Universidad de Oviedo

Programa de Doctorado en Materiales

Type IIB S-folds: an exceptional approach

S-folds de Tipo IIB: un enfoque excepcional

TESIS DOCTORAL

Colin Sterckx

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RESUMEN DEL CONTENIDO DE TESIS DOCTORAL

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RESUMEN (en español)

En esta tesis investigamos la estructura de vacíos de la supergravedad maximal gaugeada diónicamente con grupo de gauge [SO(1,1) x SO(6)] ⋉ R^{12} en cuatro dimensiones y encontramos nuevas familias de soluciones AdS_4 con supersimetría residual N=0,1 o 2, extendiendo la solución N=4 ya conocida en la literatura. Utilizando técnicas de Teoría de Campos Excepcionales (ExFT), elevamos estas soluciones a la supergravedad diez-dimensional de Tipo IIB en una geometría de la forma AdS_4 x S^1 x S^5. Las configuraciones resultantes se denominan *S-folds*. Éstas son configuraciones no-geométricas de la teoría de cuerdas que presentan una monodromía de dualidad S (de ahí el término S-fold) inducida por un elemento hiperbólico de SL(2,Z) al movernos alrededor de la S^1.

Se conjetura que los S-folds son los duales holográficos de nuevas CFT_3's fuertemente acopladas, estrechamente relacionadas con las interfaces localizadas en SYM_4. Con el fin de caracterizar su espectro de operadores y explorar la posible existencia de una variedad conforme de tales CFT_3's, estudiamos el espacio de módulos y el espectro de masas de los S-folds. A continuación demostramos que todos los S-folds con simetrías residuales continuas admiten deformaciones marginales exactas que rompen algunas o todas las (super)simetrías residuales. Estas deformaciones se generan en cuatro dimensiones al activar campos axiónicos que denominamos "deformaciones planas".

Al embeber estas deformaciones en la supergravedad Tipo IIB, éstas se clasifican en términos del "mapping torus" y se muestra que codifican una monodromía geométrica de la S^5 sobre la S^1. Centrándonos en las deformaciones planas del S-fold con N=4 supersimetría, establecemos la existencia de un espacio de módulos de soluciones no supersimétricas, pero perturbativamente estables. También examinamos la estabilidad no perturbativa de las soluciones no supersimétricas y no encontramos ningún canal de decaimiento. Estos resultados desafían la conjetura de la no existencia de soluciones AdS no supersimétricas y estables propuesta en la literatura.

Posteriormente consideramos los flujos de renormalización (RG flows) holográficos que terminan, en el IR, en las soluciones de S-folds utilizando técnicas numéricas y semi-analíticas. Mostramos que los S-folds son los puntos fijos en el IR desencadenados por deformaciones anisótropas de SYM_4 (colocada en una S^1), en concordancia con la interpretación de interfaz de sus CFT_3's duales. Además, presentamos un flujo de renormalización que conecta la solución N=1 en UV con la solución N=2 en IR.

Finalmente, investigamos la existencia de S-folds posiblemente más genéricos cuya descripción cuatridimensional efectiva no puede ser capturada por una supergravedad maximal sino sólo por una supergravedad semi-maximal. Esto nos lleva a investigar una clase de gaugeos del grupo ISO(3) x ISO(3) en supergravedad N=4. Dentro de esta clase, encontramos una red de vacíos que preservan supersimetría N=2 con un punto especial asociado a una solución exótica con supersimetría N=4. La realización en teoría de cuerdas de esta solución, si es que la hay, es aún desconocida.

Universidad de Oviedo

RESUMEN (en Inglés)

In this thesis we investigate the vacuum structure of the dyonic $[SO(1,1) \times SO(6)] \ltimes R^{12}$ gauged maximal supergravity in four dimensions and find new families of AdS_4 solutions with residual N=0,1 or 2 supersymmetry, extending a well-known N=4 solution in the literature. Using techniques from Exceptional Field Theory (ExFT), we uplift these solutions to Type IIB supergravity on geometries of the form $AdS_4 \times S^1 \times S^5$. The resulting Type IIB backgrounds are referred to as *S-folds*. These are non-geometric string backgrounds because they feature an S-duality monodromy (hence the term S-fold) induced by a hyperbolic element of SL(2,Z) when moving along the S^1.

S-folds are conjectured to be the holographic duals of new strongly coupled CFT_3's closely related to localised interfaces in SYM_4. In order to characterise the low-lying operator content and explore the possible existence of a conformal manifold of such CFT_3's, we study the mass spectrum and moduli space of the S-fold solutions. We then prove that all the S-folds with continuous residual symmetries admit exactly marginal deformations breaking some, or all, of the residual (super)symmetries. These deformations are generated in four dimensions by turning on axionic fields which we dub “flat deformations”.

The Type IIB uplift of these flat deformations are then classified in terms of mapping tori and shown to encode a geometric monodromy of the S^5 over the S^1. Focussing on the flat deformations of the original N=4 S-fold, we establish the existence of a moduli space of non-supersymmetric, yet perturbatively stable, solutions. We also examine the non-perturbative stability of the non-supersymmetric solutions and do not find any decay channel. These results challenge a non-SUSY AdS conjecture existing in the literature.

Next, we construct holographic RG-flows ending, in the IR, at the S-fold solutions using both numerical and semi-analytical techniques. We show that the S-folds are the IR fixed points of anisotropic deformations of SYM_4 (placed on S^1), in line with the interface-like interpretation of their CFT_3 duals. Moreover, we present an RG-flow connecting the N=1 solution in the UV to the N=2 solution in the IR.

Finally, we investigate the existence of possibly more generic S-folds whose effective four dimensional description cannot be captured by a maximal supergravity but only by a half-maximal one. This leads us to investigate a class of gaugings of ISO(3) x ISO(3) in N=4 supergravity. Within this setup, we find a rich web of AdS_4 solutions preserving N=2 supersymmetry with a special point of symmetry enhancement to an exotic N=4 solution. The string theoretic realisation of this solution, if any, is still lacking.

SR. PRESIDENTE DE LA COMISIÓN ACADÉMICA DEL PROGRAMA DE DOCTORADO EN ________________

Quand j'en ai assez de l'ombre,
je prends un livre dans une salle
pour voir un peu de ciel.

Alain Damasio

Abstract

We investigate the vacuum structure of the dyonic $[\mathrm{SO}(1,\,1) \times \mathrm{SO}(6)] \ltimes \mathbb{R}^{12}$ gauged maximal supergravity in four dimensions and find new families of AdS_4 solutions with residual $\mathcal{N} = 0$, 1 or 2 supersymmetry, extending the known $\mathcal{N} = 4$ solution. Using techniques from Exceptional Field Theory (ExFT), we uplift these solutions to Type IIB supergravity on $\mathrm{AdS}_4 \times S^1 \times S^5$. The resulting backgrounds are referred to as *S-folds* because they feature an S-duality monodromy induced by an hyperbolic element of $SL(2,\,\mathbb{Z})$ around the S^1.

S-folds are conjectured to be the holographic duals of new strongly coupled CFT_3's closely related to localised interfaces in SYM_4. In order to characterise the low-lying operator content and explore the possible existence of a conformal manifold of such CFT_3's, we study the mass spectrum and moduli space of the S-fold solutions. We prove that all S-folds with continuous residual symmetries admit exactly marginal deformations breaking some, or all, of the residual (super)symmetries. These deformations are generated in four dimensions by turning on axionic fields which we dub "flat deformations".

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Finally, we investigate the existence of possibly more generic S-folds whose effective four dimensional description cannot be captured by a maximal supergravity but only by a half-maximal one. This leads us to investigate a class of gaugings of $\mathrm{ISO}(3) \times \mathrm{ISO}(3)$ in $\mathcal{N} = 4$ supergravity. Within this setup, we find a web of AdS_4 solutions preserving $\mathcal{N} = 2$ supersymmetry with a special point of symmetry enhancement to an exotic $\mathcal{N} = 4$ solution. The string theoretic realisation of this solution, if any, is still lacking.

This thesis is based on the work published in [1, 2, 3, 4, 5, 6, 7].

Resumen

En esta tesis investigamos la estructura de vacíos de la supergravedad maximal gaugeada diónicamente con grupo de gauge $[SO(1,1) \times SO(6)] \ltimes R^{12}$ en cuatro dimensiones y encontramos nuevas familias de soluciones AdS4 con supersimetría residual $\mathcal{N} = 0$, 1 o 2, extendiendo la solución $\mathcal{N} = 4$ ya conocida en la literatura. Utilizando técnicas de Teoría de Campos Excepcionales (ExFT), elevamos estas soluciones a la supergravedad diez-dimensional de Tipo IIB en una geometría de la forma $AdS_4 x S^1 x S^5$. Las configuraciones resultantes se denominan S-folds. Éstas son configuraciones de la teoría de cuerdas que presentan una monodromía de dualidad S (de ahí el término S-fold) inducida por un elemento hiperbólico de $SL(2, \mathbb{Z})$ al movernos alrededor de la S^1.

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Esta tesis está basada en los trabajos publicados [1, 2, 3, 4, 5, 6, 7].

Résumé

Nous étudions la structure des solutions vides de la théorie de supergravité maximale en quatre dimensions jaugée dyoniquement par le groupe $[\mathrm{SO}(1,\,1)\times\mathrm{SO}(6)]\ltimes\mathbb{R}^{12}$. Nous trouvons de nouvelles familles de solutions AdS_4 avec une supersymmétrie résiduelle de $\mathcal{N}=0$, 1 ou 2, qui s'ajoutent à la solutions déjà connue avec $\mathcal{N}=4$ supersymétries. En utilisant des techniques de la théorie des champs exceptionelle (ExFT), nous réinterprétons ces solutions comme des solutions de la supergravité de type IIB sur $\mathrm{AdS}_4\times S^1\times S^5$. Ces solutions sont appellées "S-folds" car elles présentent une monodromie de S-dualité, par un élément hyperbolique du groupe $SL(2,\,\mathbb{Z})$, autour du S^1.

Ces S-folds ont été proposé comme les duaux holographiques de nouvelles théories des champs conformes, fortement couplées et pouvant être interprétées comme des interfaces dans SYM_4. Afin de caractériser les opérateurs de ces CFT et d'étudier l'existence d'une possible variété conforme associée, nous avons étudié le spectre de masse et l'espace des moduli des solutions de type "S-fold". Nous avons prouvé que tous les "S-folds" avec des symmétries résiduelles continues admettent des déformations exactement marginales brisant tout, ou une partie, des (super)symétries résiduelles. Ces déformations sont générées, en quatre dimensions, en activant des champs axioniques et nous les avons appelées "déformations plates".

Les solutions de Type IIB ainsi induites sont classifiées en termes d'un "tore d'application" et encodent une monodromie géométrique de S^5 sur S^1. Les déformations plates du $\mathcal{N}=4$ "S-fold" établissent l'existence d'un moduli de solutions non-supersymétriques et pourtant pertubativement stables. Nous avons aussi examiné la stabilité non-perturbative de ces solutions sans trouver de canaux de désintégration. Ces résultats remettent en cause la conjecture concernant les solutions AdS non-supersymétriques existant dans la littérature.

Ensuite, nous considérons, holographiquement, les flots de renormalisations se terminant, dans l'infrarouge, aux solutions "S-fold". Nous utilisons à la fois des méthodes numériques et semi-analytiques. Nous montrons que les S-folds sont les points fixes dans l'IR de déformations anisotropiques de SYM_4 dans l'ultraviolet, ce qui était attendu étant donné l'interprétation de leurs CFT duales. Nous présentons aussi un flot de renormalisation connectant la solution avec $\mathcal{N}=1$ dans l'UV à la solutions avec $\mathcal{N}=2$ dans l'IR.

Enfin, nous explorons des S-folds qui pourraient être plus génériques et dont la description effective en quatre dimensions ne serait pas capturée par une supergravité maximale mais seulement par une supergravité demi-maximale. Nous étudions donc une famille de théories de supergravité demi-maximale avec un groupe de jauge $\mathrm{ISO}(3)\times\mathrm{ISO}(3)$. Dans ce modèle, nous trouvons un réseaux de vides de type AdS_4 préservant $\mathcal{N}=2$ supersymétries avec une augementation de supersymétrie à un point exotique de $\mathcal{N}=4$. La réalisation en théorie des cordes de ce vide, si elle existe, est encore manquante.

Cette thèse est basée sur les articles publiés dans [1, 2, 3, 4, 5, 6, 7].

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Contents

Chapter 1

Introduction

The modern interpretation of the laws of Nature is based on two pillars: quantum mechanics and general relativity. Quantum mechanics, and its generalisation to relativistic theories: quantum field theory (QFT), is the modern framework to understand physics at subatomic scales. A QFT describes how fundamental particles interact and propagate at the quantum level. It is the tool that allowed the physics community to build the standard model of particle, a model describing all known particles and their interactions in the absence of gravity. This has been a massive undertaking spanning several decades whose predictions have been successfully tested up to energies of the order of the TeV. However, it is a descriptive model, with several parameters that must be adjusted to fit with experiments.

A fundamental prediction of QFT concerns the *renormalisation group*. It describes how QFT's do behave differently at different scales. This is what makes it difficult to work with them as a QFT that is well understood in its UV or IR limits might have a completely different behaviour at other scales. The only mathematical tool that reliably produced predictions, for the standard model for example, is perturbative QFT. It consists in taking a well understood QFT, in general a free QFT, and perturbing it with a certain operator, encoding interactions. Then any quantity in the QFT can be expressed as a Taylor expansion around the free QFT in terms of Feynman's diagrams. This method is only valid up to a certain scale, at which the perturbative expansion breaks down. It has thus been a long-standing problem to understand QFT outside of their perturbative regimes.

The other pillar of theoretical physics concerns the largest scales of our universe and gravity. We have a good understanding of the classical description of gravity since the beginning of the 20th century through the theory of general relativity. This theory gives small but measurable corrections to Newtonian gravity by postulating that space and time are not static objects but dynamical ones. In this framework, the "force" of gravity that we experience in everyday life is just a consequence of the curvature of space-time itself. This theory has produced incredible predictions as the existence of black-holes or the big-bang theory, and even real-world applications in the GPS.

The issue that we now face is that those two facets of Nature are fundamentally incompatible. The issue is that the perturbative QFT approach, applied to GR, breaks down. This is because GR is a non-renormalisable theory. This is in part why building a unified theory of Quantum Gravity is so hard. However, it is certain that a unified theory of quantum gravity should encompass both general relativity and quantum field theory. The common lore is to understand both the standard model and general relativity as effective description of a more fundamental quantum theory of gravity emerging at energies of the order of Planck mass $\sim 10^{15}$ TeV.

Currently, the best[1] known candidate for such a fundamental description of our

[1] Some people might disagree, but we will largely ignore them for the purpose of this thesis.

universe is string theory. String theory postulates that fundamental particles are not just points but are extended objects, *strings*, propagating in space-time. If these strings are small enough, their low energy description would be that of point-like particles interacting i.e. a QFT. These strings can be excited along different modes, each of these modes corresponding to a specific particle of a certain charge, mass, spin etc. Amongst the possible excitations of the string, there is a spin-2 excitation which would propagate the gravitational interactions. As such, string theory can describe both quantum field theories and gravity in a unified framework. Moreover, for consistency reasons (anomaly cancellations), string theory depends only on a unique parameter.

Since nothing comes for free in life, string theory also has some drawbacks. First, for a string theory to describe both bosonic and fermionic particles, which are present in the standard model, consistency requires some amounts of *supersymmetry*. This supersymmetry, which is an extension of usual Poincaré invariance, is a symmetry exchanging bosonic and fermionic degrees of freedom. Although it has been used to a great extend to produce results beyond the perturbative regime in QFT, it has not, at the moment, been observed in Nature. This implies that there must be a mechanism for supersymmetry breaking embedded in string theory at low energies. Secondly, supersymmetric string theory, in its simplest regime, predicts the existence of ten space-time dimensions whereas, as we are writing this thesis, we only observe four. We must thus propose a mechanism that removes these extra-dimensions in the low energy limit.

Supergravities

The incompatibility between string theory and our experiment might be resolved by finding a low-energy description of string theory. Such descriptions exist at weak coupling and are called *supergravities*. Supergravities naturally appear when we consider the gaugings of supersymmetry. They are theories invariant under *local* supersymmetry transformations and also describe gravity. There is a large zoo of such theories labelled by their dimension, field content, number of supersymmetry transformations, extra gauge symmetries,... and they will be the main subject of this thesis.

Two of these supergravities have a clear interpretation as low energy limits of superstring theory. They are called Type IIA and Type IIB supergravity. While the first one is non-chiral, the second is and they both appear as the perturbative limit of their corresponding theory of superstrings. They describe a theory of gravity in ten dimensions, invariant under two local supersymmetry transformation (which the maximal amount of supersymmetry possible in ten dimensions). A third maximal supergravity exists in eleven dimensions. This last one offers a description of Type IIA superstrings in the strong coupling limit. We will refer to them here as the *high dimensional* supergravities.

Supergravities, even if they were not related to string theory, are worth studying because they have some very desirable properties. For example, extended supergravities (with more than one supersymmetry transformations) are highly constrained theories. For example, the Type IIB and 11d supergravities depend on only one parameter, the gravitational coupling constant. As such these supergravities could be a path to a grand unified theory, encoding the field content, gauge groups and interactions of the standard model in a unified manner. However, it should be noted that this approach has not met much success so far.

Another very important aspect of supergravities is that they enjoy a better behaviour under perturbative quantum corrections. Whereas pure supergravity already has divergences at the 1-loop order, maximal supergravity in four dimensions has been proven to not have such divergences before the 5-loops corrections [8].

Finally, supergravity is exceptionally useful in the context of the gauge/gravity correspondence [9, 10, 11]. This correspondence relates the strong coupling limit of gauged quantum field theories to solutions of string theory, in its weak coupling limit. This implies that one can infer a lot of information on strongly coupled field theories from solutions of supergravities. Finding new solutions of supergravity of gravity and studying their properties such as masses, moduli spaces, RG-flow,... is of paramount importance to better understand the properties of gauge theories.

Compactification and ExFT

The easiest way to obtain a four dimensional description of the high dimensional supergravities, following the original idea of Kaluza and Klein, is to compactify them on tori. For example, let us consider Type IIB supergravity on a space of the form $M_4 \times T^6$. The four-dimensional manifold M_4 is referred to as the "external" space while the six-torus is referred to as the "internal" space. This splitting of coordinates allows one expand the different fields of Type IIB supergravity in their Fourier modes on the torus. Performing this expansion, one gets a four dimensional theory with infinite towers of fields, called "Kaluza-Klein" modes. To obtain a honest four-dimensional theory, with a finite number of fields, one can truncate (i.e. set to zero) all the higher modes. The result of this procedure is a four-dimensional ungauged $\mathcal{N} = 8$ supergravity where $\mathcal{N}$ refers to the number of independent supersymmetry transformations. This theory describes a subsector of the full Type IIB supergravity on T^6.

Surprisingly, the $\mathcal{N} = 8$ $d = 4$ ungauged maximal supergravity enjoys a global $E_{7(7)}$ symmetry. This observation suggests that it is possible to reexpress the field content and the equations of motion of Type IIB supergravity in a way that is not explicitly invariant under the $SO(1, 9)$ Lorentz group, but under the $E_{7(7)}$ group. This reformulation exists and is called $E_{7(7)}$-Exceptional Field Theory (ExFT). This theory allows one to perform consistent truncations of Type IIB supergravity on more generic spaces (e.g. spheres, hyperbolic spaces,...). Moreover, it also gives us access to the masses of the higher KK modes, which can be hard to compute directly in the Type IIB language. The resulting four-dimensional field theories obtained with this method are *gauged* maximal supergravity.

The goal of this thesis will be to use this four-dimensional description of Type IIB supergravity to produce new solutions and apply these results to the holographic correspondence. More precisely, we are going to study the vacuum solutions of dyonically gauged maximal supergravity in 4d with gauge group $[SO(1,1) \times SO(6)] \ltimes \mathbb{R}^{12}$. The vacua of this theory uplift to solutions of Type IIB supergravity on $\mathrm{AdS}_4 \times S^1 \times S^5$ featuring an S-duality monodromy induced by an hyperbolic element of $SL(2, \mathbb{Z})$ around the S^1.

Plan of the thesis

In chapter 2, we review basic results concerning quantum field theories. First, we focus on four dimensional $\mathcal{N} = 1$ supersymmetric field theories, we review its action, gaugings and supermultiplet structure. We then move on to conformal field theories which are the IR and UV fixed point of RG-flows. We present their possible deformations and review the notion of Zamolodchikov metric on their moduli spaces. Finally, we combine these results to study superconformal field theories with a focus on $\mathcal{N} = 2$ in three-dimensions. We present some known results concerning their operator contents and their exactly marginal deformations.

Chapter 3 is a review of selected supergravities and their gaugings. To introduce the subject, we start by presenting the $\mathcal{N} = 1$ $d = 4$ supergravity and its gaugings. This supergravity illustrates some of the difficulties that comes with the gauging procedure in supergravity. We then describe the construction of extended supergravities. To do so we must first introduce some generic geometric constructions. We then construct the gauged maximal supergravity in four dimensions. This construction is based on the embedding tensor formalism which encodes the gauge coupling constant and structure constant. We also discuss several aspects of four-dimensional electromagnetism duality in this context. We present briefly the main aspects of half-maximal supergravity and how it connect to truncations of maximal supergravity. Finally, we review the bosonic sector of Type IIB supergravity and its equations of motion. This chapter concludes with a presentation of the holographic dictionary in the $\text{AdS}_4/\text{CFT}_3$ context.

In chapter 4, we define the four-dimensional supergravity under investigation in this thesis. This theory is the dyonic $[SO(1,1) \times SO(6)] \ltimes \mathbb{R}^{12}$ gauged maximal supergravity. We study its AdS_4 vacua structure. To do so, we focus on a $\mathbb{Z}_2^3$ invariant subsector of this theory. In this subsector, we find four families of solutions presented in Table 1.1.

SUSY	number of moduli	max. res. sym.	min. res. sym.	stability
0	3	$SO(6)$	$U(1)^3$	unstable
1	2	$SU(3)$	$U(1)^2$	stable
2	1	$SU(2) \times U(1)$	$U(1)^2$	stable
4	0	$SO(4)$	$SO(4)$	stable

Table 1.1: Summary of the solutions found in the $\mathbb{Z}_2^3$ invariant sector of the dyonically gauged $[SO(1,1) \times SO(6)] \ltimes \mathbb{R}^{12}$ maximal supergravity. We present the amount of residual supersymmetry, the dimension of the moduli space, the maximum and minimum number of residual symmetry on the moduli space and the perturbative stability of these solutions.

For each of these families, we compute the mass spectrum of the scalars, the vectors, and the gravitini.

In chapter 5, to relate the vacua we found in four dimensions to solutions of Type IIB supergravity, we review the $E_{7(7)}$-Exceptional Field Theory. This allows us to reformulate Type IIB supergravity in an $E_{7(7)}$ covariant way. Then, we present the generalised Scherk-Schwarz ansatz that truncates the full Type IIB supergravity down to a 4d gauged supergravity. In particular, we review how this ansatz relates our dyonically gauged supergravity to Type IIB compactified on $S^1 \times S^5$. Using this ansatz, we uplift the solutions found in the previous chapter to ten-dimensions. These solutions feature an S-duality monodromy around the S^1. The presence of this monodromy, generated by an hyperbolic element of $SL(2, \mathbb{Z})$, shows that they are part of families of solutions called “S-folds”. S-folds are extremal limits of solutions known as “Janus solutions”. We present the connection between the $\mathcal{N} = 4$ Janus solutions, its CFT_3 duals, and our $\mathcal{N} = 4$ solution.

In chapter 6, we investigate systematically the presence flat direction in the scalar potential around our S-folds. Using the $E_{7(7)}$ duality group of maximal supergravities, we prove that all the S-folds with a continuous residual symmetry group belong to moduli space of solutions. This moduli space has the same dimension as the rank of the residual symmetry group. Not only does this construction allows us to build the AdS counterpart of the moduli spaces of CFT's, it can also be used to break some or all of the residual (super)-symmetries of the seed solution. This would imply the existence of moduli spaces of CFT, even in the absence of supersymmetry.

To expand on this idea, we present an equivalent result for solutions of supergravity with continuous isometries and a S^1 factor in their geometries. We construct explicitly the flat deformations of our S-folds in 10d which allows us to reinterpret these deformations as "mapping tori". Focussing on the $\mathcal{N} = 4$ S-fold, we show that it is a point of supersymmetry enhancement in a 2 dimensional non-supersymmetric moduli space. Moreover, the non-supersymmetric solutions are perturbatively stable at all order in their KK spectrum and have passed several test of non-perturbative stability. This challenges the non-SUSY AdS conjecture.

In chapter 7, we go back to the 4d description of our theory and build, numerically and semi-analytically, RG-flow connecting our S-folds in the IR to anisotropic deformations of the D3-brane solutions, dual to the SYM_4 SCFT. We also present RG-flows between the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ solutions.

In the final chapter of this thesis, we present the first steps to, perhaps, build new S-fold solutions. In particular, we want to build S-folds which do not admit a description in a maximal supergravity theory but admit one in a half maximal supergravity theory. To build such S-folds, we start studying the $\mathbb{Z}_2$ invariant sector of the dyonic $SO(1,1) \times SO(6) \ltimes \mathbb{R}^{12}$ gauged maximal supergravity. This procdure results in a specific $\text{ISO}(3) \times \text{ISO}(3)$ in $\mathcal{N} = 4$ supergravity specified by its associated $\mathcal{N} = 4$ embedding tensor. By deforming the embedding tensor of this supergravity, without changing the gauge group, we show that there exists a web of solutions preserving $\mathcal{N} = 2$ supersymmetry. This web contains a special point of symmetry enhancement to an exotic $\mathcal{N} = 4$ solution. The string theoretic realisation of this solution, if any, is still lacking.

Chapter 2

Superconformal field theories

The d-dimensional quantum field theories we will study are always Poincaré invariant. This means that they are invariant under the action of Lorentz transformations, generated by the rotations and boosts, $M^{[\mu\nu]}$, and the translations, P^μ. These span the algebra $\mathfrak{so}(1,d-1) \ltimes \mathbb{R}^{1,3}$ with the usual commutation relations

$$\begin{aligned} &[P^\mu,\, P^\nu] = 0\,, \qquad [M^{\mu\nu},\, P^\rho] = i\,2\,P^{[\mu}\,\eta^{\nu]\rho}\,, \\ &[M^{\mu\nu},\, M^{\rho\sigma}] = i\,2\,\left(M^{\mu[\sigma}\,\eta^{\rho]\nu} - M^{\nu[\sigma}\,\eta^{\rho]\mu}\right)\,. \end{aligned} \tag{2.1}$$

This algebra can be extended by a global internal symmetry algebra $\mathfrak{g}$. A celebrated theorem by Coleman and Mandula [12] states that in order for the S-matrix of a theory to be non-trivial, the two algebras must commute, i.e. $\mathfrak{g}$ must transform as a scalar. Thus, the total symmetry group must be of the form $\left(\mathfrak{so}(1,d-1) \ltimes \mathbb{R}^{1,3}\right) \times \mathfrak{g}$, notice the direct product between $\mathfrak{g}$ and the factor corresponding to the Poincaré algebra. This means that $\mathfrak{g}$ can only act on internal degrees of freedom. As such, it does not seem possible to get more insight on QFTs by requiring invariance under extended symmetry groups, others than usual gauge and global symmetry groups, while preserving Poincaré invariance. Fortunately, there are two very important exceptions to this theorem: supersymmetric field theory and conformal field theories[1].

