. \tag{33} $$

Proof. Using Euler's transformation formula (9) we obtain

$$ {}_2F_1\left(\begin{array}{c} a+n, b \\ a \end{array} \middle| z \right) = (1-z)^{-b}\, {}_2F_1\left(\begin{array}{c} -n, b \\ a \end{array} \middle| \frac{z}{z-1} \right). $$

Applying the definitions given in (8) and (4), as well as the property (5), we arrive at the desired result. □

5. Conclusions

Herein, we calculated reduction formulas for the generalized hypergeometric functions

$$ {}_{m+1}F_{m+2}(a, b_1, \ldots, b_m; b_1+1, \ldots, b_m+1; z) $$

and

$$ {}_{m+2}F_{m+1}(a+n, b, c_1, \ldots, c_m; a, c_1+1, \ldots, c_m+1; z) $$

in terms of elementary functions and incomplete beta $\mathrm{B}_z(\alpha, \beta)$ functions. As corollaries, we derived expressions for these generalized hypergeometric functions for $z = 1$ in terms of elementary and beta $\mathrm{B}(\alpha, \beta)$ functions. For both purposes, we calculated the n-th order derivative of the function $z^\gamma \mathrm{B}_z(\alpha, \beta)$ for arbitrary argument z, as well as for $z = 1$. For the completeness of the present note, we included a simple proof of the reduction formula of ${}_2F_1(a+n, b; a; z)$ in terms of elementary functions. As a future research direction, it would be interesting to investigate the singular cases of (25) and (31) for certain equal parameters (i.e., for $i \neq j$, $b_i = b_j$ and $c_i = c_j$, respectively). Also, the extension of the results presented in this paper to the corresponding basic hypergeometric functions ${}_r\phi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q; z)$ deserves future investigations. Finally, we would like to highlight that all the results presented in this paper were numerically checked with MATHEMATICA. This program is available at https://shorturl.at/grsBH (accessed on 1 August 2023).

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