In this chapter, we will start by studying *global* $\mathcal{N} = 1$ SUSY theories in four dimensions. To do so, we will study representations of the $\mathcal{N} = 1$ superalgebra and build invariant actions for two multiplets: the chiral multiplet, containing matter fields such as scalars and spin-1/2 fermions, and the vector multiplet, containing vector fields. We will then study the gaugings of global symmetries in SUSY theories to illustrate how non-trivial this procedure is, even in very simple setups. This will be put to good use when studying theories with *local* SUSY invariance: supergravities.

Then, we will present generalities about conformal field theories and their deformations. This will become useful in light of the AdS/CFT correspondence. This correspondence conjectures that vacua in theories of gravity of the form $\mathrm{AdS}_d \times \mathcal{M}_{\mathrm{int}}$ should be dual to conformal field theories in $d-1$ dimensions. This conjecture relates for example masses at a vacuum of the gravity theory to the conformal dimensions of operators in the CFT. This will naturally lead us to the study of exactly marginal deformations which are deformations preserving the conformality of a theory. These deformations are better understood in supersymmetric theories where several theorems constrain the type of allowed exactly marginal deformations. With this in mind, we will focus on the study of deformations of three-dimensional supersymmetric theories which should be dual to vacua of four-dimensional supergravities.

[1] Other exceptions concern higher-form symmetries and non-invertible symmetries, but this is not the subject of this thesis.

2.1 $\mathcal{N} = 1$ supersymmetry in $D = 4$

In the case of supersymmetry, on top of requiring Poincaré invariance, we require invariance under a certain superalgebra exchanging bosonic and fermionic degrees of freedom. The UV behaviour of supersymmetric theories is generally tamer than that of a generic QFT. There even exists some nonrenormalisation theorem [13] stating that some quantities in supersymmetric field theories do not get renormalised. This can naively be understood in perturbative QFT as the fact that for each loop diagram with bosonic particles there will be a corresponding diagram with fermionic particles whose masses and coupling constant compensate that of the bosons.

The supersymmetry superalgebra avoids the Coleman-Mandula theorem because it is not an algebra and is generated, in part, by fermionic generators Q^I_α. Here α is a spinor index in four dimensions and the index $I = 1, \ldots, \mathcal{N}$ is the number of supersymmetries. For simplicity, we will first focus on $\mathcal{N} = 1$ and suppress the $I = 1$ index. The Q_α is a four-component Majorana spinor[2]. These new operators obey *anti-commutation* relations between themselves

$$\left\{Q_\alpha,\, \bar{Q}^\beta\right\} = -\frac{1}{2}(\gamma_\mu)_\alpha{}^\beta P_\mu\,, \tag{2.2}$$

and transform as spinors under the other Poincaré transformations i.e.

$$[Q_\alpha,\, M_{\mu\nu}] = -\frac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta\,. \tag{2.3}$$

The γ are the usual gamma-matrices in four-dimensions while $\bar{Q}$ is This means that we no longer have a Lie-algebra of symmetries but a superalgebra, where the brackets are commuting or anticommuting depending on the bosonic/fermionic nature of its arguments. For a theory preserving this superalgebra, its operator content must be organised in $\mathcal{N} = 1$ supermultiplets (i.e. representations of the superalgebra). For global supersymmetry there are three relevant multiplets that we will discuss: the chiral multiplet, the vector multiplet and the real multiplet. The result presented in this section are based on [14, 15].

2.1.1 The chiral multiplet

The chiral multiplet contains a complex scalar z and a Weyl spinor $P_L\chi$. To this field content we add an auxiliary field F, which is a complex scalar. This auxiliary field allows us to simplify the transformation rules of the chiral multiplet under supersymmetry and to easily write supersymmetric actions for this multiplet. The set of fields $(z,\, P_L\,\chi, F)$ constitutes a chiral multiplet while $(\bar{z}, P_R\,\chi, \bar{F})$ constitutes an anti-chiral multiplet. The supersymmetry transformations of the chiral multiplet are given by

$$\delta z = \frac{1}{\sqrt{2}}\bar{\epsilon}P_L\chi\,, \tag{2.4}$$

$$\delta P_L\chi = \frac{1}{\sqrt{2}}(\partial\!\!\!/ z + F)\epsilon\,, \tag{2.5}$$

$$\delta F = \frac{1}{\sqrt{2}}\bar{\epsilon}\partial\!\!\!/ P_L\chi\,, \tag{2.6}$$

[2]which in four dimensions is equivalent to a Weyl spinor, we will switch back and forth between the two equivalent representations

and the variations of the anti-chiral multiplets are obtained by complex conjugation. The most basic action for the chiral multiplet consists of a kinetic term of the form

$$S_{\text{kin}} = \int \mathrm{d}^4x \left[\partial^\mu \bar{z}\partial_\mu z + \bar{\chi}\partial\!\!\!/ P_L\,\chi + \bar{F}F\right] , \tag{2.7}$$

and an interaction term of the form

$$S_F = \int \mathrm{d}^4x \left[FW'(z) - \frac{1}{2}\bar{\chi}P_L W''\chi\right] , \tag{2.8}$$

for any holomorphic function $W(z)$. The complete action is then

$$S = S_{\text{kin}} + S_F + S_{\bar{F}}\,. \tag{2.9}$$

This action is invariant under the SUSY transformation but can be generalized as we will see in the next sections. Moreover, in this action, the field F can be integrated out, leaving us with a theory of a complex scalar z and its superpartner χ. This integration procedure makes clear that the theory possesses a scalar potential given in terms of the derivatives of $W(z)$. This is the reason we often refer to $W(z)$ as the *superpotential.*

2.1.2 The vector multiplet

The *vector multiplet* contains massless vector gauge fields $A_\mu{}^A$ and their superpartners λ^A, Majorana spinors called the gaugini. Once again, we add auxiliary fields D^A for simplicity, which are pseudo-real scalars. All these fields transform in the adjoint representation of a gauge group G, labelled by the index A (we sometimes suppress this index when not needed). The transformation rules for this multiplet are

$$\begin{aligned}
\delta A_\mu &= -\frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda\,,\\
\delta\lambda &= \left(\frac{1}{4}\gamma^{\mu\nu}F_{\mu\nu} + \frac{1}{2}i\gamma_\star D\right)\epsilon\,,\\
\delta D &= \frac{1}{2}i\bar{\epsilon}\gamma_\star D\!\!\!\!/_\mu\lambda\,.
\end{aligned} \tag{2.10}$$

The simplest SUSY-invariant action for this multiplet, generalizing the Yang-Mills action, can be written as

$$S_{\text{gauge}} = \int \mathrm{d}^4x \left[-\frac{1}{4}F^{\mu\nu\,A}F_{\mu\nu}{}^A - \frac{1}{2}\bar{\lambda}^A\gamma^\mu D_\mu\lambda^A + \frac{1}{2}D^A D^A\right] , \tag{2.11}$$

where

$$F = \mathrm{d}A + A\wedge A\,, \tag{2.12}$$

$$D_\mu\lambda^A = \partial_\mu\lambda^A + f_{BC}{}^A A_\mu{}^B\,\lambda^C\,, \tag{2.13}$$

where $f_{BC}{}^A$ are the structure constants of the gauge algebra $\mathfrak{g}$. Once again, the auxiliary field D can be safely integrated out.

2.1.3 Gaugings with global supersymmetry

At this stage, we have a theory with matter fields in the chiral multiplet, and a theory with vector fields in the vector multiplet. Let us introduce the gauging procedure and its challenges. The usual gauging procedure would be to specify some representation $\mathbf{R}$ in which the chiral multiplet transforms, and to replace partial derivatives by covariant derivatives. However, this simple prescription breaks supersymmetry. To preserve it, one must also modify the SUSY transformation rules and introduce couplings in the supersymmetric action to preserve supersymmetry. In the end, one gets

$$S = S_{\text{gauge}} + S_{\text{kin}} + S_F + S_{\bar{F}} + S_{\text{couplings}} \,, \tag{2.14}$$

where

$$S_{\text{couplings}} = \int \mathrm{d}^4x \left[-\sqrt{2} \left(\bar{\lambda}^A \, \bar{z} \, t_A P_L \chi - \bar{\chi} P_R \, t_A \, z \, \lambda^A \right) + i \, D^A \, \bar{z} \, t_A \, z \right] \tag{2.15}$$

and t_A are the generators of $\mathfrak{g}$ in the representation $\mathbf{R}$. This gauging procedure only requires the superpotential W and the kinetic terms of the chiral multiplet to be invariant under the gauge transformations. The transformation rules for the chiral and anti-chiral multiplets in the presence of gaugings are now

$$\delta z = \frac{1}{\sqrt{2}} \bar{\epsilon} P_L \chi \,, \tag{2.16}$$

$$\delta P_L \chi = \frac{1}{\sqrt{2}} (\not{D} z + F) \epsilon \,, \tag{2.17}$$

$$\delta F = \frac{1}{\sqrt{2}} \bar{\epsilon} P_R \left(\not{D} \chi - \sqrt{2} \lambda^A t_A z \right) \,. \tag{2.18}$$

Notice also that, upon solving for the auxiliary fields D and F, the scalar potential gets two contributions, one from S_F and one from the $(D)^2$ terms in S_{gauge}. In more complicated models, the gauging of symmetries will always have the properties we observe in this very simple model:

- For the action to be SUSY and gauge invariant, one must introduce new couplings in addition to the minimal coupling procedure: $\partial_\mu \to D_\mu$.
- The supersymmetry transformation rules must be corrected in the presence of gaugings.
- The scalar potential gets corrections from the gaugings.

2.1.4 The real multiplet

The last supermultiplet we will consider is the real multiplet. It contains two real scalars C and D, a complex scalar $\mathcal{H}$, two Majorana spinors ζ and λ and a real vector B_μ. Its transformation rules are

$$\begin{aligned}
&\delta C = \frac{1}{2} \, i \, \epsilon \, \gamma_* \, \zeta \,, \\
&\delta P_L \zeta = \frac{1}{2} \, P_L (i \mathcal{H} - \not{B} - i \, \not{\partial} C) \, \epsilon \,, && \delta P_R \zeta = \frac{1}{2} P_R (-i \bar{\mathcal{H}} - \not{B} + i \, \not{\partial} C) \, \epsilon \,, \\
&\delta \mathcal{H} = -i \, \bar{\epsilon} \, P_R (\lambda + \not{\partial} \zeta) \,, && \delta \bar{\mathcal{H}} = i \, \bar{\epsilon} \, P_L (\lambda + \not{\partial} \zeta) \,, \\
&\delta B_\mu = -\frac{1}{2} \, \bar{\epsilon} \, (\gamma_\mu \lambda + \partial_\mu \zeta) \,, && \delta \lambda = \frac{1}{2} \, (\gamma^{\rho\sigma} \, \partial_\rho B_\sigma + i \, \gamma_* \, D) \, \epsilon \,, \\
&\delta D = \frac{1}{2} \, i \, \bar{\epsilon} \, \gamma_* \, \not{\partial} \lambda \,.
\end{aligned} \tag{2.19}$$

It is possible to fix some of these fields to zero while preserving supersymmetry by shifting them in a specific manner using the components of a chiral superfield. This allows us to go to the "Wess-Zumino" gauge in which only three fields remain: a vector A_μ, a gaugino λ, and an auxiliary field D, which is precisely the field content of the vector supermultiplet. We are not going to go into the details of this procedure here. Although, as a fundamental field the real multiplet is not of much use, its transformation rules (2.19) can be used to introduce interactions in $\mathcal{N} = 1$ SUSY theories.

2.1.5 Generic $\mathcal{N} = 1$ SUSY actions

The introduction of the real multiplet allows us to build, in a systematic way, SUSY actions without gaugings. This procedure relies on the properties of the supermultiplet structure and will be very useful when discussing deformations of superconformal field theories in the next chapters. Observe that both δF in (2.4) and δD in (2.19) are total derivatives. This means that the integrals

$$S_F = \int \mathrm{d}^4x\, F \qquad \text{and} \qquad S_D = \int \mathrm{d}^4x\, D\,, \tag{2.20}$$

are invariant under supersymmetry transformations. The two terms F and D are called the "top-components" of their supermultiplets.

Let us now start with any number of elementary chiral supermultiplets $(z^\alpha,\, \chi^\alpha,\, F^\alpha)$ and vector multiplets $(A_\mu^A,\, \lambda^A,\, D^A)$. Generic actions with this field content can be built by considering certain functions as the lowest components of a supermultiplet and acting repeatedly with the SUSY transformations to identify the corresponding top-component. For example, consider a holomorphic function $W(z^\alpha)$ as the lowest component of a chiral multiplet. Its SUSY variation is

$$\delta W(z^\alpha) = \frac{1}{\sqrt{2}}\, \partial_\alpha W\, \bar{\epsilon}\, P_L \chi^\alpha\,. \tag{2.21}$$

Comparing with (2.4), we find that

$$\chi(W) = \partial_\alpha W\, \chi\,. \tag{2.22}$$

Repeating this procedure, we get that

$$F(W) = \partial_\alpha W F^\alpha - \frac{1}{2}\bar{\chi}^\alpha P_L W_{\alpha\beta}\chi^\beta\,. \tag{2.23}$$

Because of the supersymmetry algebra, $\delta F(W)$ will be a total derivative. Moreover, it happens that the definition of S_F in (2.8) is equivalent to the one we provide here with $S_F = \int \mathrm{d}^4x\, F(W)$. Such an interaction term is called an "F-term".

In the same way, starting with a real function $K(z^\alpha,\, \bar{z}^{\bar{\alpha}})$ and considering it as the C-term of a real multiplet we can define the D-term associated to it using the equations (2.19). The integral of $D(K)$, which is SUSY invariant, gives a generalisation of the kinetic term of the chiral multiplet (2.7). Kinetic terms obtained in this way are called "D-terms".

Finally, considering the lowest component of a chiral multiplet as

$$Z(f) = \frac{1}{4} f_{AB} \bar{\lambda}^A P_L \lambda^B\,, \tag{2.24}$$

for a symmetric matrix f_{AB}, the associated $F(f)$ gives a generalisation of (2.11). Although this method allows one to write very general actions for various field contents, the gauging procedure cannot be obtained directly from this method and requires more care since the minimal coupling is not sufficient to preserve supersymmetry.

2.2 Conformal field theories and exactly marginal deformations

In the case of conformal symmetry, we require invariance of the quantum field theory under conformal transformations. These are the transformations that preserve angles, and in particular, they contain rescaling. The conformal group is the extension of the Poincaré group

$$SO(2,d) \supset SO(1,d-1)\,, \tag{2.25}$$

and it is generated by the usual Poincaré generators $M_{[\mu\nu]}$ and P_μ with the addition of the dilatation D and the special conformal transformation K_μ. These generators obey the commutation relations of the Poincaré algebra supplemented by

$$[D,\, K_\mu] = -K_\mu\,, \qquad [D,\, P_\mu] = P_\mu \tag{2.26}$$

$$[K_\mu,\, P_\nu] = -2(\eta_{\mu\nu}D - i\,M_{\mu\nu})\,, \tag{2.27}$$

$$[M_{\mu\nu},\, K_\rho] = i\,2\,\eta_{\rho[\mu}K_{\nu]})\,, \tag{2.28}$$

with all the other commutators being zero.

Amongst quantum field theories, conformal field theories (CFTs) play a distinguished role. For example, CFTs are important in statistical mechanics, where they provide a description near phase transitions where there are no scales involved. Moreover, CFTs are fixed points of the renormalisation-group flow, because of their invariance under the rescaling D, thus introducing a notion of universality. Finally, because of the constraints imposed by conformal invariance, strongly-coupled CFTs can provide an insight into non-perturbative QFTs more generally. Free massless theories are an almost trivial example of CFTs and usually serve as a starting point for perturbative renormalisation, where the free theory is understood as the UV theory. On the other end of the RG flow, in the IR, CFTs are usually thought of as being non-Lagrangian theories.

They are fully characterized by their operator content $\{\mathcal{O}(x)\}$, organized in conformal multiplets (i.e. in representations of $SO(2,\, d)$) and by their n-point correlation functions, from which all other quantities in the CFTs can be computed. These conformal multiplets can be organized as follow: up to translations, they are built from an operator $\mathcal{O}(0)$ called the *conformal primary* which is annihilated by the K_μ. This is the operator of the conformal multiplet with the lowest conformal dimension Δ. This number encodes its properties under rescalings. The conformal primary is also characterized by other quantum numbers $(j_1, \dots)$, encoding its transformation under the Lorentz group in d dimensions. All the other components of the multiplet, the "descendants", are then built by repeatedly acting with P_μ on the conformal primary.[3]

The n-point functions of the operators are also constrained by the conformal symmetry, and they must respect certain consistency relations. For example, the two-points function of two conformal primary operators is always of the form

$$\langle \mathcal{O}_1(0)\, \mathcal{O}_2(x)\rangle = \delta_{\Delta_1\Delta_2}\,\frac{C_{12}}{|x|^{2\Delta_1}}\,. \tag{2.29}$$

[3]It is important to note that in the CFT language we do not make any distinction between "fundamental" fields, e.g. the boson of a chiral multiplet z, and composite fields, like the superpotential $W(z)$.

The constant coefficient C_{12} depends on the CFT and the two operators $\mathcal{O}_1$ and $\mathcal{O}_2$ of conformal dimensions Δ_1 and Δ_2 respectively. In the same way three-point functions only depend on constants C_{abc} and have specific space-time dependencies due to the conformal symmetry. The set of operators, the operators conformal dimensions, and the coefficients C_{ab} and C_{abc} must also satisfy some extra consistency relations due to unitarity or crossing symmetry of the four-point function. This is why building generic CFT is usually a hard problem and there is no recipe to understand what will be the IR fixed point of a given perturbation of a free UV theory.

One way forward, inspired by perturbation theory, is to study general deformations around a CFT. In particular we are going to focus on deformations that consist in adding a scalar operator to the action of a CFT:

$$S_\lambda = S_{CFT} + \lambda \int \mathrm{d}x^d \, \mathcal{O}(x) \,. \tag{2.30}$$

If $\mathcal{O}$ is not a primary operator, this deformation will be the integral of a total derivative and trivial. If $\mathcal{O}$ is a conformal primary, then we have a new theory S' which needs not to be a CFT. The conformal dimensions of $\mathcal{O}$ can teach us a lot about the RG-flow of the new theory S_λ:

- If $\Delta < d$, this deformation is called *relevant*. This means that the coefficient λ will grow in the IR, signalling that the initial CFT will flow in the IR to a new fixed-point where non-perturbative effects will be important.
- If $\Delta > d$, this deformation is called *irrelevant* because, as we flow to the IR, the RG-flow makes the coefficient λ disappear and we recover the initial CFT. This flow can be understood as being the endpoint of the flow from another UV theory down to the initial IR CFT.
- If $\Delta = d$, the operator is called *marginal* i.e. it is not acted upon by the RG-flow. However, it is marginal only in the initial CFT with $\lambda = 0$. In the theory with $\lambda \neq 0$, S_λ, this operator might get quantum corrections and obtain an anomalous dimension making it either marginally relevant or marginally irrelevant. The class of operators we are going to be interested in are called *exactly marginal* and have conformal dimension equal to d even in the theory S_λ with $\lambda \neq 0$.

The presence or not of exactly marginal operators in a CFT tells us whether they are isolated fixed points of the renormalisation-group flow, or if they belong to a family of CFTs, known as a conformal manifold. The conformal manifold is spanned by exactly marginal deformations of the CFT, i.e. marginal operators whose β-functions vanish exactly to all orders. Over the last decades, much insight has been gained into local properties of conformal manifolds of supersymmetric conformal field theories [16, 17, 18, 19, 20]. In particular, it is not uncommon for four-dimensional $\mathcal{N} = 1$ and three-dimensional $\mathcal{N} = 2$ CFTs to possess conformal manifolds, whose dimensions can be deduced from the symmetry of the CFTs, without the need to compute β-functions or even to have a Lagrangian description.

On the other hand, no example is known of a non-supersymmetric conformal field theory in more than two dimensions featuring a conformal manifold. Indeed, they are widely believed not to exist, since it is unclear how the precise cancellations in the β-functions will be achieved without supersymmetry. However, there are no "no-go theorems" that forbid non-supersymmetric conformal manifolds. As a result, the existence of non-supersymmetric conformal manifolds has been largely the subject of speculation, with only few systematic analyses performed recently [21, 22, 23, 24].

If we have several operators $\mathcal{O}_I$ which do not break the CFT property of S_{λ^I} then we say that the λ^I are coordinates on the *conformal manifold*, the manifold parametrising the space of connected CFTs. On this space one can define a metric called the *Zamolodchikov metric* g_{IJ} defined as

$$\langle \mathcal{O}_I \mathcal{O}_J \rangle = \frac{g_{IJ}}{|x|^{2d}} \,. \tag{2.31}$$

The Zamolodchikov metric depends on the coupling constants λ^I which play the role of coordinates while the operators $\mathcal{O}^I$ play the role of tangent vector on the conformal manifold. Moreover, this metric transforms correctly under changes of coordinates $\lambda^{I'} = \lambda^{I'}(\lambda^I)$. Notice that this construction only gives us access to the local description of the conformal manifold and not to its global structure (e.g., we cannot know from these information whether the conformal manifold is a circle or a line).

2.3 Superconformal field theories

In this section, we are going to study theories invariant under both supersymmetry and conformal transformations. This will allow us to infer exact information about the local properties of conformal manifolds. The superconformal algebra is generated by the conformal algebra, the supersymmetry transformations Q^I_α, with $I = 1, \ldots, \mathcal{N}$, and the special supersymmetry transformations S^I_α.[4] In the case of extended supersymmetry, the bosonic algebra also includes a bosonic *R-symmetry* algebra, commuting with $\mathfrak{so}(2, d)$, and acting on the index I. The specific form of these superalgebras will depend on d and $\mathcal{N}$.

A *super*conformal multiplet does not contain only one conformal primary. In fact, the conformal primaries of a given superconformal multiplet fit in a representation of the SUSY superalgebra. Still, a supermultiplet contains a unique operator called the "*super*conformal primary" which is the conformal primary of lowest scaling dimension of the multiplet. This operator is annihilated by both the K_μ and the S^I_α generators. The full conformal multiplet is then built from its descendent under the repeated action of P_μ and Q^I_α.

There is one more important subtlety: some multiplets can shorten if one of the descendents has zero norm. For example, this kind of shortening happens for conserved currents where $\partial_\mu J^\mu = 0$. This allows us to label every superconformal multiplet by the conformal dimension and the representation of its superconformal primary under Lorentz and R-symmetry, with the addition of an extra information signalling whether or not the supermultiplet shortens.

We now have the necessary tools to discuss deformations of superconformal field theories. For a deformation of the type (2.30) to preserve supersymmetry and conformality without being trivial, the deforming operator must obey three conditions:

- *Non-triviality*: it must be a conformal primary.
- *Preserve supersymmetry*: its transformations under the Q's must be a total derivative (i.e. it is the top component of a supermultiplet).
- *Marginality*: it must be of conformal dimension $\Delta = d$.

Only supermultiplets containing such operators and satisfying unitary conditions can produce exactly marginal deformations preserving supersymmetry. This does not

[4]Colloquially speaking, these generators are to the Q^I_α what the K_μ were to the P_μ.

leave a lot of room for such operators. Moreover, supersymmetry can protect the conformal dimension of an operator. For short multiplets, the shortening condition can only happen at specific values of Δ. This means that the conformal dimension of a short multiplet cannot vary on the conformal manifold unless it can combine with other multiplets to form a long multiplet (whose conformal dimensions is not constrained by supersymmetry). When such a recombination is not possible, and the conformal dimension of the deformation is d, the deformation must be *exactly* marginal. Let us illustrate this discussion with the example of $d = 3$ SCFTs.

2.3.1 $d = 3$ $\mathcal{N} = 2$ superconformal multiplets

In three dimensions, the superconformal algebra is $\mathfrak{osp}(\mathcal{N}|4)$ and contains the bosonic sub-algebra $\mathfrak{so}(3,2) \times \mathfrak{so}(\mathcal{N})_R$ ($\mathcal{N} \leq 8$ for non-trivial interacting theories). Following the general strategy we have already sketched, the Table 2.1 lists the possible deformations of theories preserving various amounts of supersymmetry [25, 26]. These deformations are named by analogy with the $d = 4$ case (2.20).

$\mathcal{N}$	Relevant	Marginal	Irrelevant
1	D-term	D-term	$\Delta_{\text{min}} > 3$
2	Flavour current, F-term	F-term	$\Delta_{\text{min}} > 3$
3	Flavour current	—	4
4	Stress Tensor, Flavour current	—	4
$5 \leq \mathcal{N} \leq 6$	Stress Tensor	—	5
8	Stress Tensor	—	6

TABLE 2.1: Supersymmetric deformations in three dimensions

Notice that there are no marginal deformations for $\mathcal{N} > 2$, so let us focus on the case $\mathcal{N} = 2$. The $d = 3$ $\mathcal{N} = 2$ superconformal algebra is $\mathfrak{osp}(2|4)$ and contains an R-symmetry algebra $\mathfrak{u}(1)$. Details concerning this particular superalgebra can be found in [27]. Operators $\mathcal{O}$ are characterized by their conformal dimension Δ, their R-charge (r) and their representation under the Lorentz group $j \in \frac{1}{2}\mathbb{N}$, denoting the representation of dimension $(2j + 1)$. This is compactly denoted $\mathcal{O} \in [j]_{\Delta}^{(r)}$. There are two supersymmetry generators:

$$Q \in [\tfrac{1}{2}]_{\frac{1}{2}}^{(-1)} \quad \text{and} \quad \bar{Q} \in [\tfrac{1}{2}]_{\frac{1}{2}}^{(1)} . \tag{2.32}$$

As a results, there are two possible types of shortening conditions, depending on whether the Q or the $\bar{Q}$ produces a zero-norm operator. For each Q and $\bar{Q}$ there are four shortening conditions listed in Table 2.2.

Putting these two information together, we get the conformal primary and unitarity bounds presented in Table 2.3 for all possible superconformal multiplets in $\mathcal{N} = 2$ $d = 3$. Some of the short multiplets can be understood in the following way:

- The $B_1\bar{B}_1[0]_0^{(0)}$ only contains the identity operator.
- The free chiral multiplets are encoded in $A_2\bar{B}_1[0]_{1/2}^{(1/2)}$ and its complex conjugate $\bar{A}_2B_1[0]_{1/2}^{(-1/2)}$.
- The $A_2\bar{A}_2[0]_1^{(0)}$ supermultiplet is a flavour current multiplet signalling the presence of global symmetries in the theory.

Name	Primary	Unitarity bound
L	$[j]_\Delta^{(r)}$	$\Delta > j - r + 1$
A_1	$[j]_\Delta^{(r)}$ and $j \geq \frac{1}{2}$	$\Delta = j - r + 1$
A_2	$[0]_\Delta^{(r)}$	$\Delta = 1 - r$
B_1	$[0]_\Delta^{(r)}$	$\Delta = -r$
Name	Primary	Unitarity bound
$\bar{L}$	$[j]_\Delta^{(r)}$	$\Delta > j + r - 1$
$\bar{A}_1$	$[j]_\Delta^{(r)}$ and $j \geq \frac{1}{2}$	$\Delta = j + r + 1$
$\bar{A}_2$	$[0]_\Delta^{(r)}$	$\Delta = 1 + r$
$\bar{B}_1$	$[0]_\Delta^{(r)}$	$\Delta = r$

TABLE 2.2: Shortening conditions for Q and $\bar{Q}$ in $d = 3$ $\mathcal{N} = 2$ superconformal theories.

	$\bar{L}$	$\bar{A}_1$	$\bar{A}_2$	$\bar{B}_1$
L	$[j]_\Delta^{(r)}$ $\Delta > j + \lvert r \rvert + 1$	$[j \geq 1/2]_\Delta^{(r>0)}$ $\Delta = j + r + 1$	$[j = 0]_\Delta^{(r>0)}$ $\Delta = r + 1$	$[j = 0]_\Delta^{(r>\frac{1}{2})}$ $\Delta = r$
A_1	$[j \geq 1/2]_\Delta^{(r<0)}$ $\Delta = j - r + 1$	$[j \geq 1/2]_\Delta^{(r=0)}$ $\Delta = j + 1$	–	–
A_2	$[j = 0]_\Delta^{(r<0)}$ $\Delta = 1 - r$	–	$[j = 0]_\Delta^{(r=0)}$ $\Delta = 1$	$[j = 0]_\Delta^{(r=1/2)}$ $\Delta = \frac{1}{2}$
B_1	$[j = 0]_\Delta^{(r<-1/2)}$ $\Delta = -r$	–	$[j = 0]_\Delta^{(r=-1/2)}$ $\Delta = \frac{1}{2}$	$[j = 0]_\Delta^{(r=0)}$ $\Delta = 0$

TABLE 2.3: Supermultiplet of $d = 3$ $\mathcal{N} = 2$ superconformal theories.

- The $A_1\bar{A}_1[1/2]_{3/2}^{(0)}$ is an extra SUSY-current, signalling the presence of supersymmetry enhancement in the theory.
- The $A_1\bar{A}_1[1]_2^{(0)}$ is the stress-tensor multiplet.

The possible deformations of $d = 3$ $\mathcal{N} = 2$ superconformal field theories presented in Table 2.1 can be read from this supermultiplet structure:

- The relevant deformations are contained in the flavour current multiplets $A_2\bar{A}_2[0]_1^{(0)}$. This multiplet contains a scalar of conformal dimension 2 which does not break supersymmetry. It is the $Q\bar{Q}$ descendent of the superconformal primary.
- The chiral multiplets $B_1\bar{L}$ and $L\bar{B}_1$, which are related by complex conjugation, each contains a deformation equivalent to the F-term in $\mathcal{N} = 1$ $d = 4$. The deforming operator is the Q^2 or $\bar{Q}^2$ descendent of the conformal primary and has conformal dimension $\Delta = 3 \pm r > \frac{3}{2}$. As such, this multiplet can contain irrelevant, relevant or marginal operators.
- Finally, the $L\bar{L}$ supermultiplet contains a supersymmetric deformation of conformal dimension $\Delta > 3$ which is irrelevant. This deformation is generated by the operator which is the top component of the supermultiplet of conformal primaries. Moreover, the conformal dimension of this operator is not fixed by supersymmetry, it is expected to vary along the RG-flow or when moving along the conformal manifold.

For the deformation in the $L\bar{B}_1[0]_2^{(2)}$ supermultiplet, the conformal dimension is fixed by superconformality to be exactly $\Delta = 3$. Naively, it would then be expected that the marginal deformation coming from that multiplet is exactly marginal. However, this multiplet can pair up with a flavour current multiplet $A_2\bar{A}_2[0]_1^{(0)}$ to make a long multiplet $L\bar{L}[0]^{(0)}$, which is not protected. The correct statement is that if the marginal deformation does not break any global symmetry, then it is exactly marginal. Indeed, in that case, the flavour current must stay conserved and cannot pair up with the marginal deformation to produce a long multiplet. This is the manifestation in three dimensions of an equivalent result for $d = 4$ $\mathcal{N} = 1$ theories (see. [18]).

This concludes our introduction to superconformal field theories in three dimensions. In the next chapter we will see, using the AdS/CFT correspondence, how to compute the multiplet structure of certain CFT_3's as well as their conformal manifolds and its associated Zamolodchikov metric.

Chapter 3

Supergravity

Supergravity theories are theories at the junctions of two fundamental ideas of mathematical physics: supersymmetry and gauge symmetries. As such, supergravities are the theories invariant under *local* supersymmetry. To gauge the supersymmetries, one must introduce gravitini: fermionic fields carrying a vector index and denoted ψ_μ. The super-partner of these gravitini is a spin-2 particles describing gravity, hence the name "supergravity". There is a whole zoo of supergravities up to eleven dimensions with different amounts of supersymmetries. When these theories present global symmetries, they can be gauged, modulo some constraints we will explore in this section. As such, a specific theory of supergravity is often labelled by its dimension, the number of supersymmetries $\mathcal{N}$ and its gauge group G. When several different matter contents are possible, we also use them to label the supergravity.

In four dimensions, we can build supergravities with an arbitrary number of gravitini $\mathcal{N} \leq 8$ each being the gauge field of a local supersymmetry transformation. In general, it is hard to build invariant actions for extended supergravities because the allowed couplings between the different fields of the theory are highly constrained. In particular, it is only possible to build a minimal model for supergravity with just a graviton and a gravitino in $\mathcal{N} = 1$ supergravity. This allows us to use techniques from SUSY field theory to add matter fields and gauge multiplets and correct the transformations rules and the action by factors proportional to the gravitational coupling constant. In the same spirit, the gauging of symmetries in $\mathcal{N} = 1$ supergravity can be done by correcting the action and the SUSY transformations order by order in the gauge coupling constant. Once again, this is only possible because the gravity, vectors and matter fields appear in different supermultiplets. For extended supergravities in four dimensions it is usually more challenging to build such supergravities and they require more advanced geometrical tools to be build.

Concerning maximal supergravities in four dimensions, the ungauged version was first built by Cremmer and Julia in 1978 [28]. This theory was obtained by compactifying eleven-dimensional supergravity on a torus. Surprisingly, when performing the dualisation between p-form and $(2-p)$-forms in four dimensions, an $E_{7(7)}$ global symmetry group appears. This hidden symmetry groups was shown to appear in [29]. Following the first ungauged supergravity theory, the $SO(8)$-gauged maximal supergravity was obtained by the compactification of 11d supergravity on a 7-sphere in [30]. The construction of these supergravities relied on building compacification of higher-dimensional supergravity. The systematic study of maximal supergravities became possible only after the introduction of the embedding tensor formalism which allowed to build the most generic gauged supergravity possible without referring to an uplift [31]. Here, we will present the modern construction of maximal supergravity.

In this chapter we will review the structure of classical theories of supergravity. As an introduction, we will start with $\mathcal{N} = 1$ supergravity in four dimensions and its gauging. We will then review some facts about coset geometry and electromagnetic duality that will prove useful to describe extended supergravities in four dimensions. We will spend some time reviewing the structure of maximal supergravity in four dimensions. Finally, we will provide a quick review of ten-dimensional bosonic sector of Type IIB supergravity and its equations of motion.

There are no original results presented in this section. The reference used for four-dimensional $\mathcal{N} = 1$ supergravity is [14] (see also the more recent work [32]). Regarding extended supergravities and their gaugings in four dimensions, see the excellent review [33]. For the duality-covariant formulation of maximal supergravity in four dimensions see [34].

3.1 The $\mathcal{N} = 1$ supergravity in four dimensions

3.1.1 The gravity sector

Pure $\mathcal{N} = 1$ supergravity describes a multiplet comprising the graviton, a spin 2 particle encoded in the vierbein, and its supersymmetric partner, a spin 3/2 field ψ_μ called the gravitino. The gravitino plays the role of gauge connection for the supersymmetry transformations. The transformation rules for the vierbein and the gravitino are

$$\delta e_\mu{}^a = \frac{1}{2}\,\bar{\epsilon}\gamma^a\psi_\mu \tag{3.1}$$

$$\delta\psi_\mu = D_\mu\epsilon = \partial_\mu\epsilon + \frac{1}{4}\,\omega_{\mu ab}\gamma^{ab}\,\epsilon \tag{3.2}$$

where the $\omega_{\mu ab}$ is the spin connection. The simplest action containing gravity that is invariant under this transformation, called the “universal part of supergravity” is

$$S = S_2 + S_{3/2} \tag{3.3}$$

$$S_2 = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\, e\, R(\omega) \qquad S_{3/2} = -\frac{1}{2\kappa^2}\int \mathrm{d}^4x\, e\,\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho \tag{3.4}$$

The action S_2 is the Einstein-Hilbert action while the action $S_{3/2}$ is the Rarita-Schwinger action. The pure supergravity action S is invariant under local supersymmetry transformations in four dimensions (although a proof for the exact cancellation, including four-fermions terms is a bit lengthy). The algebra generated by the gauge transformations 3.2 includes both diffeomorphisms and local lorentz transformation.

3.1.2 Adding matter

Now that we have an action for pure supergravity, we want to couple matter fields to this theory. The matter field content fit into a representation under the supersymmetry algebra. There are two relevant supersymmetric multiplets here: the chiral multiplet and the real multiplet. We start by reviewing the action for the chiral multiplet with $\mathcal{N} = 1$ supersymmetry.

The chiral multiplets are made of a complex scalars z_α and their supersymmetric partners, χ_α, which are spin-1/2 fields. The α is an index parametrising the different chiral multiplets. Supersymmetry imposes the complex scalars to parametrize a Kähler manifold $\mathcal{M}_K$, i.e. a complex manifold with extra structure given by an hermitian metric $K_{\alpha\bar{\beta}}$ and a compatible closed symplectic form ω. Locally, on $\mathcal{M}_K$, they can both be described by a Kähler potential $K(z,\bar{z})$ which is a real-valued function on

$\mathcal{M}_K$. In terms of the Kähler potential, the hermitian metric on $\mathcal{M}_K$ is

$$\mathrm{d}s^2 = (\partial_\alpha \partial_{\bar\beta} K)\, dz^\alpha\, d\bar z^{\bar\beta} \;=\; K_{\alpha\bar\beta}\, dz^\alpha\, d\bar z^{\bar\beta}. \tag{3.5}$$

and the closed symplectic form is

$$\omega = -iK_{\alpha\bar\beta}\, dz^\alpha \wedge d\bar z^{\bar\beta}\,. \tag{3.6}$$

Both are invariant under Kähler transformations

$$K \to K + f(z) + \bar f(\bar z) \tag{3.7}$$

specified by a holomorphic function f. Since the Kähler potential is only defined locally on $\mathcal{M}_K$, its expression on different coordinate patches are related by such Kähler transformations.

The kinetic terms of the scalars are entirely determined by the Kähler potential and read

$$\mathcal{L}_{kin} = -K_{\alpha\bar\beta}\, \partial_\mu z^\alpha \partial^\mu \bar z^{\bar\beta}\,. \tag{3.8}$$

Scalar fields can interact via a scalar potential compatible with SUSY. To specify it, one needs to specify a holomorphic function $W(z)$ called the *superpotential*. This superpotential is a scalar under holomorphic coordinate transformations in the target space and transforms under Kähler transformations as

$$W(z) \to e^{-f(z)}\, W(z)\,. \tag{3.9}$$

From this superpotential, supersymmetry allows us to build a scalar potential of the form

$$V = e^{\kappa^2 K} \left(K^{\alpha\bar\beta} D_\alpha W \bar D_{\bar\beta} \bar W - 3\,\kappa^2\, W \bar W \right) \tag{3.10}$$

where $D_\alpha = \partial_\alpha W + (\partial_\alpha K)\, W$ is the Kähler covariant derivative of W and $K^{\alpha\bar\beta}$ is the inverse of the Kähler metric. This potential is a scalar under both reparametrization of $\mathcal{M}_K$ and Kähler transformations. Moreover, it is strikingly different than the scalar potential obtained from rigid supersymmetry. This scalar potential is not positive-definite as it contains a negative piece proportional to κ^2. The κ^2 coefficient translates the fact that the negative definite term appears to compensate for the supersymmetric variations of the gravitino. Of course, in the limit where gravity decouples ($\kappa \to 0$), we recover the scalar potential of global supersymmetry.

We can already present the quadratic terms of the action for an ungauged supergravity theory with chiral multiplets:

$$\mathcal{L}_{\text{chiral}} = -K_{\alpha\bar\beta} \left(\partial_\mu z^\alpha \partial^\mu z^{\bar\beta} + \tfrac{1}{2} \bar\chi^\alpha \not{D}^{(0)} \chi^{\bar\beta} + \tfrac{1}{2} \bar\chi^{\bar\beta} \not{D}^{(0)} \chi^\alpha \right) - V + \cdots \tag{3.11}$$

where we have omitted for the moment mass terms, mixed terms and four-fermion terms. In the next subsection, we will review in detail the action and transformation rules of $\mathcal{N}=1$ supergravity in four dimensions with chiral multiplets, real multiplets, and gaugings. However, this requires an understanding of the symmetries of a Kähler manifold.

Symmetries are diffeomorphisms that preserve the metric on the target space. Since we are working with *complex* Kähler manifolds, we are only going to look at symmetries preserving the complex structure of our manifold. Infinitesimally, these symmetries are generated by holomorphic Killing vectors

$$\delta_k z^\alpha = k^\alpha(z)\,. \tag{3.12}$$

These Killing vectors must preserve the metric. Equivalently, they must also preserve the symplectic form. This implies that:

$$\mathcal{L}_k\, K_{\alpha\bar\beta} = 0 \qquad , \qquad \mathcal{L}_k\, \omega = 0\,. \tag{3.13}$$

Using the fact that ω is closed, for each infinitesimal transformation, there exists a *moment map* $\mathcal{P}$, which is a real function on $\mathcal{M}_K$ satisfying

$$i_k\, \omega = -2\, \mathrm{d}\mathcal{P}\,. \tag{3.14}$$

Notice that these moment maps are only defined up to a constant. The Killing condition on k can be re-expressed in terms of the moment maps as

$$\nabla_\alpha \partial_\beta \mathcal{P}(z,\, \bar z) = 0 \qquad \text{and} \qquad k^\alpha = -i\, K^{\alpha\bar\beta} \partial_{\bar\beta}\mathcal{P}\,, \tag{3.15}$$

where the moment maps determine the killing vectors. It is possible to invert this relationship using the Kähler potential leading to

$$\mathcal{P} = i\,(k^\alpha \partial_\alpha K(z,\, \bar z) - r(z))\ , \tag{3.16}$$

where we assumed that the Kähler potential transforms as

$$\delta_k K = \, k \cdot K(z,\, \bar z) = (r_k(z) + \bar{r_k}(\bar z))\ . \tag{3.17}$$

Indeed, although the metric is invariant under the infinitesimal transformations, the Kähler potential can change by a Kähler transformation.

If there are several Killing vectors on $\mathcal{M}_K$, we can endow them with a Lie algebra structure by using the Lie bracket. Upon choosing a basis of Killing vectors k_A for this algebra, we can compute its structure constants ${f_{AB}}^C$ as

$$[k_A,\, k_B] = {f_{AB}}^C k_C\,. \tag{3.18}$$

They can be used to fix the additive constant of the moments map $\mathcal{P}_A$ by requiring that

$$k_A \mathcal{P}_B = {f_{AB}}^C \mathcal{P}_C \tag{3.19}$$

for all simple non-abelian subgroup of the symmetry groups.

Up to this point, we have only considered the terms containing only the Kähler metric. If there is a scalar potential in the theory, parametrized by a superpotential W, we must ensure that such a superpotential transforms appropriately under (3.12) to yield an invariant scalar potential. Since the Kähler potential transforms as in (3.17), the superpotential must transform as

$$k \cdot W = -r_k\, W\,. \tag{3.20}$$

In conclusion, infinitesimal transformations of the chiral supergravity Lagrangian are generated by conformal Killing vectors satisfying (3.13) and (3.20). They are dual to moment maps $\mathcal{P}$ defined as (3.16). We have now all the tools to add real supermultiplets to our action and study its gaugings.

3.1.3 Adding vector multiplets and gaugings

The $\mathcal{N}=1$ vector multiplet contains a real vector field A_μ^A and its supersymmetric partner λ^A, the gaugino. The index A enumerates the real multiplets. We are going to use the vector fields A^A to gauge possible symmetries of the Kähler manifold spanned by the chiral scalars introduced in the previous subsection. To do so, we must specify two things: a basis of vectors k_A generating the symmetry group G of transformations of the chiral sector discussed above, and a holomorphic symmetric matrix $f_{AB}(z^\alpha)$ transforming in the symmetric representation of G i.e.

$$\delta_{(a^C k_C)} f_{AB}(z) =: a^C k_C \cdot f_{AB}(z) \hat{=} 2a^C f_{C(A}{}^D f_{B)D}(z)\,. \tag{3.21}$$

This matrix will generalise the usual metric for the kinetic term of Yang-Mills theory $\propto f_{AB} F^A \wedge \star F^B$ where $f_{AB} = \delta_{AB}$.

We start by writing the full $D=4$ $\mathcal{N}=1$ gauged supergravity action before making explicit all our definitions. The Lagrangian is

$$\begin{aligned}
\mathcal{L}e^{-1} = \; & \frac{1}{2\kappa^2}\left(R(e) - \bar\psi_\mu R^\mu\right) \\
& - K_{\alpha\bar\beta}\left[\hat\partial_\mu z^\alpha \hat\partial^\mu \bar z^{\bar\beta} + \tfrac12 \bar\chi^\alpha \not{D}^{(0)} \chi^{\bar\beta} + \tfrac12 \bar\chi^{\bar\beta} \not{D}^{(0)}\chi^\alpha\right] - V \\
& + (\mathrm{Re} f_{AB})\left[-\tfrac14 F_{\mu\nu}^A F^{\mu\nu\,B} - \tfrac12 \bar\lambda^A \not{D}^{(0)} \lambda^B\right] \\
& + \tfrac14 i\left[(\mathrm{Im} f_{AB})\, F_{\mu\nu}^A \tilde F^{\mu\nu\,B} + \left(\hat\partial_\mu\, \mathrm{Im} f_{AB}\right) \bar\lambda^A \gamma_* \gamma^\mu \lambda^B\right] \\
& + \tfrac18 (\mathrm{Re} f_{AB})\, \bar\psi_\mu \gamma^{ab}\left(F_{ab}^A + \hat F_{ab}^A\right)\gamma^\mu \lambda^B \\
& + \left[\frac{1}{\sqrt2} K_{\alpha\bar\beta}\, \bar\psi_\mu\, \hat{\not\partial} \bar z^{\bar\beta}\, \gamma^\mu \chi^\alpha + \text{h.c.}\right] \\
& + \left[\frac{1}{4\sqrt2} f_{AB\,\alpha}\, \bar\lambda^A\, \gamma^{ab} \hat F_{ab}^B\, \chi^\alpha + \text{h.c.}\right] + \mathcal{L}_m + \mathcal{L}_{mix} + \mathcal{L}_{4f}\,.
\end{aligned} \tag{3.22}$$

We have used the compact notation

$$R^\mu = \gamma^{\mu\rho\sigma}\left(\partial_\rho + \frac14 \omega_\rho{}^{ab}(e)\gamma_{ab} - \frac32 i\mathcal{A}_\rho\gamma_*\right)\psi_\sigma\,. \tag{3.23}$$

The derivative

$$\hat\partial_\mu z^\alpha = \partial_\mu z^\alpha - A_\mu^A k_A^\alpha \tag{3.24}$$

is the gauge covariant derivative of the scalar fields. The equivalent for their supersymmetric partner is

$$D_\mu^{(0)}\chi^\alpha = \left(\partial_\mu + \frac14 \omega_\mu{}^{ab}(e)\gamma_{ab} + \frac32 i\mathcal{A}_\mu\right)\chi^\alpha - A_\mu^A \frac{\partial k_A^\alpha(z)}{\partial z^\beta}\chi^\beta + \Gamma^\alpha_{\beta\gamma}\chi^\gamma\,\hat\partial_\mu z^\beta\,. \tag{3.25}$$

The Christoffel symbol $\Gamma^\alpha_{\beta\gamma}$ on a Kähler manifold $\mathcal{M}_K$ simplify to

$$\Gamma^\alpha_{\beta\gamma} = K^{\alpha\bar\delta}\partial_\beta K_{\gamma\bar\delta}\,, \tag{3.26}$$

and the Kähler connection $\mathcal{A}_\mu$ is explicitly

$$\mathcal{A}_\mu = \frac16 i\kappa^2\left[\hat\partial_\mu z^\alpha \partial_\alpha K - \hat\partial_\mu \bar z^{\bar\alpha}\partial_{\bar\alpha} K + A_\mu^A\left(r_A - \bar r_A\right)\right]\,. \tag{3.27}$$

The scalar potential V is an extension of the scalar potential in the ungauged model where we must add an extra term to compensate for the gauge transformations:

$$V = V_- + V_+ \tag{3.28}$$
$$V_- = -3\kappa^2 e^{\kappa^2 K} W\bar{W} \tag{3.29}$$
$$V_+ = e^{\kappa^2 K}\nabla_\alpha W K^{\alpha\bar{\beta}}\bar{\nabla}_{\bar{\beta}}\bar{W} + \frac{1}{2}\left(\mathrm{Re}f\right)^{-1\,AB}\mathcal{P}_A\mathcal{P}_B\,. \tag{3.30}$$

We have separated the positive and the negative definite part of the scalar potential in V_+ and V_- respectively and we recall that

$$D_\alpha W = \partial_\alpha W + \kappa^2(\partial_\alpha K)W\,. \tag{3.31}$$

Observe that we recover scalar potential induced by the F and D terms of $\mathcal{N}=1$ SUSY in the limit where $\kappa \to 0$. Finally, the kinetic terms of the real multiplets are written in terms of the quantities

$$D^{(0)}_\mu\lambda^A = \left(\partial_\mu + \tfrac{1}{4}\,\omega_\mu{}^{ab}\gamma_{ab} - \tfrac{3}{2}\,i\,\mathcal{A}_\mu\gamma_*\right)\lambda^A - A^C_\mu\lambda^B f_{BC}{}^A \tag{3.32}$$

and the supercovariant gauge curvature $\hat{F}$ reads

$$\hat{F}_{ab}{}^A = e^\mu_a e^\nu_b\left(2\partial_{[\mu}A^A_{\nu]} + f_{BC}{}^A A^B_\mu A^C_\nu + \bar{\psi}_{[\mu}\gamma_{\nu]}\lambda^A\right)\,. \tag{3.33}$$

The Hodge dual of the field strength is $\tilde{F}^A = \star F^A$.

The last three terms of the Lagrangian, $\mathcal{L}_m + \mathcal{L}_{mix} + \mathcal{L}_{4f}$, correspond, respectively, to a mass term for the fermions, a mixing term between the gravitino and the other spin-1/2 fermions and a 4 fermions interaction term. Since we will mostly be working only with the bosonic part of the Lagrangian, we only present the mass terms:

$$\mathcal{L}_m = \frac{1}{2}m_{3/2}\,\bar{\psi}_\mu\,P_R\,\gamma^{\mu\nu}\psi_\nu \tag{3.34}$$
$$-\tfrac{1}{2}m_{\alpha\beta}\,\bar{\chi}^\alpha\chi^\beta - m_{\alpha A}\,\bar{\chi}^\alpha\lambda^A - \tfrac{1}{2}m_{AB}\,\bar{\lambda}^A P_L\lambda^B + \text{ h.c.} \tag{3.35}$$

The gravitino and spin-1/2 mass matrices are

$$m_{3/2} = \kappa^2 e^{\kappa^2 K/2}\,W \tag{3.36}$$
$$m_{\alpha\beta} = e^{\kappa^2 K/2}\,D_\alpha D_\beta\,W \tag{3.37}$$
$$m_{AB} = -\tfrac{1}{2}e^{\kappa^2 K/2}\,f_{AB\,\alpha}\,K^{\alpha\bar{\beta}}\,\bar{D}_{\bar{\beta}}\bar{W} \tag{3.38}$$
$$m_{\alpha A} = i\sqrt{2}\left[\partial_\alpha\mathcal{P}_A - \tfrac{1}{4}f_{AB\,\alpha}\left(\mathrm{Re}f\right)^{-1\,BC}\mathcal{P}_C\right] = m_{A\alpha}\,. \tag{3.39}$$

As we will see, computing the mass of the gravitino will be useful to check whether or not a given vacua of the theory breaks the supersymmetry.

The gauge transformations, generated by θ^A, and the local supersymmetry transformations, generated by ϵ, act on the fields as:

$$
\begin{aligned}
\delta e_\mu^a =& \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu \\
\delta P_L\psi_\mu =& \left(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}(e)\gamma_{ab} - \frac{3}{2}i\mathcal{A}_\mu\right)P_L\epsilon + \frac{1}{2}\kappa^2\gamma_\mu e^{\kappa^2K/2}WP_R\epsilon \\
& \frac{1}{4}\kappa^2 P_L\psi_\mu\theta^A(\bar{r}_A - r_A) + \text{ cubic in fermions} \\
\delta z^\alpha =& \frac{1}{\sqrt{2}}\bar{\epsilon}\chi^\alpha + \theta^A k_A^\alpha \\
\delta\chi^\alpha =& \frac{1}{\sqrt{2}}P_L\left(\hat{\not{\partial}}z^\alpha - e^{\kappa^2K/2}g^{\alpha\bar{\beta}}\bar{\nabla}_{\bar{\beta}}\bar{W}\right)\epsilon \\
& + \theta^A\left(\frac{\partial k_A^\alpha}{\partial z^\beta}\chi^\beta + \frac{1}{4}\kappa^2(r_A - \bar{r}_A)\chi^\alpha\right) + \text{cubic in fermions} \\
\delta A_\mu^A =& -\frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^A + \partial_\mu\theta^A + \theta^C A_\mu^B f_{BC}{}^A \\
\delta\lambda^A =& \left(\frac{1}{4}\gamma^{\mu\nu}F_{\mu\nu}^A + \frac{1}{2}i\gamma_*(Re\, f)^{-1\,AB}\mathcal{P}_B\right)\epsilon \\
& + \theta^B\left(\lambda^C f_{CB}{}^A + \frac{1}{4}\kappa^2\gamma_*(\bar{r}_B - r_B)\lambda^A\right) + \text{cubic in fermions}
\end{aligned}
\tag{3.40}
$$

These transformation rules reduce to the rigid supersymmetric case in the limit where $\kappa \to 0$.

3.1.4 Supersymmetric vacua

The study of the vacua of $\mathcal{N}=1$ supergravity is a very rich subject and, provided the theory can be embedded in string theory, it has many important applications in the AdS/CFT context. To study vacua of the $\mathcal{N}=1$ theory, i.e. Lorentz-invariant solutions, we start by truncating all degrees of freedom with the exception of the metric and the scalar fields. The action then reduces to[1]

$$
\mathcal{L}e^{-1} = \tfrac{1}{2}R - K_{\alpha\bar{\beta}}\partial_\mu z^\alpha\partial^\mu z^{\bar{\beta}} - V\,. \tag{3.41}
$$

One can study minima of the scalar potential and the metric will be that of dS_4, $M^{1,3}$, or AdS_4, depending on the sign of V at the minima: V_0. The dS or AdS radius is computed from the scalar potential at the minima and reads $L^2 = \pm 3/V_0$.

Concerning residual supersymmetry at the vacua, one must turn to the supersymmetry transformation in (3.40). For Lorentz-invariant configurations, the bosonic fields are invariant under supersymmetric transformations because fermions have been turned off. Conversely, since the fermions vanish at the vacua, the corresponding supersymmetry transformations must also vanish at a SUSY vacuum. This imposes some constraints on the possible supersymmetric solutions.

For the gaugini, $\delta_{\text{SUSY}}\lambda^A \propto \mathcal{P}^B\epsilon$. Since $\delta_{\text{SUSY}}\lambda^A$ vanishes, we have that at a supersymmetric vacuum, $\mathcal{P}=0$. For the chiralini, $\delta\chi^\alpha \propto K^{\alpha\bar{\beta}}\bar{D}_{\bar{\beta}}\bar{W}$. This implies that, at a supersymmetric vacua, $D_\alpha W = 0$. The transformation of the gravitino is a bit more subtle as it gets a contribution from the curvature ${\omega_\mu}^{ab}$ and from $e^{K/2}W$ that can compensate for each other. It happens that for the maximally symmetric solution we consider, the supersymmetry is preserved exactly when the renormalized mass of the gravitino is $m_{3/2}L = 1$. Thus, supersymmetric vacua are the points in $\mathcal{M}_K$ such

[1]In this section, and most of the thesis, we work in the natural units where $\kappa = 1$.

that $\mathcal{P} = 0$ and $D_\alpha W = 0$. At such points, the scalar potential is necessarily negative or null which means that de Sitter supersymmetric vacua do not exist in a theory of supergravity.

When an embedding of the $\mathcal{N} = 1$ theory in string theory exists, the holographic dictionary relates the masses of the different excitations with the conformal dimensions of operators of the dual CFT_3. In particular, the gravitino of normalized mass equals to 1 is dual to a conserved supersymmetry current in the dual CFT.

3.2 The geometry of extended supergravities

We can extend the previous $\mathcal{N} = 1$ model by requiring invariance under more than just one supersymmetry transformation. As we add supercharges, the supermultiplets get bigger. To keep consistency, the supermultiplets should not contain particles of spin greater that 2. This restricts the number of supersymmetries in four dimensions to $\mathcal{N} \leq 8$, equivalently to at most 32 supercharges in any dimensions. The more supersymmetry we impose, the more constrained the theory becomes. In analogy with the $\mathcal{N} = 1$ case, these constraints will materialise as constraints on the field content of the theory, the allowed scalar manifold and restrictions on the possible interaction terms in the theory.

However, all the extended supergravities share some common properties. Their bosonic sector contains a graviton ${e_\mu}^a$ (equivalently a metric $g_{\mu\nu}$). It can also contain a number n_v of vector fields A_μ^Λ and a number n_s of scalar fields ϕ_s. These scalars will parametrise a non-linear σ-model from which we can read the scalar kinetic terms. The vectors kinetic terms can, in all generality, couple to the scalar in the same spirit as in $\mathcal{N} = 1$ supergravity. The most general form of ungauged supergravity will always have an action with kinetic terms of the from

$$\frac{1}{e}\mathcal{L} = \frac{R}{2} - \frac{1}{2}\mathcal{G}_{st}(\phi)\partial_\mu\phi^s\partial^\mu\phi^t + \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}(\phi)F^\Lambda_{\mu\nu}F^\Sigma_{\mu\nu} + \frac{1}{8e}\mathcal{R}_{\Lambda\Sigma}(\phi)\epsilon^{\mu\nu\rho\sigma}F^\Lambda_{\mu\nu}F^\Sigma_{\rho\sigma}\,, \tag{3.42}$$

where R is the Ricci scalar of the metric, $\mathcal{G}_{st}$ is a metric on the scalar manifold, $F^\Lambda_{\mu\nu}$ is the field strength of A_μ^Λ and the matrices $\mathcal{I}$ and $\mathcal{R}$ are symmetric and $\mathcal{I}$ is negative definite. Once again, the specifics will depend on the number of supercharges, the number of vector fields and the number of scalar fields. For $\mathcal{N} = 2$, the scalars will parametrize a direct product of a special Kähler manifold and a quaternionic Kähler manifold. When $\mathcal{N} > 2$, the scalars will parametrize homogeneous symmetric spaces.

In this section, and in preparation for the study of maximal gauged supergravity, we will study homogeneous symmetric spaces. After that, we will consider the vector sector and the electro-magnetic duality, which is very important for theories in four dimensions. Finally, we will study the global symmetry of the action (3.42).

3.2.1 Homogeneous symmetric space

A homogeneous spaces $\mathcal{M}$ is a manifold that admits a transitive action of a group G. The stabiliser of a point p is called the isotropy group $H_p \subset G$. Due to the transitive property of the G-action on $\mathcal{M}$, this group does not depend on p and allows us to write the scalar manifold as the quotient space $\mathcal{M} = G/H$. In the context of $\mathcal{N} > 2$ supergravity, we can assume that G is a semi-simple group while H is its maximal compact subgroup. We define the Cartan-Killing metric on $\mathfrak{g}$ as

$$g(t^\alpha, t^\beta) = \text{Tr}(t^\alpha t^\beta) \qquad \forall\, t^\alpha, t^\beta \in \mathfrak{g} \tag{3.43}$$

which induces a pseudo-metric on G. This metric is non-degenerate because G is semi-simple. This metric allows us to split $\mathfrak{g}$ into two orthogonal pieces $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{K}$ where $\mathfrak{h}$ is the Lie algebra generating H and $\mathfrak{K}$ is its orthogonal complement. For symmetric reducible spaces we have the property that $\mathfrak{h}$ and $\mathfrak{K}$ satisfy the conditions

$$\underbrace{[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}}_{\mathfrak{h} \text{ is a subalgebra of } \mathfrak{g}} \quad , \quad \underbrace{[\mathfrak{h}, \mathfrak{K}] \subset \mathfrak{K}}_{\text{the pair } (\mathfrak{h}, \mathfrak{K}) \text{ is reductive}} \quad \text{and} \quad \underbrace{[\mathfrak{K}, \mathfrak{K}] \subset \mathfrak{h}}_{G/H \text{ is symmetric}} \quad . \tag{3.44}$$

This implies that $\mathfrak{K}$ admits a H-representation. Then, one can canonically identify the tangent space on G/H to $\mathfrak{K}$ and the Cartan-Killing metric on G induces a Riemannian metric on G/H. In particular, this suggests that we can build the quotient space by exponentiating $\mathfrak{K}$.

Let us review useful parametrisations of G/H. The first one consists in defining an element of G/H as an element of $\exp(\mathfrak{K})$. In this parametrisation:

$$L(\phi^r) = \exp(\phi^r K_r) \tag{3.45}$$

for $\{K_r\}$ a basis of $\mathfrak{K}$. There is no group structure using this parametrisation since $\mathfrak{K}$ does not close as an algebra. However, the action of H on the scalar fields is linear. The second useful parametrisation is the *solvable* parametrisation. It is built by first selecting a maximal abelian subspace of $\mathfrak{K}$. These will generate a non-compact maximal abelian subgroup of G. Let us denote $\mathcal{S}^+$ the vector space generated by the positive roots of $\mathfrak{g}$ with respect to this choice of non-compact Cartan generators. The union of $\mathcal{S}^+$ and the Cartan generators yields a solvable algebra $\mathcal{S}$. In this parametrisation

$$L(\phi^r) = \exp(\phi^r T_r) \tag{3.46}$$

for $\{T_r\}$ a basis of $\mathcal{S}$. Since the algebra $\mathcal{S}$ closes and is solvable (i.e. it can be described as upper-triangular matrices), this parametrisation has a natural group structure. However, $\mathcal{S}$ is no longer orthogonal to $\mathfrak{h}$ and thus does not admit a natural representation of H.

The quotient of G by H has endowed the group G with a structure of H-principal bundle. Moreover, the Cartan-Killing metric has induced a natural splitting of the tangent space of G, identified with $\mathfrak{g}$, into $\mathfrak{h} \oplus \mathfrak{K}$. This allows us to decompose one-forms on G as a vertical part, coming from the $\mathfrak{h}$ factor, and a horizontal part, coming from a one-form on G/H. For example, the left-invariant one-form

$$\Omega = L^{-1}\mathrm{d}L \tag{3.47}$$

splits uniquely into two parts $\mathcal{P} \in \mathfrak{K}$ and $\mathcal{Q} \in \mathfrak{h}$ as

$$\Omega = \mathcal{P} + \mathcal{Q}\,. \tag{3.48}$$

The one-form $\mathcal{Q}$, in the horizontal space has the interpretation of a H-connection on G/H. In these notations, the metric on G/H can be interpreted as $k\,\mathrm{Tr}(\mathcal{P} \cdot \mathcal{P})$ which is invariant under both G and H action. There is thus a local symmetry-group H which will be identified with the R-symmetry group for theories with enough supercharges. The non-linear sigma model for the scalars, expressed in terms of functions $\phi : M^{1,3} \to \mathcal{M}_{scal}$ can be written in terms of the pull-back of $\mathcal{P}$ on the exterior 4 dimensional space: $\mathcal{P}_\mu$ as

$$\mathcal{L}_\sigma = -\frac{e}{2} k\, g^{\mu\nu}\mathrm{Tr}(\mathcal{P}_\mu \mathcal{P}_\nu) \tag{3.49}$$

for a specific coefficient k.

3.2.2 Vectors and E-M duality

In four dimensions, we can define a notion of electro-magnetic duality for vector fields. Let us start with Maxwell equations in four dimensions for an abelian gauge field. The equations of motion and Bianchi identities can be written as

$$dF = 0 \qquad \text{and} \qquad d\star F = 0\,, \tag{3.50}$$

where $\star$ denote the Hodge duality operator. These equations are invariant under the transformation $F \to \star F$ which is the usual electro-magnetic duality. We can rewrite these equations in a $Sp(2)$ invariant way in terms of a vector of two-forms $\mathcal{G} = (F,\, G)$. The Maxwell equations are equivalent to the equations

$$d\mathcal{G} = 0 \qquad \text{with the constraint} \qquad \star\,\mathcal{G} = \Omega\,\mathcal{G} \tag{3.51}$$

where Ω is the antisymmetric symplectic invariant matrix. In particular, the constraint, which is called the "self-duality condition", imposes that $G = \star F$.

This discussion generalizes in the context of the supergravity action (3.42). We must now consider the couplings of the scalars to the vectors in the kinetic term. We associate to the abelian electric field strengths F^Λ their magnetic dual G_Λ defined as:

$$G_\Lambda = \star\frac{\delta\mathcal{L}}{\delta F^\Lambda} = \mathcal{R}_{\Lambda\Sigma}\,F^\Sigma - \mathcal{I}_{\Lambda\Sigma}\,\star F^\Sigma\,. \tag{3.52}$$

The Maxwell case is the one obtained upon fixing $\mathcal{I} = \mathbb{1}_{n_v}$ and $\mathcal{R} = \mathbb{0}_{n_v}$. With these definitions we can restate the equations of motion for the vectors in a way that is $Sp(2n_v)$ covariant. To do so, we first define a $2n_v$ dimensional two-form

$$\mathcal{G} = \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix}\,. \tag{3.53}$$

The equations of motion read $\mathrm{d}\mathcal{G} = 0$. However, the constraint becomes the "twisted self-duality condition":

$$\star\,\mathcal{G} = -\Omega\,\mathcal{M}\,\mathcal{G} \tag{3.54}$$

where Ω is the $2n_v$ symplectic metric and

$$\mathcal{M} = \begin{pmatrix} \mathcal{R}\mathcal{I}^{-1}\mathcal{R} + \mathcal{I} & -(\mathcal{R}\mathcal{I}^{-1}) \\ -(\mathcal{I}^{-1}\mathcal{R}) & \mathcal{I}^{-1} \end{pmatrix} \tag{3.55}$$

which is a symplectic negative-definite matrix. The equation (3.54) is called the "twisted self-duality equation" and is not $Sp(2n_v,\, \mathbb{R})$ invariant unless $\mathcal{M}$ transforms in the symmetric representation of the symplectic group. However, the $\mathcal{M}$ matrix is a function of the scalars on which only G acts, not $Sp(2n_v,\, \mathbb{R})$. We can thus extend the G-symmetry group to be a symmetry of the whole action (3.42) provided that we specify an embedding

$$G \to Sp(2n_v,\, \mathbb{R})\,. \tag{3.56}$$

Such a choice is called a choice of "*symplectic frame*". If we denote R_v the fundamental representation of $Sp(2n_v)$, under which the vectors transform, the G-invariance of the equations of motion and the self-duality condition is ensured if $\mathcal{M}$ transforms as

$$\mathcal{M}(g\star\phi) = R_v[g]^{-T}\mathcal{M}(\phi)R_v[g]^{-1}\,. \tag{3.57}$$

The existence of a symplectic frame $G \to Sp(2n_v)$ as well as the action (3.57) on $\mathcal{M}$ are special properties of the scalar manifolds of extended supergravities. Mathematically, the scalar manifold is said to admit a *flat symplectic structure* which amounts to our necessary conditions for global invariance under G. When $\mathcal{N} > 2$, it implies extra properties on the parametrisation of the coset space G/H.

In particular, for a given parametrisation ϕ^r of G/H, which is equivalent to a function $L : G/H \to G : \phi^r \to L(\phi^r)$, we can parametrise G/H as a subset of $Sp(2n_v,\,\mathbb{R})$:

$$\phi^r \to R_v[L(\phi^r)]\,. \tag{3.58}$$

Then, it is always possible to find a conjugation matrix $\mathcal{S} \in Sp(2n_v,\,\mathbb{R})$ such that the equivalent representation $R'_v = \mathcal{S}^{-1}R_v\mathcal{S}$ has the property that $R_{v'}[H] \subset SO(2n_v) \cap Sp(2n_v)$. With this representation we can define the hybrid coset representative:

$$\mathbb{L}(\phi) = R_v[L(\phi)]\cdot\mathcal{S}\,. \tag{3.59}$$

In terms of this coset representative, the matrix $\mathcal{M}$ is

$$\mathcal{M} = \Omega\cdot\mathbb{L}\cdot\mathbb{L}^T\cdot\Omega\,. \tag{3.60}$$

It transforms as in (3.57) and it is invariant under the right action of H on the scalars. The coset representative carries naturally a left action under the group G and a right action under the group H which comes form the G/H coset structure. To make this clear, we can write explicitly the matrix $\mathbb{L}$ with its index structure as

$$\mathbb{L}(\phi)^M{}_{\underline{N}} \tag{3.61}$$

where the underlined index is acted upon by H and the normal index is acted upon by G. Since this matrix is an element of $Sp(2n_v)$ this means that its inverse transpose is simply $\mathbb{L}_M{}^{\underline{N}}$ where we use the symplectic matrix Ω to raise and lower indices.

Supersymmetry ensures that the matrix $\mathcal{M}$ defined this way is the same as the one coupling to the vector fields in (3.55). Moreover, we can rewrite the scalar kinetic term as

$$\mathcal{L}_\sigma = \frac{e}{8}k\,\mathrm{Tr}\left(\mathcal{M}^{-1}\partial_\mu\mathcal{M}\mathcal{M}^{-1}\partial^\mu\mathcal{M}\right)\,. \tag{3.62}$$

It is important to state that supersymmetry fixes completely the form of the action (3.42) for extended supergravities *except* for a choice of symplectic frame. Although in the ungauged theory this does not change the physical theory, in the gauged theory this choice is relevant. We will thus discuss which symplectic frames are inequivalent. The choice of a symplectic frame can be understood as a change of matrix $\mathcal{M}$ by a constant symplectic transformation. However, some of the choices of symplectic frames are trivial if they can be reabsorbed by either the global symmetry group G or by a local redefinition of the vector fields which forms a $GL(n_v,\,\mathbb{R})$ group. Symplectic frames are thus parametrised by elements of

$$Gl(n_v,\,\mathbb{R})\backslash Sp(2n_v,\,\mathbb{R})/R_v[G]^{-T}\,. \tag{3.63}$$

Finally, notice that the equations of motion are invariant under the full group G, but it is not the case for the Lagrangian itself. The reason is that to prove the

G-invariance of the equations of motion one must use the self-duality condition. In the action itself one cannot just exchange field-strength and their magnetic dual. The Lagrangian is thus invariant under the smaller group $G_{el} \subset G$ which are the element of G whose action on the fields can be parametrized by matrices of the form

$$R_v[g] = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} . \tag{3.64}$$

Under such a transformation the Lagrangian only changes by a total derivative which can be removed. This discussion must be corrected when adding fermions. Still, the main results concerning symmetries and symplectic frames remains valid.

3.3 Maximal supergravity in four dimensions

The maximal supergravity field content consists only of the gravity supermultiplet (all the other multiplets of $\mathcal{N} = 8$ supersymmetry contain particles of spin > 2). The gravity multiplet contains 70 real scalars ϕ, 56 spin-1/2 dilatini χ_{ABC}, 28 vectors A_μ^Λ, 8 gravitini ψ_μ^A and a spin-2 particle, the vierbein (or the metric). As discussed for general supergravities, we will supplement this field content with 28 auxiliary vectors $A_{\mu\,\Lambda}$, playing the role of the magnetic duals to A_μ^Λ.

Doing so, the maximal supergravity enjoys a $E_{7(7)}$ global symmetry on top of the local $\mathcal{N} = 8$ supersymmetry. The R-symmetry group is $SU(8)$ and it is the maximal compact subgroup of $E_{7(7)}$. Some standard results on $E_{7(7)}$ have been collected in Appendix A. The two groups $E_{7(7)}$ and $SU(8)$ will be used to organize the different terms in the action of the maximal supergravity. In what follows, the index for the fundamental **56** representation will be lowered and raised using the symplectic matrix Ω_{MN}[2] while the index of the adjoint representations will be lowered and raised using the Cartan-Killing metric of $\mathfrak{e}_7$.

The 70 scalar fields parametrize a non-linear σ-model on the Riemannian homogeneous symmetric manifold

$$\mathcal{M}_{\text{scal}} = \frac{E_{7(7)}}{\text{SU}(8)} . \tag{3.65}$$

The $E_{7(7)}$ global symmetry action on the scalars is realised as the left action on that manifold. We will often denote a scalar configuration by $\mathcal{V} \in E_{7(7)}/SU(8)$ with the understanding that $\mathcal{V} \in E_{7(7)} \subset Sp(56)$ for a specific choice of representative and symplectic frame.

The electric and magnetic vectors can be arranged in a vector

$$A_\mu^M = (A_\mu^\Lambda,\, A_{\mu\,\Lambda}) \tag{3.66}$$

where $M = 1, \ldots, 56$ is the index of the fundamental **56** representation of $E_{7(7)}$. It is possible to use these vectors to gauge a subgroup $G_g \subset E_{7(7)} \subset Sp(56, \mathbb{R})$ and to build a gauged maximal supergravity. The resulting gauged supergravity will have a gauge group G_g but will still transform under a duality action of the $E_{7(7)}$ group. This duality group is not an invariance of the action or the equations of motion, but it relates different theories to one another.

[2] with the NW-SE conventions.

3.3.1 Gauging and embedding tensor

As in the $\mathcal{N}=1$ case, the gauging procedure is more complicated than just replacing the derivative with covariant derivatives for two reasons. The first one, is that the covariantisation of the derivatives breaks supersymmetry. Thus, the gauged action and the transformation rules needs to be further modified to ensure gauge and supersymmetry invariance (in the case of $\mathcal{N}=1$ supergravity this was manifest in the addition of a term proportional to the moment map in the scalar potential). The second issue with the simple covariantisation of the derivatives is that vectors do not transform in the adjoint of $E_{7(7)}$ thus, they cannot be canonically identified with generators of gauge transformation. We did not have that problem in the $\mathcal{N}=1$ case because we were adding vectors to the ungauged model, whereas in this case the vectors are already part of the ungauged action and thus have specific transformation rules under $E_{7(7)}$.

To solve these two problems, we introduce an object called the *embedding tensor* which will play the role of both gauge group structure constant and gauge coupling constant. This tensor, $\Theta_M{}^\alpha$, transforms in the 56×133 representations of duality group $E_{7(7)}$ and it allows us to build covariant derivatives of the form

$$D_\mu = \partial_\mu + A_\mu^M \Theta_M{}^\alpha t_\alpha \,. \tag{3.67}$$

The embedding tensor can equivalently be rewritten in the $56\times56\times56$ representation of $E_{7(7)}$ as

$$X_{MN}{}^P = \Theta_M{}^\alpha (t_\alpha)_N{}^P \tag{3.68}$$

where the t_α are the generators of $\mathfrak{e}_7$. We also use the notation $\Theta_M = \Theta_M{}^\alpha t_\alpha = X_M$ where we suppress the last two indices.

There are three constraints on the embedding tensor which are imposed by the closure of the gauge algebra, the supersymmetry invariance, and the locality of the gauging.

- The *gauge-algebra constraint* requires Θ_M to span a Lie subalgebra of $\mathfrak{e}_7$. This implies that
$$[\Theta_M, \Theta_N] = f_{MN}{}^P \Theta_P \tag{3.69}$$
where $f_{MN}{}^P$ are the structure constant of the gauge group G_g. This can be restated as
$$[X_M, X_N] = -X_{MN}{}^P X_P \tag{3.70}$$
where $X_{[MN]}{}^P$ plays the role of structure constants. This also imposes that $X_{(MN)}{}^P X_P = 0$. Moreover, the commutator must also satisfy the Jacobi identity.

- The *supersymmetry constraint* imposes that the embedding tensor lies in a specific representation of $E_{7(7)}$: the **912** dimensional representation. The group theoretical computation:
$$\mathbf{56} \otimes \mathbf{133} = \mathbf{56} \oplus \mathbf{912} \oplus \mathbf{6480} \tag{3.71}$$
shows that this constraint is equivalent to the following equations called the *linear constraints*:
$$t_{\alpha\,M}{}^N \Theta_N{}^\alpha = 0 \qquad \text{and} \qquad (t_\beta t^\alpha)_M{}^N \Theta_N{}^\beta = -\frac{1}{2}\Theta_M{}^\alpha \,. \tag{3.72}$$

In terms of the $X_{MN}{}^P$ quantity, the linear constraints are

$$X_{M[NP]} = 0,\ X_{(MNP)} = 0,\ \text{and}\ X_{MN}{}^N = X_{MN}{}^N = 0\,. \tag{3.73}$$

- The *locality constraint* imposes that there always exists a frame in which the gauge connection is expressed using solely electric fields. This is encoded in the equation
$$\Theta_M \Omega^{MN} \Theta_N = 0\,. \tag{3.74}$$
This constraint prevents that an electric field and its magnetic duals gauge two different symmetries. This equation is called the *quadratic constraint* and shows explicitly how the gauge connection $A_\mu^M \Theta_M$, which a priori can depend on all 56 vector fields, actually only depends on at most 28 linearly independent vectors. This also imposes constraints on the possible gauge group of theory which cannot have dimension greater than 28.

Using the linear constraint, it can be shown that this last constraint is equivalent the the closure of the gauge algebra in (3.69). The condition (3.74) allows us to define a tensor $Z^{M\,\alpha} = \frac{1}{2}\Omega^{MN}\Theta_N{}^\alpha$ which is orthogonal to Θ_M in the sense that $Z^M \Theta_M = 0$ for an embedding tensor satisfying the quadratic constraints.

3.3.2 Building gauge invariant tensors

Now that we have a gauge connection A_μ^M for a group $G \subset E_{7(7)}$, we can write its variation as

$$\delta_\zeta A_\mu^M = \mathcal{D}_\mu \zeta^M = \partial_\mu \zeta^M + A_\mu^N X_{NP}{}^M \zeta^P\,. \tag{3.75}$$

The associated gauge curvature reads

$$F^M = \mathrm{d}A^M + \frac{1}{2} X_{NP}{}^M A^N \wedge A^P\,. \tag{3.76}$$

With these definitions, the first problem that arises is that the gauge curvature does not satisfy the Bianchi identity

$$DF^M = X_{(PQ)}{}^M A^P \wedge \left(dA^Q + \frac{1}{3} X_{RS}{}^Q A^R \wedge A^S \right) \tag{3.77}$$

due to a term proportional to $X_{(MN)}{}^P$. This term encodes, in some sense, the magnetic charges that we have put in our theory. The failure to solve the Bianchi identity is related to the fact that F^M is not covariant under gauge transformations i.e.

$$\delta_\zeta F^M = -X_{NP}{}^M \zeta^N F^P + X_{(NP)}{}^M (2\,\zeta^N F^P - A^N \wedge \delta A^P) \neq -X_{NP}{}^M \zeta^N F^P\,. \tag{3.78}$$

Once again the failure to transform covariantly is proportional to $X_{(MN)}{}^P$. We must thus add a Stueckelberg coupling to two-forms B_α to compensate for this failure and define new improved field strengths:

$$\mathcal{H}^M = F^M + Z^{M\alpha} B_\alpha\,. \tag{3.79}$$

We consider this particular compensating two-form because $X_{(MN)}{}^P = -Z^{P\alpha} t_{\alpha\,MN}$ and thus it has the right form to compensate the gauge transformations. For $\mathcal{H}^M$ to be covariant we have to impose

$$\delta B_\alpha = t_{\alpha\,NP}(2\,\zeta^N \mathcal{H}^P - A^N \wedge \delta A^P)\,. \tag{3.80}$$

Consistency of the theory now requires the two-forms to be gauge invariant under a 1-form symmetry parametrized by ξ_α. Our fields transform as

$$\delta_\xi B_\alpha = D\xi_\alpha - t_{\alpha\, NP} A^N \wedge \delta_\xi A^P \tag{3.81}$$

$$\delta_\xi A^M = -Z^{M\alpha}\xi_\alpha \tag{3.82}$$

$$(D\xi)_\alpha = d\xi_\alpha + \Theta_M{}^\beta (t_\beta)_\alpha{}^\gamma A^M \wedge \xi_\gamma\,. \tag{3.83}$$

The three-form field strength for B_α reads

$$\mathcal{H}_\alpha = DB_\alpha - t_{\alpha\, PQ} A^P \wedge (dA^Q + \frac{1}{3} X_{RS}{}^Q A^R \wedge A^S)\,. \tag{3.84}$$

In terms of the improved field strengths, we have the Bianchi identities

$$D\mathcal{H}^M = Z^{M\alpha}\mathcal{H}_\alpha \qquad \text{and} \qquad D\mathcal{H}_\alpha = X_{NP}{}^M \mathcal{H}^N \wedge \mathcal{H}^P\,. \tag{3.85}$$

The need for this series of corrections using higher form fields due to the fact that the embedding tensor $X_{MN}{}^P$ is not the structure constant of a Lie algebra but of a Leibniz algebra. The construction we have reviewed here is called the "tensor hierarchy" [35].

3.3.3 The bosonic action and the scalar potential

Finally, the bosonic action of the gauge supergravity reads:

$$\begin{aligned} \mathcal{L}_{bos} = &-\frac{e}{2}R - \frac{e}{2}\mathcal{G}_{st}(\phi)\mathcal{D}_\mu\phi^s\mathcal{D}^\mu\phi^t - eV(\phi) \\ &+ \frac{e}{4}\mathcal{I}_{\Lambda\Sigma}\mathcal{H}^\Lambda_{\mu\nu}\mathcal{H}^{\mu\nu\Sigma} + \frac{1}{8}\mathcal{R}_{\Lambda\Sigma}\epsilon^{\mu\nu\rho\sigma}\mathcal{H}^\Lambda_{\mu\nu}\mathcal{H}^\Sigma_{\rho\sigma} \\ &+ \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}\Theta^{\Lambda\alpha}B_{\mu\nu\alpha}\left(2\partial_\rho A_{\sigma\Lambda} + X_{MN\Lambda}A^M_\rho A^N_\sigma - \frac{1}{4}\Theta_\Lambda{}^\beta B_{\rho\sigma\beta}\right) \\ &+ \frac{1}{3}\epsilon^{\mu\nu\rho\sigma}X_{MN\Lambda}A^M_\mu A^N_\nu\left(\partial_\rho A^\Lambda_\sigma + \frac{1}{4}X_{PQ}{}^\Lambda A^P_\rho A^Q_\sigma\right) \\ &+ \frac{1}{6}\epsilon^{\mu\nu\rho\sigma}X_{MN}{}^\Lambda A^M_\mu A^N_\nu\left(\partial_\rho A_{\sigma\Lambda} + \frac{1}{4}X_{PQ\Lambda}A^P_\rho A^Q_\sigma\right) \end{aligned} \tag{3.86}$$

The third line is a topological term depending on the B fields and the last two lines are Chern-Simons terms and are required by supersymmetry. The function $V(\phi)$ is a scalar potential which is uniquely determined by the embedding tensor and the symplectic frame. It reads

$$V(\mathcal{M},\, X) = \frac{g^2}{672}\mathcal{M}^{MP}\left(X_{MN}{}^P X_{PQ}{}^S \mathcal{M}^{NQ}\mathcal{M}_{RS} + 7\, X_{MN}{}^Q X_{PQ}{}^N\right) \tag{3.87}$$

$$\text{where} \qquad \mathcal{M} = \mathcal{V}\cdot\mathcal{V}^T\,. \tag{3.88}$$

Notice that in the limit where $\Theta_M{}^\alpha \to 0$, the scalar potential vanishes. This is because the ungauged theory cannot have a scalar potential and there is superpotential-like deformations in $\mathcal{N}=8$ supergravities. The only way to deform the ungauged theory is by introducing an embedding tensor and the associated gauging.

This scalar potential is obviously invariant under the combined action of $E_{7(7)}$ on both the embedding tensor and the scalar fields as they will compensate each other. This is more visible if one re-express this scalar potential in terms of the *T-tensor*:

$$T_{\underline{M}\underline{N}}{}^{\underline{P}} = (\mathcal{V}\star X)_{\underline{M}\underline{N}}{}^{\underline{P}} = \mathcal{V}^R{}_{\underline{M}}\mathcal{V}^S{}_{\underline{N}} X_{RS}{}^T \mathcal{V}_T{}^{\underline{P}}\,. \tag{3.89}$$

This tensor transforms only under the local $SU(8)$ R-symmetry group. In term of this tensor, and dropping the underline, we can rewrite the scalar potential in a manifestly $SU(8)$ and $E_{7(7)}$ invariant way as

$$V(T) = \frac{1}{672}\, T_{MNP}\, T_{QRS}\, \delta^{MQ} (\delta^{NR}\, \delta^{PS} + 7\, \Omega^{RP}\, \Omega^{NS}) \ . \tag{3.90}$$

In the same way that the X-tensor can be written in term of the Θ tensor (3.68), the T-tensor can be written as

$$T_{MN}{}^{P} = \Xi_{M}{}^{\alpha}\, [t_\alpha]_N{}^P \qquad \text{with} \qquad \Xi_M{}^{\alpha} = \mathcal{V} \star \Theta_M{}^{\alpha} \ , \tag{3.91}$$

where the $\star$ operation denotes the action of a group element of $E_{7(7)}$ both on the fundamental and the adjoint indices. In terms of the scalar-dependent Ξ-tensor in (3.91) the scalar potential (3.90) can be rewritten as

$$V(\Xi) = \frac{1}{672}\, \Xi_{\mathbb{M}}{}^{\alpha}\, \Xi_{\mathbb{M}}{}^{\beta}\, (\delta_{\alpha\beta} + 7\, k_{\alpha\beta}) \ , \tag{3.92}$$

where we made use of a normalisation condition on the generators t_α of $\mathfrak{e}_7$:

$$\mathrm{Tr}(t_\alpha\, t_\beta^t) = \delta_{\alpha\beta} \qquad \text{and} \qquad k_{\alpha\beta} = \mathrm{Tr}(t_\alpha\, t_\beta)\, . \tag{3.93}$$

Note that the first contribution in the r.h.s of (3.92) is positive definite (sum of squares) while the second one has no definite sign due to the non-compactness of $E_{7(7)}$. We can observe that, although the scalar potential is non-linear in the scalar ϕ parametrising the coset $E_{7(7)}/SU(8)$, it is actually a quadratic function of the T and Ξ tensors. This observation is going to be very useful when building solution generating techniques in chapter 6.

The study of vacua of a particular Θ-gauged maximal supergravity is often easier to make when using the $E_{7(7)}$ duality of the $\mathcal{N} = 8$ theories. Let us consider a maximally symmetric solution of the Lagrangian maximal gauged supergravity, namely, an extremum of the scalar potential (3.92). As we observed before, the scalar potential is invariant under the simultaneous action of $E_{7(7)}$ on the scalars *and* on the embedding tensor. In other words, $\forall\, g \in E_{7(7)}$, $\mathcal{V} \in E_{7(7)}/SU(8)$ and embedding tensor $\Theta \in \mathbf{912}$ we have that

$$V(g \star \mathcal{V},\, g \star \Theta) = V(\mathcal{V},\, \Theta)\, . \tag{3.94}$$

This means that if $\mathcal{V}_0$ is an extremum of the scalar potential with embedding tensor Θ then $g \cdot \mathcal{V}_0$ is an extremum of the scalar potential with embedding tensor $g \star \Theta$ i.e.

$$\partial_{\phi^s}\, V(\mathcal{V},\, g \star \Theta)\Big|_{\mathcal{V}=g\star\mathcal{V}_0} = 0 \quad \Leftrightarrow \quad \partial_{\phi^s}\, V(\mathcal{V},\, \Theta)\Big|_{\mathcal{V}=\mathcal{V}_0} = 0\, . \tag{3.95}$$

In other words, upon modifying the embedding tensor, we can always place a vacuum of the theory at any point of the scalar coset space. This observation will prove itself useful when studying the vacua of gauged maximal supergravities.

3.3.4 Fermions, T-tensor and masses

For the gauged supergravities, the SUSY transformation rules for the fermions are encoded in two *fermion shift* matrices. These are complex matrices transforming in specific representations of $SU(8)$:

$$A_{AB} \in \mathbf{36} \qquad \text{and} \qquad A^{D}{}_{ABC} \in \mathbf{420} \tag{3.96}$$

where

$$A_{AB} = A_{BA} \quad \text{and} \quad A^D{}_{ABC} = A^D{}_{[ABC]} \quad \text{with} \quad A^D{}_{DBC} = 0 \tag{3.97}$$

We will denote $A^{AB} = \bar{A}_{AB}$ and $A_D{}^{[ABC]} = \bar{A}^D{}_{[ABC]}$. These matrices transform under the local isotropy group $SU(8)$ and not under the $E_{7(7)}$ duality group that acts on the bosonic degrees of freedom as seen above.

The T-tensor determines these two matrices. Explicitly, under the branching $E_{7(7)} \to SU(8)$, we have

$$\mathbf{912} \to \mathbf{36} \oplus \mathbf{420} \oplus \text{ c.c } . \tag{3.98}$$

In terms of $SU(8)$ indices this means that

$$\begin{aligned}
(T^{AB})^{CD}{}_{EF} &= 4\delta^{[C}_{[E} T_{F]}{}^{D]AB} \,, \\
T_C{}^{DAB} &= \frac{1}{\sqrt{2}} A_C{}^{DAB} + \sqrt{2}\, A^{D[A} \delta^{B]}_C \,. \\
(T_{AB})^{CDEF} &= 4\sqrt{2}\, \delta^{[C}_{[A} A_{B]}{}^{DEF]} \,.
\end{aligned} \tag{3.99}$$

We can inverse these formuli to obtain

$$A^{AB} = \frac{\sqrt{2}}{21} (T^{AD})^{BC}{}_{CD} \quad \text{and} \quad A_A{}^{BCD} = \frac{1}{\sqrt{2}} (T_{FA})^{F[BCD]} \,. \tag{3.100}$$

The two matrices A_{AB} and $A_A{}^{BCD}$ not only determine the supersymmetric transformations of the fermionic fields. They can also be used to compute the scalar potential and its derivatives. In particular, they allow us to check if a point extremise V or to compute the mass matrices. We collect here the relevant formuli. The scalar potential is

$$V = \left(\frac{1}{24} |A^A{}_{BCD}|^2 - \frac{3}{4} |A_{AB}|^2 \right) \,. \tag{3.101}$$

The gradient of the scalar potential is

$$\partial_{\phi^s} V = -\frac{1}{12} \mathcal{P}_s^{BEFG} \left(\mathcal{C}_{BEFG} + c.c. \right) \tag{3.102}$$

where

$$\mathcal{C}_{BEFG} = A^A{}_{[BEF} A_{G]A} + \frac{3}{4} A^A{}_{D[BE} A^D{}_{FG]A} \,. \tag{3.103}$$

A point in the coset space will extremise V if and only if $\mathcal{C}$ is anti-self dual i.e.

$$\mathcal{C}_{ABCD} + \frac{1}{24} \epsilon_{ABCDEFGH} \mathcal{C}^{EFGH} = 0 \,. \tag{3.104}$$

Concerning the mass matrices, we have

- The gravitni mass matrix:

$$M^{3/2}_{AB} = \sqrt{2} A_{AB} \,. \tag{3.105}$$

- The spin 1/2 mass matrix:

$$\begin{aligned}
M^{1/2}_{ABC,\,EFG} = \frac{\sqrt{2}}{12} \Bigg(& \epsilon_{ABC\,HIJ\,[EF} A_{G]}{}^{HIJ} \\
& + \frac{4}{3} \sum_{G,\,H}{}' \left(\frac{A}{|A|^2 + \frac{V_0}{6} \mathbb{1}} \right)_{GH} A^G{}_{ABC} A^H{}_{EFD} \Bigg) .
\end{aligned} \tag{3.106}$$

where the sum $\sum'$ over G and H only goes over the broken supersymmetries (i.e. when A_{AB} is diagonalised, these are the entries for which $|A|^2{}_{GH} \neq -\frac{V_0}{6}$).

- The scalars mass matrix:

$$\begin{aligned} M_{[ABCD]}{}^{[EFGH]} = & \, 6\left(A_A{}^{IJE}A^F{}_{IJB} - \tfrac{1}{4}A_I{}^{JEF}A^I{}_{JAB}\right)\delta^{GH}_{CD} \\ & + \left(\tfrac{5}{24}A_I{}^{JKL}A^I{}_{JKL} - \tfrac{1}{2}A_{IJ}A^{IJ}\right)\delta^{EFGH}_{ABCD} \\ & - \tfrac{2}{3}A_A{}^{EFG}\,A^H{}_{BCD}\,, \end{aligned} \tag{3.107}$$

  where the antisymmetrisation on the indices $[ABCD]$ and $[EFGH]$ is implied on the l.h.s.

- The vector masses are more easily computed from the gauge transformations and gives the $E_{7(7)}$ covariant expression:

$$M_{v\,M}{}^{N} = -\frac{1}{24}\Big(\mathrm{Tr}(X_M X_P) + \mathrm{Tr}\big(\mathcal{M}^{-1}X_M\mathcal{M}(X_P)^T\big)\Big)\mathcal{M}^{PN}\,. \tag{3.108}$$

  This matrix has 28 zero eigenvalues because of the locality constraint. The other eigenvalues are the masses of the physical vector fields. This can of course be reexpressed in terms of A matrices.

These formuli will be the only informations we will need. The full $\mathcal{N}=8$ action is a relatively long action. This action and the SUSY transformations can be found explicitly in [33].

3.4 Aspects of $\mathcal{N}=4$ gauged supergravity

The half-maximal $\mathcal{N}=4$ gauged supergravity in four dimensions was constructed in [36] where the reader can find all the details about its construction. In this section, we will review the main aspects of the bosonic sector of half-maximal supergravity and compare wiht the possible truncation from maximal supergravity performed in [37].

The bosonic sector of half-maximal supergravity consists of the gravity supermultiplet, containing the metric, six vectors as well as two real scalars. This field content can be enlarged by adding n vector multiplets, each containing a vector and six scalars. As in maximal supergravity, the half maximal supergravity is characterised by a specific duality group $\mathcal{G} = \mathrm{SL}(2)\times\mathrm{SO}(6,n)$ where n is the number of vector multiplets to which the supergravity multiplet is coupled. Promoting a subgroup $\mathrm{G}\subset\mathcal{G}$ from global to local, *i.e.* performing a so-called *gauging*, the duality group $\mathcal{G}$ is generically broken and only the local gauge symmetry G is left in the gauged supergravity. Still, the commutant (if any) of G inside $\mathcal{G}$ remains as a global symmetry of the theory after the gauging procedure.

In this section, we will consider the case $n=6$. This is the largest value for which the duality group of half-maximal supergravity can be embedded into the one of maximal supergravity, *i.e.* $\mathrm{SL}(2)\times\mathrm{SO}(6,6)\subset\mathrm{E}_{7(7)}$, and the half-maximal supergravity Lagrangian can be embedded into the maximal one provided certain quadratic constraints on the couplings in the theory hold (see eq.(3.112) below).

In the duality-covariant formulation of [36], the bosonic field content of the half-maximal supergravity consists of the metric $g_{\mu\nu}$, 12 (electric) plus 12 (magnetic) vector fields $A_\mu{}^{\alpha M}$, and $2+36$ scalar fields ϕ's serving as coordinates in the coset space geometry

$$\mathcal{M}_{\text{scal}} = \frac{\text{SL}(2)}{\text{SO}(2)} \times \frac{\text{SO}(6,6)}{\text{SO}(6)\times\text{SO}(6)} \ , \tag{3.109}$$

and being parameterised by a coset representative $\mathcal{V}(\phi)$. Already at this level we have introduced a fundamental $\text{SL}(2)$ index $\alpha=\pm$ as well as a fundamental $\text{SO}(6,6)$ index M. These are raised/lowered using the $\epsilon_{\alpha\beta}$ and η_{MN} invariant tensors of $\text{SL}(2)$ and $\text{SO}(6,6)$, respectively.

Having set the bosonic field content of the theory, all the interactions compatible with $\mathcal{N}=4$ supersymmetry and induced by gaugings are encoded into two embedding tensors $f_{\alpha\,[MNP]}$ and $\xi_{\alpha\,M}$ which have the same role as the embedding tensor of maximal supergravity. In this thesis, we will not need to use the ξ embedding tensor, as such we will fix it to zero and only study the half maximal supergravity with $f_{\alpha\,[MNP]}$ gaugings. This tensor lives in the $(\mathbf{2},\mathbf{220})$ irreducible representation (irrep) of $\mathcal{G}$. It also specifies the non-Abelian gauge structure of the half-maximal gauged supergravity, namely,

$$[\,T_{\alpha M}\,,\,T_{\beta N}\,] = f_{\alpha MN}{}^{P}\,T_{\beta P} \ , \tag{3.110}$$

where $T_{\alpha M}$ are the generators of $\mathcal{G}$ that couple to the vector fields $A_{\mu}{}^{\alpha M}$ in the gauge connection. Consistency of the gauging procedure requires a set of quadratic constraints on the embedding tensor of the form [36]

$$(\mathbf{3},\mathbf{495}):\quad f_{\alpha[MN}{}^{R}\,f_{\beta PQ]R}=0 \qquad \text{and} \qquad (\mathbf{1},\mathbf{66}{+}\mathbf{2079}):\quad \epsilon^{\alpha\beta}\,f_{\alpha MN}{}^{R}\,f_{\beta PQR}=0\,, \tag{3.111}$$

where we have included the irrep of $\mathcal{G}$ where each constraint lives. In addition, there are two additional constraints given by

$$(\mathbf{3},\mathbf{1}):\quad f_{\alpha MNP}\,f_{\beta}{}^{MNP}=0 \qquad \text{and} \qquad (\mathbf{1},\mathbf{462'}):\quad \epsilon^{\alpha\beta}\,f_{\alpha[MNP}\,f_{\beta QRS]}\Big|_{\text{SD}}=0\ , \tag{3.112}$$

where SD refers to the self-dual projection of the $\text{SO}(6,6)$ six-form. These two additional constraints (3.112) are *not* required by $\mathcal{N}=4$ supersymmetry but must hold for the half-maximal Lagrangian to be embeddable into an $\mathcal{N}=8$ maximal gauged supergravity [37]. Whenever these two additional constraints are satisfied, the half-maximal supergravity can be viewed as a subsector of the maximal gauged supergravity.

As a consequence of the gauging procedure, the fermions in the theory develop scalar-dependent mass terms and supersymmetry requires to introduce a non-trivial scalar potential. This is given by

$$\begin{aligned} V \;=\; & \tfrac{1}{64}\,f_{\alpha MNP}\,f_{\beta QRS}M^{\alpha\beta}\left[\tfrac{1}{3}\,M^{MQ}\,M^{NR}\,M^{PS}+\left(\tfrac{2}{3}\,\eta^{MQ}-M^{MQ}\right)\eta^{NR}\eta^{PS}\right] \\ & -\tfrac{1}{144}\,\epsilon^{\alpha\beta}\,f_{\alpha MNP}\,f_{\beta QRS}\,M^{MNPQRS}\ , \end{aligned} \tag{3.113}$$

where

$$M_{\alpha\beta} = \frac{1}{\text{Im}z_7}\begin{pmatrix} |z_7|^2 & \text{Re}z_7 \\ \text{Re}z_7 & 1\end{pmatrix} \in \text{SL}(2)\ , \tag{3.114}$$

encodes a complex scalar z_7 spanning the $\text{SL}(2)/\text{SO}(2)$ factor in (3.109). Together with this, the potential depends on additional scalars spanning the $\text{SO}(6,6)/(\text{SO}(6)\times$

SO(6)) factor in (3.109). These are 36 real scalars which can be assembled in a matrix

$$M_{MN} = \begin{pmatrix} G^{-1} & -G^{-1}B \\ B\,G^{-1} & G - B\,G^{-1}B \end{pmatrix} \in \mathrm{SO}(6,6) \ , \tag{3.115}$$

where G and B are arbitrary symmetric and anti-symmetric 6×6 matrices accounting for 21 and 15 scalars, respectively. As for any $\mathcal{N} > 2$ supergravity, the kinetic terms for the scalar fields serving as coordinates on the scalar geometry (3.109) are constructed using standard coset techniques and read

$$\mathcal{L}_{\text{kin}} = \frac{1}{8}\,\partial_\mu M_{\alpha\beta}\,\partial^\mu M^{\alpha\beta} + \frac{1}{16}\,D_\mu M_{MN}\,D^\mu M^{MN} \ , \tag{3.116}$$

where $M^{\alpha\beta}$ and M^{MN} are the inverse of $M_{\alpha\beta}$ in (3.114) and M_{MN} in (3.115), respectively. The general covariant derivatives are

$$D_\mu M_{MN} = \partial_\mu M_{MN} + 2\,A_\mu{}^{\alpha P}\,f_{\alpha P(M}{}^{Q}\,M_{N)Q} \ . \tag{3.117}$$

Lastly, the scalar potential (3.113) depends on a specific SO(6, 6) six-form[3]

$$M_{MNPQRS} = \epsilon_{mnpqrs}\,\mathcal{V}_M{}^m\,\mathcal{V}_N{}^n\,\mathcal{V}_P{}^p\,\mathcal{V}_Q{}^q\,\mathcal{V}_R{}^r\,\mathcal{V}_S{}^s \ , \tag{3.118}$$

that is constructed from the $\mathrm{SO}(6,6)/(\mathrm{SO}(6)\times\mathrm{SO}(6))$ coset representative $\mathcal{V}_M{}^N$ such that $M_{MN} = (\mathcal{V}\,\mathcal{V}^t)_{MN}$ (see [36] for more details).

3.5 Type IIB supergravity

Type IIB supergravity is the low energy description of Type IIB superstrings. It is a ten-dimensional theory with chiral $(2,0)$ supersymmetry (i.e. both gravitini have the same chirality). The bosonic massless spectrum of ten-dimensional (chiral) type IIB supergravity contains the universal NS-NS sector: the metric G, a two-form B_2 with field strength $H_3 = dB_2$, and the dilaton Φ. It also contains a R-R sector consisting of a set of even p-forms i.e. a fourth-rank antisymmetric self-dual tensor C_4, a two-form C_2 and a scalar C_0.

The bosonic part of the type IIB supergravity action in the Einstein's frame consists of the three terms:

$$S_{bos} = S_{\text{NS-NS}} + S_{\text{R-R}} + S_{\text{CS}} \ . \tag{3.119}$$

It contains the $S_{\text{NS-NS}}$ term accounting for the fields in the universal sector, namely,

$$S_{\text{NS-NS}} = \frac{1}{2\kappa^2}\int d^{10}x\,\sqrt{-G}\,\left(R - \tfrac{1}{2}\,\partial^M\Phi\partial_M\Phi - \tfrac{1}{2}e^{-\Phi}|H_3|^2\right) \ . \tag{3.120}$$

The $S_{\text{R-R}}$ term in the action controlling the dynamics of the R-R fields C_0, C_2 and C_4 is given by

$$S_{\text{R-R}} = \frac{1}{2\kappa^2}\int d^{10}x\,\sqrt{-G}\,\left(-\tfrac{1}{2}\,e^{2\Phi}|F_1|^2 - \tfrac{1}{2}\,e^{\Phi}|\widetilde{F}_3|^2 - \tfrac{1}{4}\,|\widetilde{F}_5|^2\right) \ , \tag{3.121}$$

[3] Due to the ϵ_{mnpqrs} tensor with $m,n,\ldots = 1,\ldots,6$ in the definition of the SO(6, 6) six-form (3.118), the components $\mathcal{V}_M{}^n$ entering (3.118) must be extracted from the coset representative $\mathcal{V}_M{}^N$ using a Lorentzian basis (for the column index N) of SO(6, 6) where $\eta_{MN} = \mathrm{diag}(-\mathbb{I}_6, \mathbb{I}_6)$.

where the tilded field strengths are defined as

$$\begin{array}{rcl} \widetilde{F}_3 &=& F_3 - C_0 \wedge H_3 \ , \\ \widetilde{F}_5 &=& F_5 + \frac{1}{2}\left(B_2 \wedge F_3 - C_2 \wedge H_3\right) \ , \end{array} \tag{3.122}$$

in terms of the standard ones $F_{n+1} = dC_n$. Additionally, the self-duality condition

$$\widetilde{F}_5 = \star \widetilde{F}_5 \ , \tag{3.123}$$

with $(\star\widetilde{F})^{MNOPQM} \equiv \dfrac{1}{5!\sqrt{-G}}\,\epsilon^{MNOPQM'N'O'P'Q'}\,\widetilde{F}_{M'N'O'P'Q'}$ has to be supplemented by hand in order to have the correct number of bosonic degrees of freedom. The type IIB theory also incorporates a topological Chern-Simons term S_{CS} in the action given by

$$S_{\text{CS}} = -\frac{1}{4\,\kappa^2}\int C_4 \wedge H_3 \wedge F_3 \ . \tag{3.124}$$

The equations of motion that follow from the action (3.119) with the various contributions in (3.120), (3.121) and (3.124) are given by

$$\begin{array}{rcl} d\star\widetilde{F}_5 &=& \frac{1}{2}\,\epsilon_{\alpha\beta}\,\widetilde{\mathbb{H}}^{\alpha}\wedge\widetilde{\mathbb{H}}^{\beta} \ , \\ d\star\left(e^{-\Phi}\,H_3 - e^{\Phi}\,C_0\,\widetilde{F}_3\right) &=& -\widetilde{F}_5 \wedge \left(\widetilde{F}_3 + C_0\,H_3\right) \ , \\ d\star\left(e^{\Phi}\,\widetilde{F}_3\right) &=& \widetilde{F}_5 \wedge H_3 \ , \\ \nabla^{M}\left(e^{2\Phi}\,\partial_M C_0\right) &=& -\frac{1}{3!}\,e^{\Phi}\,H_{MNP}\,\widetilde{F}^{MNP} \ , \\ \Box\Phi &=& e^{2\Phi}\,|F_1|^2 - \frac{1}{2}\,e^{-\Phi}\,|H_3|^2 + \frac{1}{2}\,e^{\Phi}\,|\widetilde{F}_3|^2 \ , \end{array} \tag{3.125}$$

where $\widetilde{\mathbb{H}}^{\alpha} = (H_3, \widetilde{F}_3)$ and $\Box\Phi \equiv \nabla^M\partial_M\Phi$, together with the Einstein equation

$$\begin{array}{rcl} R_{MN} &=& \frac{1}{2}\,\partial_M\Phi\,\partial_N\Phi + \frac{1}{2}\,e^{2\Phi}\,\partial_M C_0\,\partial_N C_0 \\ &+& \frac{1}{4}\,\frac{1}{4!}\left(\widetilde{F}_{MP_1\cdots P_4}\,\widetilde{F}_N{}^{P_1\cdots P_4} - \frac{1}{10}\,\widetilde{F}_{P_1\cdots P_5}\,\widetilde{F}^{P_1\cdots P_5}\,G_{MN}\right) \\ &+& \frac{1}{4}\,e^{-\Phi}\left(H_{MP_1P_2}\,H_N{}^{P_1P_2} - \frac{1}{12}\,H_{P_1P_2P_3}\,H^{P_1P_2P_3}\,G_{MN}\right) \\ &+& \frac{1}{4}\,e^{\Phi}\left(\widetilde{F}_{MP_1P_2}\,\widetilde{F}_N{}^{P_1P_2} - \frac{1}{12}\,\widetilde{F}_{P_1P_2P_3}\,\widetilde{F}^{P_1P_2P_3}\,G_{MN}\right) \ . \end{array} \tag{3.126}$$

In addition, the set of Bianchi identities for the various gauge potentials reads

$$dH_3 = 0 \quad , \quad dF_1 = 0 \quad , \quad d\widetilde{F}_3 = -F_1 \wedge H_3 \quad , \quad d\widetilde{F}_5 = H_3 \wedge F_3 \ . \tag{3.127}$$

Note the equivalence between the first equation of motion in (3.125) and the last Bianchi identity in (3.127) for the self-dual C_4 potential.

The action (3.119) has a global $\text{SL}(2,\mathbb{R})$ invariance which becomes manifest when combining the axion C_0 and the dilaton Φ into a coset representative $\mathcal{V}_2 \in \text{SL}(2,\mathbb{R})/\text{SO}(2)$ such that the axion-dilaton matrix $m_{\alpha\beta}$ reads

$$m_{\alpha\beta} = (\mathcal{V}_2\,\mathcal{V}_2{}^t)_{\alpha\beta} = e^{\Phi}\begin{pmatrix} e^{-2\Phi} + C_0{}^2 & -C_0 \\ -C_0 & 1 \end{pmatrix} \ . \tag{3.128}$$

In terms of this matrix the second and third equations of motion in (3.125) are re-expressed in an $\text{SL}(2,\mathbb{R})$ covariant form

$$d\star\left(m_{\alpha\beta}\,\mathbb{H}^{\beta}\right) = -\epsilon_{\alpha\beta}\,\widetilde{F}_5 \wedge \mathbb{H}^{\beta} \ , \tag{3.129}$$

where $\mathbb{H}^\alpha = (H_3, F_3)$.

This $SL(2, \mathbb{R})$ is the supergravity remnant of the type IIB superstrings "S-duality" group: SL(2, $\mathbb{Z}$). todo.

3.6 The holographic dictionary

In the previous chapter, we discussed how superconformal symmetry constrains the possible deformations and the field content of a SCFT. In particular, given the set of all operators in 3d $\mathcal{N} = 2$ SCFT, we can compute the number of exactly marginal deformations explicitly. But how can we gain access to the operator content of a SCFT, especially in the strong coupling regime? In the same spirit, when a CFT admits a conformal manifold, how can we compute the Zamolodchikov metric on such a manifold? In this thesis we will provide answer to these questions using the AdS/CFT correspondence [9, 10, 11].

This correspondence maps CFT_{d-1} to vacua of string theory on AdS_d, at least in the "large-N limit" where the rank of the gauge group of the CFT is taken to be large. Taking the low energy limit of string theory to describe the gravity side of this correspondence, there is an holographic dictionnary relating the masses of excitations at a given AdS solution of a supergravity to the conformal dimensions of operators in the dual CFT. Moreover, the correspondence maps the conformal manifold of a CFT to a continuous family of AdS solutions of string theory known as the "moduli space". The metric on the conformal manifold can also be obtained from the supergravity approximation of the gravity dual.

More precisely, the relation between the normalised mass mL of a supergravity field of spin $[j]$ in a given AdS_4 solution and the conformal dimension Δ of the dual operator in the CFT_3 is given by

$$\begin{aligned} [\tfrac{3}{2}] \quad &: \quad mL = \Delta - \tfrac{3}{2} \ , \\ [1] \quad &: \quad m^2L^2 = (\Delta-2)(\Delta-1) \ , \\ [\tfrac{1}{2}] \quad &: \quad mL = \Delta - \tfrac{3}{2} \ , \\ [0] \quad &: \quad m^2L^2 = \Delta(\Delta-3) \ , \end{aligned} \tag{3.130}$$

with the graviton $[j] = [2]$ being massless. Concerning scalar operators, this means that excitations with normalised mass $m^2L^2 < 0$ correspond to relevant operators, $m^2L^2 = 0$ to marginal operators while $m^2L^2 > 0$ correspond to irrelevant operators. As such, using the effective four-dimensional supergravity approach, we can explore the landscape of CFT_3's. In particular, the set of operators can be arranged into superconformal multiplets of the $\mathfrak{osp}(\mathcal{N}|4)$ superconformal algebra for AdS_4 solutions preserving $\mathcal{N}$ supersymmetry. This allows us to compute the dimension of the supersymmetric conformal manifold and predict patterns of symmetry breaking.

In the same way, the Zamolodchikov metric on the conformal manifold can be obtained from the supergravity data. Since the dual of a CFT_3 is an AdS_4 solution, a conformal manifold is dual to a moduli space of solutions in supergravity. Since the scalar manifold, in supergravity, is endowed with a metric it is possible to take the pull back of this scalar metric on the moduli space of solutions. This pullback is the holographic dual of the Zamolodchikov metric.

Chapter 4

The $[SO(1,\,1) \times SO(6)] \ltimes \mathbb{R}^{12}$ model

In this thesis, the strategy to find new solutions of Type IIB supergravity is to first study vacua of maximal supergravity in four dimensions. For large classes of these supergravities, it is actually possible to uplift their solutions to IIA, IIB or 11d supergravities using tools of Exceptional Field Theory that we will describe in the next chapter. In this chapter, we will study solutions of the $[SO(1,\,1) \times SO(6)] \ltimes \mathbb{R}^{12}$ dyonically gauged maximal supergravity.

To motivate this specific choice of gauging, we will firstly review known solutions of maximal supergravity with different gauge groups in section 4.1. In section 4.2, we will build the specific gauging we are interested in by writing its embedding tensor. This completely determines the theory. In order to find vacua, we will study consistent truncations of the theory, i.e. subsectors where some of the scalar field are set to zero. In particular, section 4.3 describes a $\mathbb{Z}_2^3$ consistent truncation of this maximal supergravity which can be expressed as a $\mathcal{N} = 1$ model with seven chiral multiplets. Finally, in the last section, we will present the solutions we have found in this truncation as well as the masses of the excitations around these vacua.

4.1 Gaugings and vacua of maximal supergravity

Using the $E_{7(7)}$ duality, it is always possible to put a vacuum at the origin of the scalar manifold. This procedure might however change the embedding tensor. One can thus study solutions of all gauged supergravities by studying which embedding tensors admit vacua at the origin, possibly preserving certain properties. For example, in [38], the authors studied all the gauging admitting a vacua with a residual $\mathcal{N} > 2$ supersymmetry. This allowed to classify all such solutions for all gauged maximal supergravities in four dimensions. They found three families of such solutions. One with $\mathcal{N} = 8$, which is just a point, as well as one with $\mathcal{N} = 4$ and one with $\mathcal{N} = 3$. There is no solutions with exactly $4 < \mathcal{N} < 8$ residual supersymmetry.

The $\mathcal{N} = 8$ AdS$_4$ point is a solution of the de Wit-Nicolai $SO(8)$ gauged supergravity [30]. It appears as a compactification of 11d supergravity on the round S^7. This was expected since the only maximally supersymmetric solutions of the form AdS$_4 \times M^7$ is the standard Freund-Rubin solution of 11d supergravity [39, 40]. This solution was identified as the holographic dual to the ABJM theory [41].

The $\mathcal{N} = 3$ solutions appear for the gaugings $SO(8)_\omega$, $SO(1,\,7)$ and $ISO(7)$. This family of solution is parametrised by an angle φ which determines the gauge group and the symplectic frame of the theory.

- For $\varphi < \pi/6$ we get a family of $SO(8)_\omega$ gaugings where both electric and magnetic vectors are used to gauge $SO(8)$. The ω-family of $SO(8)$ gaugings was first described in [42]. The simple $SO(8)$ gauging, coming from the S^7 compactification of 11d supergravity, appears at $\omega = 0$ and admits the $\mathcal{N} = 8$

solution. However, the value $\omega=0$ in not reached in the $\mathcal{N}=3$ family of solution. When $\omega\neq 0$, there exists no-go theorem [43, 44] proving that there is no uplift to 11d supergravity (at least using the usual methods of generalised Scherk–Schwarz reductions). If such an uplift were to exists, it would be non-geometrical, meaning that it would appear as the compactification of a theory in more than 11 dimensions.

- For $\varphi>\pi/6$ the gauge group is $SO(1,7)$, this family of gaugings has not been studied extensively although it is known to uplift to M-theory compactified on the hyperboloid $H^{(1,7)}$ [45].
- Finally, at $\varphi=\pi/6$, The gauge group is ISO(7). This gauging has been shown to uplift to massive type IIA supergravity compactified on $AdS_4\times S^6$ [46]. Moreover, the holographic dual of this family has been identified as a Chern-Simons theory at level k. This level is related to the Roman mass of the Type IIA uplift.

The $\mathcal{N}=4$ solutions are found in theories of maximal supergravities with gaugings $SO(1,7)$ or $[SO(1,1)\times SO(6)]\ltimes\mathbb{R}^{12}$. These gaugings are parametrised by a variable $\varphi\in[0,2\pi[$. When $0<\varphi<2\pi$, these are solutions of gaugings of the group $SO(1,7)$. As in the $\mathcal{N}=3$ case, these solutions can be understood as compactifications of 11d supergravity on $H^{(1,7)}$. At $\varphi=2\pi$, the gauge group degenerates and reduces to $[SO(1,1)\times SO(6)]\ltimes\mathbb{R}^{12}$.

This shows that $\mathcal{N}=4$ solutions in maximal supergravity are quite rare and thus deserve to be studied in more details. Moreover, the position of the $[SO(1,1)\times SO(6)]\ltimes\mathbb{R}^{12}$ gaugings in the $\mathcal{N}=4$ family of solutions mirror the position of the $ISO(7)$ point in the $\mathcal{N}=3$ family of solution. Since the solutions for the $ISO(7)$ gauging could be uplifted and have a precise CFT_3 dual, it is interesting to see whether or not the $\mathcal{N}=4$ solution and other solutions of the same theory have these properties. For these reasons we will study the vacua of this theory and their uplift as well as possible holographic interpretations.

4.2 The $[SO(1,1)\times SO(6)]\ltimes\mathbb{R}^{12}$ embedding tensor

The $[SO(1,1)\times SO(6)]\ltimes\mathbb{R}^{12}$ dyonic gauging of supergravity is part of a large family of gaugings of the form

$$[SO(p,q)\times SO(p',q')]\ltimes N\,, \tag{4.1}$$

which are subgroups of $SL(8)\subset E_{7(7)}$. This class of gaugings can be built by branching representations of $E_{7(7)}$ under $SL(8)$. There are three representations of particular interest: the fundamental **56** (index M), the adjoint **133** (index α), and the **912** (the representation of the embedding tensor). They decompose as

$$\mathbf{56}\rightarrow\mathbf{28}+\mathbf{28}'\,, \tag{4.2}$$

$$\mathbf{133}\rightarrow\mathbf{63}\oplus\mathbf{70}\,, \tag{4.3}$$

$$\mathbf{912}\rightarrow\mathbf{36}\oplus\mathbf{420}\oplus\mathbf{36}'\oplus\mathbf{420}'\,. \tag{4.4}$$

In terms of fundamental SL(8) indices $A=1,\dots,8$ one has that

$$\mathcal{A}^M=(\mathcal{A}^{[AB]}\,,\,\mathcal{A}_{[AB]})\qquad\text{and}\qquad t_\alpha=(t_A{}^B\,,\,t_{[ABCD]})\,, \tag{4.5}$$

with $t_A{}^A = 0$ and where the square brackets denote antisymmetrisation. We have collected in the Appendix A various results regarding the $\mathrm{E}_{7(7)}$ generators in the SL(8) basis. For the gauge group (4.1), the embedding tensor transforms in the $\mathbf{36}\oplus\mathbf{36}'$ and can be expressed in terms of two 8×8 symmetric matrices:

$$\Theta_{[AB]}{}^C{}_D = 4\sqrt{3}\,\delta^C_{[A}\,\theta_{B]D} \qquad \text{and} \qquad \Theta^{[AB]C}{}_D = 4\sqrt{3}\,\delta^{[A}_D\,\tilde\theta^{B]C}\,. \tag{4.6}$$

This shows that where θ_{AB} turns on gaugings using the electric vectors of the theory, $\tilde\theta^{AB}$ uses the magnetic vectors. An explicit computation of the quadratic constraints (3.74) shows that they are satisfied if

$$\theta_{AB}\,\tilde\theta^{BC} = 0\,. \tag{4.7}$$

Using an $SL(8)$ duality transformations, the θ and $\tilde\theta$ matrices can be set to a scalar multiple of their signature. The quadratic constraints are solved for

$$\theta_{AB} = g\,\mathrm{diag}(\overbrace{1,\,\ldots,\,1}^{p},\overbrace{-1,\,\ldots,\,-1}^{q},0,\,\ldots,\,0) \tag{4.8}$$

$$\tilde\theta^{BC} = c\,\mathrm{diag}(0,\,\ldots,\,0,\underbrace{1,\,\ldots,\,1}_{p'},\underbrace{-1,\,\ldots,\,-1}_{q'}) \tag{4.9}$$

with $p+q+p'+q' \leq 8$ leading to the gauge groups (4.1). The dyonic gauging of $\mathrm{G} = [\mathrm{SO}(1,1)\times\mathrm{SO}(6)]\ltimes\mathbb{R}^{12}$ amounts to a choice of embedding tensor $\Theta_M{}^\alpha$ determined by the matrices

$$\theta_{AB} = g\,\mathrm{diag}(\,0\,,\,\mathbb{I}_6\,,\,0\,) \qquad \text{and} \qquad \tilde\theta^{AB} = c\,\mathrm{diag}(\,-1\,,\,0_6\,,\,1\,)\;. \tag{4.10}$$

To better understand this gauging, let us further branch the $E_{7(7)}$ irreps in (4.2)-(4.4) under the group $\mathrm{SL}(6)\times\mathrm{SL}(2)\times\mathrm{SO}(1,1)\subset\mathrm{SL}(8)$. The result is

$$\begin{aligned}\mathbf{56} &\to \mathbf{28}\oplus\mathbf{28}' \\ &\to [(\mathbf{6},\,\mathbf{2})_{-1}\oplus(\mathbf{15},\,\mathbf{1})_{1}\oplus(\mathbf{1},\,\mathbf{1})_{-3}]\oplus[(\mathbf{6}',\,\mathbf{2})_{1}\oplus(\mathbf{15}',\,\mathbf{1})_{-1}\oplus(\mathbf{1},\,\mathbf{1})_{3}]\;,\end{aligned} \tag{4.11}$$

$$\begin{aligned}\mathbf{133} &\to \mathbf{63}\oplus\mathbf{70} \\ &\to [(\mathbf{35},\mathbf{1})_0\oplus(\mathbf{6},\mathbf{2})_2\oplus(\mathbf{6}',\mathbf{2})_{-2}\oplus(\mathbf{1},\mathbf{3})_0\oplus(\mathbf{1},\mathbf{1})_0] \\ &\qquad \oplus[(\mathbf{15},\mathbf{1})_{-2}\oplus(\mathbf{20},\,\mathbf{2})_0\oplus(\mathbf{15}',\mathbf{1})_2]\;,\end{aligned} \tag{4.12}$$

$$\begin{aligned}\mathbf{912} &\to \mathbf{36}\oplus\mathbf{36}'\oplus\mathbf{420}\oplus\mathbf{420}' \\ &\to [(\mathbf{21},\mathbf{1})_1\oplus(\mathbf{6},\mathbf{2})_{-1}\oplus(\mathbf{1},\mathbf{3})_{-3}]\oplus[(\mathbf{21}',\mathbf{1})_{-1}\oplus(\mathbf{6}',\mathbf{2})_1\oplus(\mathbf{1},\mathbf{3})_3] \\ &\quad\oplus[(\mathbf{35},\mathbf{1})_{-3}\oplus(\mathbf{84},\mathbf{2})_{-1}\oplus(\mathbf{6},\mathbf{2})_{-1}\oplus(\mathbf{105},\mathbf{1})_1\oplus(\mathbf{15},\mathbf{3})_1\oplus(\mathbf{15},\mathbf{1})_1\oplus(\mathbf{20},\mathbf{2})_3] \\ &\quad\oplus[(\mathbf{35},\mathbf{1})_3\oplus(\mathbf{84}',\mathbf{2})_1\oplus(\mathbf{6}',\mathbf{2})_1\oplus(\mathbf{105}',\mathbf{1})_{-1}\oplus(\mathbf{15}',\mathbf{3})_{-1}\oplus(\mathbf{15}',\mathbf{1})_{-1}\oplus(\mathbf{20},\mathbf{2})_{-3}]\;.\end{aligned} \tag{4.13}$$

Under this decomposition the electric component of the embedding tensor in $\mathbf{36}$ corresponds to $(\mathbf{21},\mathbf{1})_1\subset\mathbf{36}$ whereas the magnetic one in $\mathbf{36}'$ corresponds to $(\mathbf{1},\mathbf{3})_3\subset\mathbf{36}'$. Similarly, the fundamental SL(8) index decomposes as $A=(i\,,\,a)$ with $i=2,\ldots,7$ being a fundamental SL(6) index and $a=1,8$ a fundamental SL(2) index. As a consequence

$$[AB] = ([18]\,,\,[ai]\,,\,[ij])\;, \tag{4.14}$$

and, from the embedding tensor components in (4.6)-(4.10), one gets

$$\Theta_M{}^\alpha t_\alpha \quad \rightarrow \quad \Theta_{ij} = 2\sqrt{3}\, g\, (t_i{}^j - t_j{}^i) \quad , \quad \Theta^{18} = 2\sqrt{3}\, c\, (t_1{}^8 + t_8{}^1) \quad , \\ \Theta_{ai} = 2\sqrt{3}\, g\, t_a{}^i \quad , \quad \Theta^{ai} = 2\sqrt{3}\, c\, t_a{}^i \; . \tag{4.15}$$

Then the compact SO(6) factor of the gauge group is gauged by electric vectors A^{ij} whereas the non-compact SO(1,1) factor is gauged by the magnetic vector A_{18} and the $\mathbb{R}^{12}$ translations are gauged dyonically, i.e. use both electric and magnetic vectors, provided $gc \neq 0$.

4.3 Vacua and consistent truncations

To study maximally symmetric vacua of the $\mathcal{N} = 8$ theory, one can truncate away the vectors and the fermionic fields resulting in a Lagrangian of Einstein-scalar type

$$\mathcal{L} = \left(\frac{R}{2} - V\right) * 1 + \frac{1}{96} \mathrm{Tr}\left(d\mathcal{M} \wedge * d\mathcal{M}^{-1}\right) \; , \tag{4.16}$$

where V is the scalar potential in (3.87). It will be convenient to use the solvable parametrisation of the scalar coset representative

$$\mathcal{V} = \exp\left(\chi^{\hat{\mathtt{e}}}\, t_{\hat{\mathtt{e}}}\right) \cdot \exp\left(\phi^{\mathtt{h}}\, t_{\mathtt{h}}\right) \; , \tag{4.17}$$

where $t_{\mathtt{h}}$ are the 7 generators of the maximal non-compact torus $\mathbb{R}^7 \subset SL(8) \subset \mathrm{E}_{7(7)}$ (Cartan subalgebra) and $t_{\hat{\mathtt{e}}}$ are the 63 positive roots of $\mathrm{E}_{7(7)}$ (computed with respect to a choice of basis for the maximal torus). We will refer to the associated scalars as dilatons $\phi^{\mathtt{h}}$ and axions $\chi^{\hat{\mathtt{e}}}$.

The coset representative $\mathcal{V}$ is a non-linear function of the 70 scalar fields used to parametrise our coset space $E_{7(7)}/SU(8)$. As such, even if the scalar potential is just a quadratic function of the T-tensor, it is a complicated function of the scalars. To find vacua, of (4.16), one would have to extremise this scalar potential i.e. solve

$$\partial_{\phi^s} V = 0 \, . \tag{4.18}$$

For general solutions, this is out of computational reach even if some progress has been made in that direction using techniques from machine learning, see for example [47, 48, 49].

To find extrema of the scalar potential, we will first perform a consistent truncation of the theory to reduce the number of scalars degrees of freedom. The "truncation" part consists in setting to zero some scalar fields, before solving the equations of motion. However, one must be careful when truncating away degrees of freedom because a solution of the truncated theory might not be a solution of the full theory. This can happen if a field set to zero is sourced by a field kept in the truncation. Truncations which do not suffer this problem are called *consistent* truncations. To ensure that the truncation is consistent, we will choose a group in the symmetry group of the theory and focus uniquely on the fields which are invariant under this group, following the original idea in [50]. This ensures that the solutions of the truncated theory are solutions of the full theory. For maximal supergravities, after the gauging procedure, the residual symmetry group is the group leaving the embedding tensor invariant. We will thus use such groups to produce our consistent truncations.

The first truncation we could focus on is a truncation by a $\mathbb{Z}_2$ subgroup of $SL(8)$ acting on its fundamental index as

$$\mathbb{Z}_2^* : \quad V^A \to (-1)^{A+1}\, A^A \ . \tag{4.19}$$

This truncation has been studied in [37] and describes a half-maximal supergravity coupled to six vector multiplets. We will study in more details this truncation and deformation thereof in chapter 8, following [7].

4.3.1 The $\mathbb{Z}_2^3$-invariant model

The $\mathcal{N} = 4$ model can be further truncated to a $\mathbb{Z}_2^3$-invariant enlarging the $\mathbb{Z}_2^*$ described in (4.19) with two other generators acting on the fundamental representation of $SL(8)$ as

$$\begin{aligned} \mathbb{Z}_2^{(1)} &= \{\mathbb{1}, \text{diag}(1, \ \ 1, \ \ 1, -1, -1, -1, -1, 1)\} \ , \\ \mathbb{Z}_2^{(2)} &= \{\mathbb{1}, \text{diag}(1, -1, -1, \ \ 1, \ \ 1, -1, -1, 1)\} \ . \end{aligned} \tag{4.20}$$

The resulting $\mathbb{Z}_2^3$ invariant sector describes a $\mathcal{N} = 1$ supergravity coupled to seven chiral multiplets (and no vector multiplets). These chiral multiplets parametrize the Kähler space

$$\mathcal{M}_{\text{scal}} = \left(\frac{SL(2)}{SO(2)}\right)^7 \subset \frac{E_{7(7)}}{SU(8)}. \tag{4.21}$$

Following the solvable parametrisation of the scalar coset space (3.46), we associate the fourteen real spinless fields with generators $t_A{}^B$ (scalars) and $t_{[ABCD]}$ (pseudo-scalars) of $\text{E}_{7(7)}$ in the SL(8) basis. The former have associated generators of the form

$$\begin{aligned} g_{\varphi_1} &= -t_1{}^1 - t_2{}^2 - t_3{}^3 + t_4{}^4 + t_5{}^5 + t_6{}^6 + t_7{}^7 - t_8{}^8 \ , \\ g_{\varphi_2} &= -t_1{}^1 + t_2{}^2 + t_3{}^3 - t_4{}^4 - t_5{}^5 + t_6{}^6 + t_7{}^7 - t_8{}^8 \ , \\ g_{\varphi_3} &= -t_1{}^1 + t_2{}^2 + t_3{}^3 + t_4{}^4 + t_5{}^5 - t_6{}^6 - t_7{}^7 - t_8{}^8 \ , \\ g_{\varphi_4} &= t_1{}^1 - t_2{}^2 + t_3{}^3 + t_4{}^4 - t_5{}^5 + t_6{}^6 - t_7{}^7 - t_8{}^8 \ , \\ g_{\varphi_5} &= t_1{}^1 + t_2{}^2 - t_3{}^3 - t_4{}^4 + t_5{}^5 + t_6{}^6 - t_7{}^7 - t_8{}^8 \ , \\ g_{\varphi_6} &= t_1{}^1 + t_2{}^2 - t_3{}^3 + t_4{}^4 - t_5{}^5 - t_6{}^6 + t_7{}^7 - t_8{}^8 \ , \\ g_{\varphi_7} &= t_1{}^1 - t_2{}^2 + t_3{}^3 - t_4{}^4 + t_5{}^5 - t_6{}^6 + t_7{}^7 - t_8{}^8 \ , \end{aligned} \tag{4.22}$$

whereas the latter correspond with generators given by

$$\begin{aligned} g_{\chi_1} &= t_{1238} \ , & g_{\chi_4} &= t_{2578} \ , & & \\ g_{\chi_2} &= t_{1458} \ , & g_{\chi_5} &= t_{4738} \ , & g_{\chi_7} &= t_{8246} \ . \\ g_{\chi_3} &= t_{1678} \ , & g_{\chi_6} &= t_{6358} \ , & & \end{aligned} \tag{4.23}$$

Exponentiating (4.22) and (4.23) with coefficients φ_i and χ_i as

$$\mathcal{V} = \text{Exp}\left[-12 \sum_{i=1}^{7} \chi_i \, g_{\chi_i}\right] \text{Exp}\left[\frac{1}{4} \sum_{i=1}^{7} \varphi_i \, g_{\varphi_i}\right] \ , \tag{4.24}$$

yields a parameterisation of the matrix $\mathcal{M} = \mathcal{V}\,\mathcal{V}^t$ on the coset space $[\text{SL}(2)/\text{SO}(2)]^7$.

This sector of the theory can be reexpressed as a $\mathcal{N}=1$ theory because only one gravitino is invariant under the $\mathbb{Z}_2^3$ action. The kinetic terms in the resulting $\mathcal{N}=1$ sector follow from (3.62) and (4.24), and are given by

$$\mathcal{L}_{kin} = -\frac{1}{4}\sum_{i=1}^{7}\left[(\partial\varphi_i)^2 + e^{2\varphi_i}\,(\partial\chi_i)^2\right] \ . \tag{4.25}$$

These match the standard kinetic terms $\mathcal{L}_{kin} = -(\partial^2_{z_i,\bar{z}_j}K)\,dz_i \wedge *d\bar{z}_j$ for a set of seven chiral fields

$$z_i = -\chi_i + e^{-\varphi_i} \tag{4.26}$$

with Kähler potential

$$K = -\sum_{i=1}^{7}\log[-i(z_i - \bar{z}_i)] \ . \tag{4.27}$$

Lastly, when restricted to the $\mathbb{Z}_2^3$ invariant sector entering (4.24), the scalar potential, as computed from (3.87), can be recovered from a holomorphic superpotential

$$W = 2\,g\,[\,z_1z_5z_6+z_2z_4z_6+z_3z_4z_5+(z_1z_4+z_2z_5+z_3z_6)\,z_7\,]+2\,g\,c\,(1-z_4z_5z_6z_7)\ , \tag{4.28}$$

using the standard $\mathcal{N}=1$ formula in (3.28). Note that only the last term in the superpotential (4.28) turns out to be sensitive to the electromagnetic parameter c. Moreover, this model is invariant under the discrete group S_4 generated by the three permutations acting on the index $i=1,\ldots,7$ as

$$f:(1,\,2)(4,\,5) \quad , \quad g:(1,\,2,\,3)(5,\,7,\,6) \quad , \quad h:(1,\,3)(4,\,5,\,6,\,7)\,. \tag{4.29}$$

4.4 Vacua in the $\mathbb{Z}_2^3$-invariant sector

In the $\mathbb{Z}_2^3$-invariant sector, we are going to focus on four different families of vacua. Each of these families is labelled by the most symmetric point of its moduli.

4.4.1 $\mathcal{N}=0$ $SO(6) \to SU(3)\times U(1) \to SU(2)\times U(1) \to U(1)^3$

There is a three-parameter family of $\mathcal{N}=0$ solutions that preserves in general $\mathrm{U}(1)^3$ and is located at

$$z_{1,2,3} = c\left(-\chi_{1,2,3} + i\,\frac{1}{\sqrt{2}}\right) \qquad \text{and} \qquad z_4=z_5=z_6=z_7=i\ , \tag{4.30}$$

with $\chi_{1,2,3}$ being arbitrary (real) parameters. This family of solutions has a vacuum energy given by

$$V_0 = -2\sqrt{2}\,g^2\,c^{-1}\ . \tag{4.31}$$

It proves convenient to introduce a set of four variables ω_A with $A=1,\ldots,4$ such that

$$\chi_i = \omega_j + \omega_k \quad (i\neq j\neq k) \qquad \text{and} \qquad \omega_1+\omega_2+\omega_3+\omega_4 = 0\ . \tag{4.32}$$

In these variables, the scalar masses can be written very symmetrically as

$$
\begin{aligned}
m^2L^2 \;\;=\;\; & 6\;\;(\times 2)\;\;\;,\;\;\;-3\;\;(\times 2)\;\;\;,\;\;\;0\;\;(\times 28)\;,\;\;\;-\tfrac{3}{4}+6\,\omega_A^2\;\;(\times 2)\;, \\
& -\tfrac{3}{4}+\tfrac{3}{2}(\omega_A+\omega_B)^2\;\;\;,\;\;\;-3+6(\omega_A+\omega_B)^2\;\;\;, \\
& -3+\tfrac{3}{2}\,(\omega_A-\omega_B)^2\;\;(\text{with } A\neq B)\;,
\end{aligned}
\tag{4.33}
$$

where $L^2 = -3/V_0$ is the AdS$_4$ radius. This family of solutions is perturbatively unstable due to the mass eigenvalue -3 lying below the Breitenlohner-Freedman bound for stability in AdS$_4$ [51] . The computation of the vector masses yields

$$
\begin{aligned}
m^2L^2 \;\;=\;\; & 0\;\;(\times 3)\;\;\;,\;\;\;6\;\;(\times 1)\;,\;\;\tfrac{9}{4}+\tfrac{3}{2}\,(\omega_A+\omega_B) \\
& \tfrac{3}{2}\,(\omega_A-\omega_B)^{\,2} \qquad (\text{with } A\neq B\,)\;.
\end{aligned}
\tag{4.34}
$$

Note that a generic solution in this family preserves a $\text{U}(1)^3$ symmetry as three vectors are generically massless. Therefore, out of the 28 massless scalars in (4.33), only 3 of them correspond to physical directions in the scalar potential. Imposing a pairwise identification between the free axions $\chi_{1,2,3}$ results in a symmetry enhancement to $\text{SU}(2)\times\text{U}(1)^2$. A further identification $\chi_1=\chi_2=\chi_3\neq 0$ implies a symmetry enhancement to $\text{SU}(3)\times\text{U}(1)$. Lastly, setting $\chi_{1,2,3}=0$ enhances the symmetry to $\text{SU}(4)\sim\text{SO}(6)$. This SO(6) symmetric solution was first found in [52] and was latter extended to the full $(\chi_1,\,\chi_2,\,\chi_3)$ moduli in [1].

4.4.2 $\mathcal{N}=1$ vacua with $\text{U}(1)^2\rightarrow\text{SU}(2)\times\text{U}(1)\rightarrow\text{SU}(3)$ symmetry

There is a two-parameter family of $\mathcal{N}=1$ supersymmetric AdS$_4$ solutions that preserves $\text{U}(1)^2$ and is located at

$$
z_{1,2,3}=c\left(-\chi_{1,2,3}+i\,\frac{\sqrt{5}}{3}\right) \qquad \text{and} \qquad z_4=z_5=z_6=z_7=\frac{1}{\sqrt{6}}(1+i\,\sqrt{5})\;, \tag{4.35}
$$

subject to the constraint

$$
\chi_1+\chi_2+\chi_3=0\;. \tag{4.36}
$$

This family of AdS$_4$ solutions has a vacuum energy given by

$$
V_0=-\frac{162}{25\sqrt{5}}\,g^2\,c^{-1}\;, \tag{4.37}
$$

and a spectrum of normalised scalar masses of the form

$$
\begin{aligned}
m^2L^2 \;\;=\;\; & 0\;\;(\times 28)\;\;\;,\;\;\;4\pm\sqrt{6}\;\;(\times 2)\;\;\;,\;\;\;-2\;\;(\times 2)\;, \\
& -\tfrac{14}{9}+5\chi_i^2\pm\tfrac{1}{3}\sqrt{4+45\chi_i^2}\;\;(\times 2) \qquad i=1,2,3\;, \\
& -\tfrac{14}{9}+\tfrac{5}{4}\chi_i^2\pm\tfrac{1}{6}\sqrt{16+45\chi_i^2}\;\;(\times 2) \qquad i=1,2,3\;, \\
& \tfrac{7}{9}+\tfrac{5}{4}\chi_i^2\;\;(\times 2) \qquad i=1,2,3\;, \\
& -2+\tfrac{5}{4}\,(\chi_i-\chi_j)^{\,2}\;\;(\times 2) \qquad i<j\;,
\end{aligned}
\tag{4.38}
$$

where $L^2=-3/V_0$ is the AdS$_4$ radius. The computation of the vector masses yields

$$\begin{aligned} m^2L^2 \;=\;& 0\ (\times 2)\ ,\ 6\ (\times 1)\ ,\ 2\ (\times 1)\ ,\\ & \tfrac{16}{9}+\tfrac{5}{4}\chi_i^2\pm\tfrac{1}{6}\sqrt{64+45\chi_i^2}\ (\times 2) \qquad i=1,2,3\ ,\\ & \tfrac{25}{9}+\tfrac{5\chi_i^2}{4}\ (\times 2) \qquad i=1,2,3\ ,\\ & \tfrac{5}{4}\left(\chi_i-\chi_j\right)^2\ (\times 2) \qquad i<j\ . \end{aligned} \tag{4.39}$$

The rescaled gravitini masses are

$$m^2L^2=1\ ,\ 4\ ,\ \frac{16}{9}+\frac{5\chi_i}{4}\ (\times 2) \qquad i=1,\,2,\,3\,. \tag{4.40}$$

Note that a generic solution in this family preserves $\mathrm{U}(1)^2$ because only two vectors are generically massless. Therefore, out of the 28 massless scalars in (4.38), only 2 of them correspond to physical directions in the potential. The residual symmetry gets enhanced to $\mathrm{SU}(2)\times\mathrm{U}(1)$ when imposing a pairwise identification between the axions $\chi_{1,2,3}$ so that a total of four vectors become massless. Finally there is a symmetry enhancement to $\mathrm{SU}(3)$ when setting $\chi_{1,2,3}=0$ so that a total of eight vectors become massless. The $SU(3)$ symmetric solution and its moduli was first found in [1] in a $U(1)^2$-invariant sector of the theory. Similar vacua had been found earlier in $SO(8)$ or $ISO(7)$ gaugings of the $\mathcal{N}=8$ theory see [53, 54].

4.4.3 $\mathcal{N}=2$ vacua with $\mathbf{U}(1)^2\to\mathbf{SU}(2)\times\mathbf{U}(1)$ symmetry

There is a one-parameter family of $\mathcal{N}=2$ supersymmetric AdS$_4$ solutions that preserves $\mathrm{U}(1)^2$ and is located at

$$z_1=-\bar{z}_3=c\left(-\chi+i\,\frac{1}{\sqrt{2}}\right)\ ,\ z_2=i\,c\ ,\ z_4=z_6=i\ \text{ and }\ z_5=z_7=\frac{1}{\sqrt{2}}(1+i)\,. \tag{4.41}$$

This family of AdS$_4$ solutions has a vacuum energy given by

$$V_0=-3\,g^2\,c^{-1}\ , \tag{4.42}$$

and a spectrum of normalised scalar masses of the form

$$\begin{aligned} m^2L^2 \;=\;& 0\ (\times 30)\ ,\ 3\pm\sqrt{17}\ (\times 2)\ ,\ -2\ (\times 4)\ ,\ 2\ (\times 6)\ ,\ -2+4\chi^2\ (\times 6)\\ & -1+4\chi^2\pm\sqrt{16\chi^2+1}\ (\times 2)\ ,\ \chi^2\pm\sqrt{\chi^2+2}\ (\times 8)\ , \end{aligned} \tag{4.43}$$

where $L^2=-3/V_0$ is the AdS$_4$ radius. The computation of the vector masses yields

$$\begin{aligned} m^2L^2 \;=\;& 0\ (\times 2)\ ,\ 6\ (\times 2)\ ,\ 4\ (\times 2)\ ,\ 2\,(\times 4)\ ,\\ & 4\chi^2\ (\times 2)\ ,\ 2+\chi^2\pm\sqrt{\chi^2+2}\ (\times 8)\ . \end{aligned} \tag{4.44}$$

The gravitni masses are

$$m^2L^2=1\ (\times 2)\ ,\ 4\ (\times 2)\ ,\ 2+\chi^2\ (\times 4) \qquad . \tag{4.45}$$

Note that a generic solution in this family preserves $\mathrm{U}(1)^2$ as only two vectors are generically massless. Therefore, out of the 30 massless scalars in (4.43), only 4 of them correspond to physical directions in the scalar potential. However, the residual symmetry gets enhanced to $\mathrm{SU}(2) \times \mathrm{U}(1)$ when $\chi = 0$ and two additional vectors become massless.

4.4.4 $\mathcal{N} = 4$ vacuum with SO(4) symmetry

This family is just a point with $\mathcal{N} = 4$ and $SO(4)$. It is located at

$$z_1 = z_2 = z_3 = i\,c \qquad z_4 = z_5 = z_6 = -\bar{z}_7 = \frac{1+i}{\sqrt{2}} \,. \tag{4.46}$$

This AdS_4 solution has a vacuum energy given by

$$V_0 = -3\, g^2\, c^{-1} \,. \tag{4.47}$$

as for the previous solution, and a spectrum of normalised scalar masses of the form

$$m^2L^2 \;=\; 0\,(\times 48) \quad , \quad 10\,(\times 1) \quad , \quad 4\,(\times 10) \quad , \quad -2\,(\times 11)\,, \tag{4.48}$$

where $L^2 = -3/V_0$ is the AdS_4 radius. The computation of the vector masses yields

$$m^2L^2 \;=\; 0\,(\times 6) \quad , \quad 6\,(\times 7) \quad , \quad 2\,(\times 15)\,, \tag{4.49}$$

This $\mathcal{N} = 4$ solution was first reported in [38], and then uplifted to a ten-dimensional family of type IIB S-fold backgrounds in [55].

4.5 Remarks

In the next chapters, we will study the uplift of these solutions to Type IIB supergravity. We will start by uplifting the most symmetric points of each moduli in section 5.4. We will then study in more details the axionic deformations and their uplifts in chapter 6. This discussion about the uplift will allow us to make some conjecture about the would be CFT_3 duals of these solutions. However, before presenting these results let us make two remarks, one concerning a family of solutions not presented in the main text and one concerning the moduli of our solutions parametrized by axions χ's.

The φ-family of solutions

We have not presented an important family of solutions present in our $\mathbb{Z}_2^3$ invariant sector. This family is parametrized by a parameter $\varphi \in \mathbb{R}$ and was found in [56]. In this φ-family of solutions, the scalars take the vev:

$$\begin{aligned} z_1 = z_2 &= \frac{i\,c\,\sqrt{\varphi^2+1}}{\sqrt{2}} \quad , \quad z_3 = i\,c \\ z_4 = z_5 &= \frac{1+i}{\sqrt{2}} \qquad\quad , \quad z_6 = -\bar{z}_7 = \frac{-\varphi+i}{\sqrt{\varphi^2+i}} \,. \end{aligned} \tag{4.50}$$

This family interpolate between the $\mathcal{N} = 4$ $SO(4)$ solution (4.46) at $\varphi = 1$ and the $\mathcal{N} = 2$ $SU(2) \times U(1)$ solution of (4.41) at $\varphi = 0$ (up to the action of an S_4 symmetry as presented in (4.29)). The whole family preserves $\mathcal{N} = 2$ supersymmetry as well as a $U(1)^2$ global symmetry. This family of solution is stable, and its spectrum has been computed and organized as $\mathfrak{osp}(2|4)$ supermultiplets. However, it has not been uplifted to Type IIB supergravity yet. The issue with the uplift is not a conceptual one but a computational one. It would be worth making some efforts in this direction because, although the moduli seems to be non-compact in four dimensions, this behavior might change from a ten-dimensional perspective (e.g. see the flat deformations of chapter 6). From unpublished preliminary works, we expect that, as $\varphi \to \infty$, the solution should reach some (de)compactification limit where the supergravity approximation of this solution would break down, as the swampland distance conjecture would suggest [57].

Axionic deformations

The $\mathcal{N} = 0, 1, 2$ vacua all include axionic deformations generated by scalars associated to generators of $\mathfrak{e}_7$ of the form $\chi^{ij} t_{1ij8}$. These deformations induce a pattern of symmetry breaking

$$SU(n) \to SU(n-1) \times U(1) \to \cdots . \tag{4.51}$$

In the first two solutions, $\mathcal{N} = 0$ and $\mathcal{N} = 1$, there are as many axionic deformations as the rank of the residual symmetry group. At this point, this observation might seem accidental since this is not the case of the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ solutions. However, since we are working in a consistent truncation of maximal supergravity, it is possible that there are axionic deformations in the full theory which are not present in our truncation. As we will see in chapter 6, this is the case. For each solution $\mathcal{V}_0$ with a symmetry group of rank r, we will be able to build an r-dimensional family of deformations.

Chapter 5

Consistent compactifications through exceptional field theory

In this section, we will review how to build consistent Kaluza-Klein (KK) compactification of Type IIB supergravity down to four dimensions. A KK compactification of any theory relies first on a splitting of the coordinates in a set of internal coordinates and a set of external coordinates

$$X^A \to (\underbrace{x^\mu}_{\text{external}}, \underbrace{y^m}_{\text{internal}}) . \tag{5.1}$$

From this splitting, one provides an ansatz describing how the fields in the higher dimensional theory can depend on the internal coordinates. For example, the relevant ansatz to study compactifications on a circle would be the decomposition in Fourier modes

$$\phi(x^\mu, y) = \sum_n \phi_n(x^\mu) e^{i\,n\,y} . \tag{5.2}$$

With this ansatz, the theory is described by an infinite number of fields ϕ_n depending only on the external coordinates x^μ. If one is simply interested in an *effective* description of the full theory, it is possible to truncate away massive modes and keep only massless modes. This description would have a finite number of degrees of freedom but it would only be valid in a low-energy limit because massless fields could source massive fields. This means that solutions of the truncated theory might not uplift to solutions of the full theory.

What we are interested in when uplifting low-dimensional solutions are *consistent* truncations of supergravities. A truncation is consistent if the solutions of the low-dimensional theory uplift to solutions of the higher dimensional theory at all orders. To do this, the ansatz and the truncation procedure must conspire in such a way that all dependencies in the internal coordinates factor out of the equations of motion. For the compactification on a circle, this is what happens when keeping only the $U(1)$-invariant modes, i.e. the massless modes.

In a generic setting, finding the right truncation and ansatz for a consistent compactification on a general manifold is a surprisingly hard problem. A lot of progress has been made in the case of consistent compactifications of type IIB and 11d supergravity. This progress stems from the $E_{d(d)}$ global symmetries appearing when compactifying these theories on tori. These symmetry groups indicate that there should be a way to understand higher dimensional theories in an $E_{d(d)}$ covariant way. The higher dimensional origin of these global symmetry groups is not completely clear because the exceptional symmetry groups only appear after the dualisation of certain degree of freedom (e.g. two-form to scalar in four dimensions). Since some of these symmetries are not visible in the higher dimensional setting we refer to them in general as *hidden symmetries*. Such hidden symmetries appear in a host of compactification, the prototypical example being the compactification of pure 4d gravity on circle which becomes $SL(2)$-invariant instead of the $GL(1) \times \mathbb{R}$-invariance expected from geometric considerations (see [58] for a recent review on hidden symmetries in the context of maximal supergravity).

The goal of Exceptional Field Theory (ExFT) is to reorganize the degrees of freedom and the equations of motion of the higher-dimensional supergravity in a way that is compatible with these exceptional symmetry groups. The emergence of ExFT mirrors that of Double Field Theory (DFT), which describe heterotic string theory studied theories with a $SO(D, D)$ duality group.

The mathematical premises of DFT arises in the works [59, 60] devoted to the study of generalized complex geometry. There, the authors studied the geometry of vector bundles which are extensions of the tangent bundle of the form $TM \oplus T^*M$. These admit an $SO(D, D)$ structure on M. The reduction of this structure bundle to an $SO(D) \times SO(D)$ bundle is equivalent to the presence of a generalized metric $\mathcal{M}$. This metric encodes the internal geometry of the NS-NS sector of string theory compactified on M in a $SO(D, D)$ covariant way. The equations of motion in the internal manifold then reduces to the vanishing of a generalized curvature for the metric $\mathcal{M}$. Then, the DFT shows how to make this metric dynamical with respect to the external space [61, 62, 63]. To do so, one writes a pseudo action on a formally $2D$ internal manifold and then imposes section constraints. These constraints reduce the effective number of dimensions on the internal manifold to D. Upon imposing these section constraints, the DFT is a reorganisation of the full NS-NS sector of string theory in a $SO(D, D)$ covariant way.

For the exceptional groups, the exceptional generalized geometry encodes the NS-NS and R-R sectors of superstrings [64, 65]. The field theory of this geometry, ExFT, was introduced in [66]. Now, some more refined setups exists which can encode both generalized geometry and exceptional geometry in different dimensions in terms of G-algebroid [67]. However, there is not yet an unified framework for the field theory of G-algebroid encompassing both ExFT and DFT.

This chapter is organised as follows. In section 5.1, we will review $E_{7(7)}$-ExFT and its generalised diffeomorphism structure. In section 5.2, we will show how Type IIB fields are encoded in $E_{7(7)}$-ExFT by solving the "section constraints" of ExFT. In section 5.3, we will study how to consistently compactify ExFT down to a maximal gauged supergravity in four dimensions using *generalised Scherk-Schwarz* compactifications. We will focus on this procedure for the $[SO(1, 1) \times SO(6)] \ltimes \mathbb{R}^{12}$ gauged maximal supergravity, which is the compactification of Type IIB supergravity on $S^1 \times S^5$. Finally, in section 5.4, we will uplift the solutions we found in the $\mathbb{Z}_2^3$ invariant sector of maximal supergravity.

5.1 $E_{7(7)}$-Exceptional Field Theory

The $E_{7(7)}$-ExFT is a theory on a $4 + 56$ dimensional space-time. The first four coordinates, x^μ, parametrise the *external space* while the others, Y^M, parametrise the *internal space*. The internal coordinates Y^M are naturally in the **56**-dimensional representation of $E_{7(7)}$. The field content of the ExFT is

$$\left\{g_{\mu\nu},\, \mathcal{M}_{MN},\, A_\mu{}^M,\, B_{\mu\nu\,\alpha},\, B_{\mu\nu\,M}\right\} \tag{5.3}$$

where $g_{\mu\nu}$ is a metric on the external space, $\mathcal{M}_{MN}$ is an internal generalised metric parametrising the coset space $E_{7(7)}/SU(8)$, $A_\mu{}^M$ will be a gauge connection and the two-forms $B_{\mu\nu\,\alpha}$ and $B_{\mu\nu\,M}$ transform in the adjoint and the fundamental representations of $E_{7(7)}$ respectively.

What separates ExFT from usual field theories is the presence of *section constraints*. They require that

$$(t_\alpha)^{MN}\partial_M\partial_N A = 0\,, \qquad \Omega^{MN}\partial_M A\partial_N B = 0\,, \qquad (t_\alpha)^{MN}\partial_M A\partial_N B = 0\,, \tag{5.4}$$

for any fields A and B of the theory. The symbols t_α are the generators of $\mathfrak{e}_7$ and Ω^{MN} is the symplectic metric of $Sp(56,\,\mathbb{R})$. This severely restricts the possible dependencies of the fields in the internal coordinates. We will show later how to solve these section constraints, the main result being that they admit only two inequivalent solutions. In the first solution, the fields depend non-trivially on exactly 6 coordinates, corresponding to a formulation of Type IIB supergravity. In the other solution, the field depend on 7 coordinates, corresponding to a formulation of 11d supergravity.

The bosonic equations of motion of ExFT are completely determined by requiring the invariance under the generalised diffeomorphism in both the external and internal coordinates. The equations of motions can be computed from the pseudo-action

$$\begin{aligned} S = \int \mathrm{d}^4x\,\mathrm{d}^{56}Y\,e\,\Big(\hat{R} + \frac{1}{48}g^{\mu\nu}\mathcal{D}_\mu\mathcal{M}^{MN}\mathcal{D}_\nu\mathcal{M}_{MN} \\ - \frac{1}{8}\mathcal{M}_{MN}F^{\mu\nu\,M}F_{\mu\nu}{}^N + e^{-1}\mathcal{L}_{top} - V\Big)\,, \end{aligned} \tag{5.5}$$

supplemented with the self-duality equations

$$\mathcal{F}_{\mu\nu}{}^M = -\frac{1}{2}e\,\epsilon_{\mu\nu\rho\sigma}\,\Omega^{MN}\,\mathcal{M}_{NK}\,\mathcal{F}^{\rho\sigma K}\,, \tag{5.6}$$

for a field-strength $\mathcal{F}_{\mu\nu}{}^M$ we will define later. Notice how the ExFT action is reminiscent of the action of $\mathcal{N}=8$ supergravity in 4 dimensions (3.86).

The generalised diffeomorphisms are defined in terms of a generalised Lie derivative with respect to a vector parameter Λ^M. A vector V^M of weight λ transforms as:

$$\delta_\Lambda V^M = \mathbb{L}_\Lambda V^M = \Lambda^K\partial_K V^M - \mathbb{P}_{133}{}^M{}_N{}^K{}_L\partial_K\Lambda^L V^N + \lambda\partial_P\Lambda^P V^M\,. \tag{5.7}$$

This is the usual definition of the Lie derivative of a vector of weight λ up to two important discrepancies. The first one is that, after solving the section constraints, the derivatives ∂_K are non-trivial only for a subset of the 56 possible values of K. The second is the presence of the projector $\mathbb{P}_{133}$:

$$\mathbb{P}_{133}{}^K{}_M{}^N{}_L = (t_\alpha)_M{}^K(t^\alpha)_N{}^L\,, \tag{5.8}$$

which projects on the 133-dimensional representation of $\mathfrak{e}_7$. The definition of the generalized Lie derivative can be extended to an arbitrary number of fundamental indices.

The section constraints implies that some gauge transformations parametrized by a vector Λ^N are actually trivial. For example, this is the case for all gauge parameters of the form

$$\Lambda^M = (t^\alpha)^{MN}\,\partial_N\chi_\alpha \quad \text{ or } \quad \Lambda^M = \Omega^{MN}\chi_N \tag{5.9}$$

for χ_N satisfying triviality conditions of the form

$$(\mathbb{P}_{1+133})^{MN}\chi_M\partial_N = 0 = (\mathbb{P}_{1+133})^{MN}\chi_M\,\chi_N\,. \tag{5.10}$$

The triviality of the gauge transformations of the type (5.9) follows directly from the section constraints. These trivial gauge transformations are what allows the algebra of generalised diffeomorphisms to close. We can check this by computing that, up to the section constraints, we have

$$[\delta_{\Lambda_1},\,\delta_{\Lambda_2}] = \delta_{[\Lambda_1,\,\Lambda_2]_E} \tag{5.11}$$

where the "E-bracket" is defined as

$$\begin{aligned}[\Lambda_1,\,\Lambda_2]_E^M = 2\,\Lambda_{[1}^K\partial_K\Lambda_{2]}^M + 12\,(t_\alpha)^{MN}(t^\alpha)_{KL}\,\Lambda_{[1}^K\partial_N\Lambda_{2]}^L \\ -\frac{1}{4}\,\Omega^{MN}\Omega_{KL}\,\partial_N(\Lambda_{[1}^K\Lambda_{2]}^L)\,.\end{aligned} \tag{5.12}$$

This E-bracket, although similar to the Lie bracket, does not satisfy the Jacobi identity. It satisfies a weaker identity implying that the failure to satisfy Jacobi identity is a term producing trivial gauge transformations. In other words, we have that

$$[\,[\delta_{[\Lambda_1},\,\delta_{\Lambda_2}],\delta_{\Lambda_3]}] = 0 \tag{5.13}$$

only after applying the section constraints and despite the fact that

$$[[\Lambda_{[1},\,\Lambda_2]_E,\,\Lambda_{3]}]_E \neq 0\,. \tag{5.14}$$

This is still sufficient for the internal diffeomorphism algebra to close.

Let us now discuss the action of the generalized Lie derivative on the adjoint representation. We define:

$$\mathbb{L}_\Lambda W_\alpha = \Lambda^K\partial_K W_\alpha + 12\,f_{\alpha\beta}{}^\gamma\,(t^\beta)_L{}^K\,\partial_K\Lambda^L\,W_\gamma + \lambda\,\partial_K\Lambda^K\,W_\alpha\,, \tag{5.15}$$

where $f_{\alpha\beta}{}^\gamma$ are the $\mathfrak{e}_7$ structure constant and $(t^\beta)_N{}^M$ are the $\mathfrak{e}_7$ generators acting on the fundamental indices. Because of the constructions we are going to study below, we also want to study how the following quantity transforms:

$$V^M = (t^\alpha)^{MN}\partial_N W_\alpha\,. \tag{5.16}$$

This quantity does not transform as a vector in the fundamental representation under the generalised Lie derivative. To compensate, we have to add a field W_M subject to the "triviality constraints":

$$(\mathbb{P}_{1+133})^{MN}W_M\partial_N = 0 = (\mathbb{P}_{1+133})^{MN}W_M\,W_N\,. \tag{5.17}$$

Then, if W_α has weight $\lambda = 1$, the sum

$$\hat{V}^M = (t^\alpha)^{MN}\partial_N W_\alpha + \frac{1}{24}\Omega^{MN}W_N \tag{5.18}$$

transforms as a vector of weight $\lambda = \frac{1}{2}$, provided that the compensating field W_M transforms as

$$\delta_\Lambda W_M = \mathbb{L}_\Lambda W_M - 24(t^\alpha)_L{}^K W_\alpha\partial_M\partial_K\Lambda^L\,. \tag{5.19}$$

These results will be useful when describing the transformations of B_α and B_M.

From the generalised Lie derivative, we can now define a covariant derivative w.r.t. the gauge connection ${A_\mu}^M$ defined as

$$\mathcal{D}_\mu = \partial_\mu - \mathbb{L}_{A_\mu} . \tag{5.20}$$

By definition, the covariant derivative should transform covariantly, this imposes the gauge transformations of the connection to be

$$\delta_\Lambda {A_\mu}^M = \mathcal{D}_\mu \Lambda^M \tag{5.21}$$

where the gauge transformation parameter Λ^M is a tensor of weight $\lambda = \frac{1}{2}$. This allows to define the Yang-Mills field strength associated to ${A_\mu}^M$ as

$${F_{\mu\nu}}^M = 2\partial_{[\mu} {A_{\nu]}}^M - [A_\mu,\, A_\nu]_E^M . \tag{5.22}$$

However, this field strength is not covariant under generalised diffeomorphisms. To compensate for this, we define a fully covariantized field strength as

$${\mathcal{F}_{\mu\nu}}^M = {F_{\mu\nu}}^M - 12(t^\alpha)^{MN}\partial_N B_{\mu\nu\alpha} - \frac{1}{2}\Omega^{MK} B_{\mu\nu\, K} \tag{5.23}$$

where we recognize the quantity $\hat{B}^M$ as the compensator. Importantly, the two-form $B_{\mu\nu\, M}$ is also constrained by the triviality constraints:

$$(\mathbb{P}_{1+133})^{MN} B_{\mu\nu\, M}\partial_N = 0 = (\mathbb{P}_{1+133})^{MN} B_M\, B_N . \tag{5.24}$$

If one defines the gauge transformations of the two-forms to be

$$\begin{aligned} \Delta_\Lambda B_{\mu\nu\,\alpha} &= (t_\alpha)_{KL}\Lambda^K {\mathcal{F}_{\mu\nu}}^L \\ \Delta_\Lambda B &= -\Omega_{KL}\left({\mathcal{F}_{\mu\nu}}^K\partial_M\Lambda^L - \Lambda^L\partial_M{\mathcal{F}_{\mu\nu}}^K\right) \end{aligned} \tag{5.25}$$

then, the field strength $\mathcal{F}$ transforms covariantly as a vector of weight $\frac{1}{2}$. Finally the two two-forms carry their own gauge transformations under which fields transform as

$$\begin{aligned} \delta {A_\mu}^M &= 12(t^\alpha)^{MN}\partial_N \Xi_{\mu\,\alpha} + \frac{1}{2}\Omega^{MN}\,\Xi_{\mu\, N} \\ \Delta B_{\mu\nu\,\alpha} &= 2\mathcal{D}_{[\mu}\Xi_{\nu]\alpha} \\ \Delta B_{\mu\nu\, M} &= 2\mathcal{D}_{[\mu}\Xi_{\nu]M} + 48{(t^\alpha)_L}^K\left(\partial_K\partial_M {A_{[\mu}}^L\right)\Xi_{\nu]\alpha} \end{aligned} \tag{5.26}$$

where $\Xi_{\mu\alpha}$ is of weight 1 and $\Xi_{\mu\, M}$ is of weight $\frac{1}{2}$.

We now have all the information required to build the action of exceptional field theory. First, we build the kinetic terms. These are given by a covariantized Einstein-Hilbert term build out of an improved Riemann tensor:

$${\hat{R}_{\mu\nu}}{}^{ab} = {R_{\mu\nu}}^{ab}(\omega) + {\mathcal{F}_{\mu\nu}}^M e^{a\rho}\partial_M e^b_\rho \tag{5.27}$$

where ${R_{\mu\nu}}^{ab}$ is the curvature of the spin connection given in term of the vielbein with all derivatives being covariantized i.e.

$$\mathcal{D}_\mu {e_\nu}^a = \partial_\mu {e_\nu}^a - {A_\mu}^M\partial_M {e_\nu}^a - \frac{1}{2}\partial_M {A_\mu}^M {e_\nu}^a . \tag{5.28}$$

Considering the generalised metric $\mathcal{M}_{MN}$ as a weight zero tensor, the first three terms in the action of (5.5) are defined, giving a generalized Einstein-Hilbert action, a generalized σ-model kinetic term and a generalized Yang-Mills action. The topological term in the ExFT action is more easily understood as the boundary term of a manifestly gauge-invariant exact form in five dimensions:

$$\begin{aligned}\mathcal{S}_{top} &= -\frac{1}{24}\int_{\Sigma_5} \mathrm{d}^5 x \int \mathrm{d}^{56} Y \, \epsilon^{\mu\nu\rho\sigma\tau} \mathcal{F}_{\mu\nu}{}^M \mathcal{D}_\rho \mathcal{F}_{\sigma\tau\, M} \\ &= \int_{\partial\Sigma_5} \mathrm{d}^4 x \int \mathrm{d}^{56} Y \mathcal{L}_{top}\end{aligned} \tag{5.29}$$

We still need to specify the "potential" V, which is

$$\begin{aligned}V = &-\frac{1}{48}\mathcal{M}^{MN}\partial_M \mathcal{M}^{KL}\partial_N \mathcal{M}_{KL} + \frac{1}{2}\mathcal{M}^{MN}\partial_M \mathcal{M}^{KL}\partial_L \mathcal{M}_{NK} \\ &- \frac{1}{2} g^{-1}\partial_M g \partial_N \mathcal{M}^{MN} - \frac{1}{4}\mathcal{M}^{MN} g^{-1}\partial_M g g^{-1}\partial_N g - \frac{1}{4}\mathcal{M}^{MN}\partial_M g^{\mu\nu}\partial_N g_{\mu\nu} \,.\end{aligned} \tag{5.30}$$

We used the nomenclature of "potential" for this last term because the only derivative present differentiate w.r.t. the internal directions. As such, if we can write an ansatz for which the internal coordinates factorize, we would have a true scalar potential, which do not depend on the derivatives of the fields.

The various pieces of the exceptional field theory action have been build by enforcing an invariance under generalised internal diffeomorphism. Now, the relative coefficients of these pieces are determined by the generalised external diffeomorphism that we are not going to review here. This bosonic action can be extended to include fermions [68]. The fact that there is a completion of this action compatible with supersymmetry is very surprising. The theory we have written is completely fixed by the external and $E_{7(7)}$ generalised diffeomorphism and knows, *a priori*, nothing of the supersymmetry of Type IIB or 11d supergravity it is supposed to encode. However, a supersymmetric completion seems to be built in the structure of the exceptional group.

5.2 Solving the section constraints

Now that we have a theory which is manifestly invariant under both external and internal generalised diffeomorphisms, we want to relate it to the 10 and 11 dimensional supergravities. This is done by solving the section constraints. It was shown that there are two solutions of the section constraints of $E_{7(7)}$-ExFT [69, 70] corresponding to Type IIB and 11d supergravities. We will focus here on Type IIB interpretation of $E_{7(7)}$-ExFT. Since Type IIB supergravity is 10 dimensional, we expect to solve the section constraints by imposing that the fields depend on only six specific internal coordinates:

$$\phi(x^\mu, Y^M) \to \phi(x^\mu, y^m) \tag{5.31}$$

for a judicious choice of $y^m \subset Y^M$. In this way, the internal derivatives ∂_M are effectively zero for $M \neq m$. To identify the six relevant coordinates y^m, we first study the branching of $E_{7(7)}$ under its subgroup $GL(6) \times SL(2)$ which encodes the diffeomorphisms on the internal manifold and the SL(2) global symmetry of Type IIB supergravity. Under this branching we have

$$
\begin{aligned}
\mathbf{56} &\to (6,\,1)_{+2} + (6',\,2)_{+1} + (20,\,1)_0 + (6,\,2)_{-1} + (6',\,1)_{-2} \\
\mathbf{133} &\to (1,2)_{+3} + (15',1)_{+2} + (15,2)_{+1} + (35+1,1)_0 + \\
&\quad (1,2)_{-3} + (15,1)_{-2} + (15',2)_{-1}
\end{aligned}
\tag{5.32}
$$

where the index denotes the $GL(1) \subset GL(6)$ charge of each irrep. This branching corresponds to the splitting of coordinates

$$
\{Y^M\} \to \{y^m,\, y_{m\,a},\, y_{kmn},\, y^{m\,a},\, y_m\}\,. \tag{5.33}
$$

Selecting the coordinates y^m to be the coordinates of highest $GL(1)$ weight solves the section constraints. Indeed, we have that $\Omega^{MN}\partial_M A \partial_N B = 0$ because Ω_{MN} is composed of off-diagonal blocks, thus $\Omega^{mn} = 0$. Moreover, $(t_\alpha)^{MN}\partial_M A\, \partial_N B = 0$ because

$$
(t_\alpha)^{mn} = 0\,. \tag{5.34}
$$

Indeed, $(t_\alpha)^{MN}\partial_M\partial_N$ transforms in the **133** of $\mathfrak{e}_7$. Thus, for $(t_\alpha)^{mn}\partial_m\partial_n$ to be non-zero, $(t_\alpha)^{mn}$ should be of charge +4, to compensate for the charge −2 of ∂_m. We can check that there is no generators of charge +4 in the decomposition of the adjoint representation. Hence, the coordinates y^m solve the section constraints.

Once the IIB solution of the section constraints is enforced, we must express the degree of freedom of the ExFT in terms of SUGRA fields. For example, the generalized metric $\mathcal{M} = \mathcal{V}\mathcal{V}^T$ is parametrized in terms of the fields of the IIB supergravity following the splitting of the adjoint representation of $E_{7(7)}$. To recover the equations of motions of Type IIB supergravity we define

$$
\mathcal{V} = \exp\left(\phi\, t_{(0)}\right) \mathcal{V}_6 \mathcal{V}_2 \exp\left(c_{mn\,a}\, t_{(+1)}^{mn\,a}\right) \exp\left(\epsilon^{klmnpq} c_{klmn}\, t_{(+2)\,pq}\right) \exp\left(c_a\, t_{(+3)}^{a}\right)\,. \tag{5.35}
$$

As an illustration, the scalar field $c_{mn\,a}$ has its origin in the internal component of the two-forms doublet whereas c_a originates from the dual of the two-forms: a six form in six dimensions. Obtaining the explicit identification between Type IIB fields and ExFT fields is not as straightforward and extra re-definitions and dualities need to be made to recover the Type IIB equations of motion.

As a comment, notice that the decomposition of the ExFT vector ${A_\mu}^M$ encodes the the KK vectors ${A_\mu}^m$, coming from the metric, as well as part of the higher dimensional forms $A_{\mu\,ma}$, $A_{\mu\,kmn}$ and ${A_\mu}^{m\,a}$. A similar reasoning holds for the two form $B_{\mu\nu\,\alpha}$. The discussion is a bit more subtle when considering the two-form $B_{\mu\nu\,M}$. Indeed, $B_{\mu\nu\,M}$ is constrained by the same equations as the section constraints. As such, only the $B_{\mu\nu\,m}$ components are non-vanishing once the section constraints are imposed. Finally, we note that the self-duality equation with index $M = m$ have no interpretation in the context of Type IIB supergravity as these modes correspond to off-diagonal modes of the ten-dimensional graviton.

Setting the vectors and the two-forms to zero (which can be done consistently), the dictionary between Type IIB fields and ExFT fields is

$$
\mathcal{M}^{mn} = G^{-1/2} G^{mn} \tag{5.36}
$$

$$
{\mathcal{M}^m}_{n\alpha} = G^{-1/2}\, G^{mk}\, B^\beta_{kn} \epsilon_{\beta\alpha} \tag{5.37}
$$

$$
\mathcal{M}_{m\alpha\, n\beta} = G^{-1/2}\, G_{mn}\, m_{\alpha\beta} + G^{-1/2}\, G^{kl}\, B^\gamma_{mk}\, B^\delta_{nl} \epsilon_{\alpha\gamma}\epsilon_{\beta\delta} \tag{5.38}
$$

$$
{\mathcal{M}^\rho}_{lmn} = -2\, G^{-1/2}\, G^{\rho k}\left(C_{klmn} - \frac{3}{2}\,\epsilon_{\alpha\beta} B^\alpha_{k[l}\, B^\beta_{mn]}\right) \tag{5.39}
$$

These have to be supplemented by the self-duality condition for the four-form which determine the external part of the four-form $C_{\mu\dots}$. Of course, these equations can be inverted to obtain the Type IIB fields in terms of the ExFT generalized metric see (5.64).

5.3 Generalized Scherk-Schwarz reduction

Using the results of the previous sections, we have all the background necessary to build consistent compactifications of Type IIB supergravity. In ExFT, there is a natural splitting of coordinates into internal and external coordinates and a formulation of Type IIB which invariant under generalised diffeomorphisms. We still need an ansatz for the compactification. The ansatz we are going to study is called a *generalized Scherck-Schwarz* (gSS) ansatz. Its construction and properties were studied in [45]. Guided by the covariance of the fields under $E_{7(7)}$, we postulate that:

$$g_{\mu\nu}(x,Y) = \rho^{-2}(Y)\, g_{\mu\nu}(x) \tag{5.40}$$

$$\mathcal{M}_{MN}(x,Y) = U_M{}^K(Y)\, U_N{}^L(Y)\, M_{KL}(x) \tag{5.41}$$

$$\mathcal{A}_\mu{}^M(x,Y) = \rho^{-1}\, A_\mu{}^N(x)\, (U^{-1})_N{}^M(Y) \tag{5.42}$$

$$\mathcal{B}_{\mu\nu\,\alpha}(x,Y) = \rho^{-2}(Y)\, U_\alpha{}^\beta(Y)\, B_{\mu\nu\,\beta}(x) \tag{5.43}$$

$$\mathcal{B}_{\mu\nu\,M}(x,Y) = -2\,\rho^{-2}(Y)\,(U^{-1})_S{}^P(Y)\,\partial_M U_P{}^R(Y)\, B_{\mu\nu\,\alpha}(x)\,(t^\alpha)_R{}^S \tag{5.44}$$

The entire dependency in the internal coordinates is encoded in a $E_{7(7)}$-valued matrix $U_M{}^N(Y)$, called the *twist* matrix, and a scalar field $\rho(Y)$ called the *scale factor*. This ansatz will produce consistent compactifications only if, after solving the section constraints, the twist matrix and the scale factors satisfy the following set of differential equations:

$$\begin{aligned}\left[(U^{-1})_M{}^P(U^{-1})_N{}^Q\partial_P U_Q{}^K\right]_{\mathbf{912}} &= \frac{1}{7}\,\rho\,\Theta_M{}^\alpha\,(t_\alpha)_N{}^K \\ \partial_N(U^{-1})_M{}^N - 3\rho^{-1}\partial_N\rho(U^{-1})_M{}^N &= 2\,\rho\,\vartheta_M\end{aligned} \tag{5.45}$$

where Θ and ϑ are constant vectors and $[\cdot]_{\mathbf{912}}$ is the projector on the **912** representation of $E_{7(7)}$. These equations are called the *consistency constraints*. If the gSS ansatz satisfies such constraints, the action of generalized diffeomorphisms with parameter $\Lambda^M = \rho^{-1}\Lambda^P(x)(U^{-1})_P{}^M$ acts as gauge transformations on the 4d fields. Moreover, if these equations are satisfied, the dependencies in the internal coordinates Y^M factorise out of the equations of motion of the ExFT theory.

The remaining x^μ-dependent equations reduce to the ones of maximal supergravity in four dimensions. The field content of maximal supergravity is precisely the set

$$\left\{g_{\mu\nu}(x),\, M_{MN}(x),\, A_\mu{}^M(x),\, B_{\mu\nu\,\alpha}(x)\right\} \tag{5.46}$$

that appears in the gSS ansatz (5.40) and the gauging is encoded in the embedding tensor $\Theta_M{}^\alpha$ appearing in the condition (5.45).[1] All this detour by ExFT has allowed to reduce the problem of finding a consistent compactification of Type IIB supergravity to the one of finding a triplet (ρ, U, Θ) satisfying (5.45). This problem, although it is simpler, is not simple. Indeed, given an embedding tensor $\Theta_M{}^\alpha$ in maximal

[1] The vector ϑ_M is related to "trombone" gaugings and have to do with an extra possible GL(1) gauging. We set it to zero in this work.

supergravity, it is not always possible to find a twist matrix satisfying the consistency constraints (5.45) *and* the section constraints (5.4). This was studied in [44].

The simplest example of un-upliftable theories is the class of $SO(8)_\omega$-gaugings for $\omega \in [0,\, \pi/4[$ we discussed in the introduction of chapter 4. At $\omega = 0$, there exists a twist matrix U solving both consistency and section constraints to 11d supergravity. At $\omega \neq 0$, the solutions of (5.45) always require more than seven extra-coordinates to be solved [44, 43] and thus do not have a geometric interpretation as the compactification of a supergravity. Whether or not these theories have an interpretation in string theory remains an open question.

5.3.1 Solutions for dyonic gaugings

There is a large class of gaugings for which there exists twist-matrices U solving the consistency condition as well as the section constraints [55]. These gaugings are the

$$[SO(p,\, q) \times SO(p',\, q')] \ltimes N \;\subset\; SL(8) \;\subset E_{7(7)} \tag{5.47}$$

gaugings studied in section 4.2. Their embedding tensors are built in terms of two symmetric 8×8 matrices θ and $\tilde\theta$ -see (4.6)- of the form

$$\theta_{AB} = g\,(\overbrace{1,\,\dots,\,1}^{p},\overbrace{-1,\,\dots,\,-1}^{q},0,\,\dots,\,0) \tag{5.48}$$

$$\tilde\theta^{BC} = c\,(0,\,\dots,\,0,\underbrace{1,\,\dots,\,1}_{p'},\underbrace{-1,\,\dots,\,-1}_{q'})\,. \tag{5.49}$$

These gaugings are in general dyonic: the vectors gauging the $SO(p,\, q)$ factor are electric while the vectors gauging the $SO(p',\, q')$ factor of the gauge group are magnetic. The nilpotent group N is in general gauged by both types of vectors.

Since the gauge group is a subgroup of $SL(8)$, it is natural to decompose the 56 internal coordinates in a $SL(8)$ basis as

$$Y^M \to Y^{[AB]} \oplus Y_{[AB]} \text{ with } A,\, B = 1,\,\dots,\,8\,. \tag{5.50}$$

We will assume that there exists coordinates $y^i \subset Y^{[AB]}$, called *electric* coordinates, and $y_a \subset Y_{[AB]}$, called *magnetic* coordinates, which solve the section constraints. These coordinates can be taken to be

$$y^i = Y^{i8} \qquad y_a = Y_{a7} \qquad \text{for } \; i = 1,\,\dots,\,p+q-1 \text{ and } a = p+q,\,\dots,\,6\,, \tag{5.51}$$

in the SL(8) basis. The twist matrix U will be an element of $SL(8) \subset E_{7(7)}$ that can be expressed as

$$(U^{-1})_A{}^B = (\dot\rho\hat\rho^{-1})^{1/2}\begin{pmatrix} \dot W_a{}^b & 0 & 0 & \dot\rho^{-2}\dot W_a{}^0 \\ 0 & \hat V_i{}^j & \hat\rho^2\hat V_i{}^0 & 0 \\ 0 & \hat\rho^2\hat V_0{}^j & \hat\rho^4 & 0 \\ \dot\rho^{-2}\dot W_0{}^b & 0 & 0 & \dot\rho^{-4}\left(1+\dot u\dot K(\dot u,\,\dot v)\right) \end{pmatrix} \tag{5.52}$$

in the fundamental representation of $SL(8)$. We have used the notations:

$$\hat V_0{}^i = \eta_{ij}y^j\,\hat K(\hat u,\,\hat v) \qquad \hat V_i{}^0 = \eta_{ij}y^j \qquad \hat V_i{}^j = \delta^{ij} + \eta_{ik}y^k\,\eta_{jl}y^l\,\hat K(\hat u,\,\hat v) \tag{5.53}$$

$$\dot W_0{}^a = -\tilde\eta^{ab}y_b \qquad \dot W_a{}^0 = y_a\,\dot K(\dot u,\,\dot v) \qquad \dot W_a{}^b = \delta_a^b \tag{5.54}$$

where η_{ij} and $\tilde{\eta}^{ab}$ are the invariant metrics of $SO(p-1,\, q)$ and $SO(p'-1,\, q')$ and

$$\hat{u} = y^i y^i\,, \qquad \hat{v} = y^i \eta_{ij} y^j\,, \qquad \dot{u} = y_a y_a\,, \qquad \dot{v} = y_a \tilde{\eta}^{ab} y_b\,. \tag{5.55}$$

In the same way that the twist matrix U factorizes, i.e. has the block diagonal form of (5.52), the scale factor ρ factorizes as

$$\rho = \dot{\rho}(y^m)\,\hat{\rho}(y_a) \quad \text{where} \quad \begin{cases} \hat{\rho} = (1-\hat{v})^{1/4} \\ \dot{\rho} = (1-\dot{v})^{1/4} \end{cases} \tag{5.56}$$

The ansatz (5.52)-(5.56), solves the section constraints and the consistency constraints provided that $\hat{K}(\hat{u},\, \hat{v}) = K_{p,\,q}(\hat{u},\, \hat{v})$ and $\dot{K}(\dot{u},\, \dot{v}) = K_{p',\,q'}(\dot{u},\, \dot{v})$, where $K_{p,\,q}(u,\, v)$ is the solution of a differential equation depending on p and q. For $q \neq 0$ and $p \neq 1$ the equation to solve is

$$2\left(1 - r^2 \sinh\varphi\right)\partial_\varphi K_{p,\,q} = \Big((1+p-q)(1-r^2\sinh\varphi) - r^2\cosh\varphi\Big) K_{p,\,q} - 1 \tag{5.57}$$

for the variables $u = r^2\cosh\varphi$ and $\hat{v} = r^2\sinh\varphi$. For $q = 0$, $K_{p,\,0}$ is defined as:

$$K_{p,\,0} = -{}_2F_1\left(1,\,(p-2)/2,\,1/2,\,1-\hat{u}\right) \tag{5.58}$$

and for $p = 1$, we have

$$K_{1,\,q} = q^{-1}(1+\hat{u}(1-q)_2F_1\left(1,\,(1+q)/2,\,1/2,\,1+\hat{u}\right)\,. \tag{5.59}$$

The solution (5.52) is given in the fundamental representation of $SL(8)$. The explicit form of U in the fundamental representation of $E_{7(7)}$ is

$$U_M{}^N = \begin{pmatrix} U_{[AB]}{}^{[CD]} & 0 \\ 0 & U^{[AB]}{}_{[CD]} = U^{-1}{}_{[CD]}{}^{[AB]} \end{pmatrix} \tag{5.60}$$

where

$$U_{AB}{}^{CD} = U_A{}^C U_B{}^D - U_B{}^C U_A{}^D\,. \tag{5.61}$$

There is a final subtlety when working with this gSS reduction. The coordinates $\{y^m,\, y_a\}$ are given in an $SL(8)$ basis of $E_{7(7)}$ whereas the interpretation of ExFT in terms of SUGRA fields is done solving the section constraints using a $GL(6)\times SL(2)$ or $GL(7)$ basis of $E_{7(7)}$. There are thus extra changes of variables which are needed to use (5.39). For $p+q$ even, it is possible to embed the solution (5.51) in the solution (5.33) which means that if $p+q$ is even, it is possible to uplift any solution of these maximal supergravities to Type IIB supergravity.

5.3.2 The $S^5 \times S^1$ reduction

According to the gSS ansatz we have just build, all the solutions in chapter 4 can be uplifted to Type IIB supergravity. In this subsection we will give more details on the uplift in the case $(p,\, q,\, p',\, q') = (6,\, 0,\, 1,\, 1)$. First of all, we have to identify the coordinates solving the section constraints and transforming under $GL(6)\times SL(2)$ in terms of the $SL(8)$ coordinates (5.51). We identify both by their transformations rules under their common $SL(5)\times SL(2)$ subgroup. The explicit identification is:

$GL(6)\times SL(2)$	y^m		$y_{m\alpha}$		Y^{kmn}		$y^{m\alpha}$		y_m	
$SL(5)\times SL(2)$	y^1	y^i	$y_{1\alpha}$	$y_{i\alpha}$	y^{ijk}	y^{1ij}	$y^{1\alpha}$	$y^{i\alpha}$	y_1	y_i
$SL(8)$	Y_{18}	Y^{i7}	$\epsilon_{ab}Y^{b7}$	Y_{ia}	$\epsilon^{ijki'j'}Y_{i'j'}$	Y^{ij}	$\epsilon^{ab}Y_{b7}$	Y^{ia}	Y^{18}	Y_{7i}

In these notations, $m = 2, \ldots, 7$ is the fundamental index of $GL(6)$, $\alpha = 1, 8$ is the fundamental $SL(2)$ index and $i = 2, \ldots, 6$ is the fundamental $SL(5)$ index. The magnetic coordinate $\tilde{y}$ is identified with Y_{18} while the electric coordinates y^i are identified with Y^{17}. This allows to read the SUGRA-ExFT correspondence (5.39) in terms of the $SL(8)$ coordinates. Finally, the functions $\hat{K}$, $\dot{K}$ and the matrices η and $\tilde{\eta}$ are

$$\eta_{ij} = \mathbb{1}_5 \qquad\qquad \tilde{\eta}_{ab} = -1 \tag{5.62}$$
$$\hat{K} = -{}_2F_1\left(1,\, 2,\, 1/2,\, 1 - y_l y^l\right) \qquad \dot{K} = 1 \tag{5.63}$$

Building new solutions of type IIB supergravity can thus be done by finding matrices $\mathcal{M}(x)$ solving the 4d equations of motions. Then, using the gSS ansatz (5.40), one can build the generalized metric $\mathcal{M}(x,\, y)$. Using the solution of the section constraints, one can use the dictionnary (5.39), to get the Type IIB fields. This dictionary can be inverted to give the Type IIB fields in terms of $\mathcal{M}(x,\, y)$:

$$\begin{aligned}
G^{mn} &= G^{\frac{1}{2}}\,\mathcal{M}^{mn}\,, \\
\mathbb{B}_{mn}{}^{\alpha} &= G^{\frac{1}{2}}\,G_{mp}\,\epsilon^{\alpha\beta}\,\mathcal{M}^{p}{}_{n\beta}\,, \\
C_{klmn} - \tfrac{3}{2}\,\epsilon_{\alpha\beta}\,\mathbb{B}_{k[l}{}^{\alpha}\,\mathbb{B}_{mn]}{}^{\beta} &= -\tfrac{1}{2}\,G^{\frac{1}{2}}\,G_{k\rho}\,\mathcal{M}^{\rho}{}_{lmn}\,, \\
m_{\alpha\beta} &= \tfrac{1}{6}\,G\left(\mathcal{M}^{mn}\,\mathcal{M}_{m\alpha\,n\beta} + \mathcal{M}^{m}{}_{k\alpha}\,\mathcal{M}^{k}{}_{m\beta}\right)\,.
\end{aligned} \tag{5.64}$$

This is the method we used to build S-fold solutions of Type IIB supergravity of the next section.

5.4 S-fold solution of Type IIB supergravity

In this section, we present the uplifts of the most symmetric point in each family of solutions constructed in section 4.4. Although we will not deal with the uplift of the φ parameter in this thesis[2], we will provide an uplift of the χ deformations and a detailed discussion on their nature in the next sections. Since all the solutions we are going to uplift are solutions of the same gauged maximal supergravity, it is not surprising that they share some commonalities. Our uplifts, called S-folds, are solutions of the source-less equations of motion and Bianchi identities of type IIB supergravity in which the geometry is of the form $\text{AdS}_4 \times \text{S}^1 \times S^5$. The five electric coordinates span the S^5 while the magnetic coordinate parametrise the circle S^1.

Their main distinctive feature is that the non-trivial dependence on the coordinate $\eta = \sinh Y_{18}$ along S^1 is totally encoded in an $\text{SL}(2,\mathbb{R})$ twist matrix

$$A^{\alpha}{}_{\beta}(\eta) = \begin{pmatrix} e^{-\eta} & 0 \\ 0 & e^{\eta} \end{pmatrix}\,. \tag{5.65}$$

[2]This uplift can be done using the method we have described. The uplift is rather lengthy and of little interest as long as a geometric interpretation of the φ parameter is not understood.

The ten-dimensional metric is given by

$$ds_{10}^2 = \Delta^{-1}\left[\tfrac{1}{2}ds^2_{\mathrm{AdS}_4} + ds_6^2\right] \,, \tag{5.66}$$

with Δ being the warping factor. The AdS_4 radius L^2 is fixed by the $\mathcal{N}=8$ vacua and is in general proportional to $\frac{c}{g^2}$ which has been fixed to 1 in this section for simplicity. A shift along η must be an isometry of the metric (5.66) as the Einstein-frame metric field is a singlet under S-duality. By the same token, the four-form potential C_4 cannot depend on η either. However, the two-form potentials $\mathbb{B}^\alpha = (B_2, C_2)$ and the axion-dilaton matrix

$$m_{\alpha\beta} = \begin{pmatrix} e^{-\Phi} + e^{\Phi}\, C_0^2 & -e^{\Phi}\, C_0 \\ -e^{\Phi}\, C_0 & e^{\Phi} \end{pmatrix} \,, \tag{5.67}$$

have a non-trivial dependence on η as they transform under S-duality. The entire dependence of these type IIB fields on η is encoded in the $SL(2,\mathbb{R})$ twist matrix (5.65). More concretely one has

$$\mathbb{B}^\alpha = A^\alpha{}_\beta\, \mathfrak{b}^\beta \qquad \text{and} \qquad m_{\alpha\beta} = (A^{-t})_\alpha{}^\gamma\, \mathfrak{m}_{\gamma\delta} (A^{-1})^\delta{}_\beta \,, \tag{5.68}$$

with $A^{-t} \equiv (A^{-1})^t$.

The coordinate η can be taken to be periodic with period T. However, due to the $\mathrm{SL}(2,\mathbb{R})$ twist in (5.65), there is a non-trivial monodromy

$$\mathfrak{M}_{\mathrm{S}^1} = A^{-1}(\eta)\, A(\eta + T) = \begin{pmatrix} e^{-T} & 0 \\ 0 & e^{T} \end{pmatrix} \,, \tag{5.69}$$

of hyperbolic type when making a loop around the S^1. This renders the S-fold backgrounds locally geometric but globally non-geometric. The monodromy in (5.69) can be brought into a generic hyperbolic monodromy $J_k \in \mathrm{SL}(2,\mathbb{Z})$ of the form

$$J_k = \begin{pmatrix} k & 1 \\ -1 & 0 \end{pmatrix} = -\mathcal{S}\,\mathcal{T}^k \quad \text{with} \quad k \in \mathbb{N} \quad \text{and} \quad k \geq 3 \,, \tag{5.70}$$

provided the period T becomes k-dependent and given by

$$T(k) = \log(k + \sqrt{k^2 - 4}) - \log(2) \,. \tag{5.71}$$

This procedure fixes the global $SL(2,\mathbb{R})$ symmetry of the initial solution and imposes, in general, that both the axion and the dilaton run along η.

The quotient by the above type of dualities prevents the S-fold solutions from entering the non-perturbative regime [71]. In this paper, the authors explain that higher derivative corrections of Type IIB supergravity are suppressed if $R_{(s)}$ and $g^{\mu\nu}_{(s)}\nabla_\mu\phi\nabla_\nu\phi$, for the metric $g_{(s)}$ in the string frame $g_{(s)} = e^{\phi}\, g$, are small. These corrections are bounded by a term proportional to $\left(\frac{\sinh T}{L^4}\right)^{1/2}$. The supergravity approximation is thus valid in the limit where T is finite and L large. A more intuitive way to understand this is to think of the string coupling as given by $g_s = e^{\Phi} \propto e^{-2\eta}\, \mathfrak{m}_{22}$, there always exists a frame in which $g_s \ll 1$ after repeated use of the J_k action. This ensures that one can safely work in a perturbative regime of type IIB string theory.

5.4.1 $\mathcal{N}=4$ and SO(4) symmetric S-fold

This S-fold solution appeared originally in [55] and has later been reinterpreted in [5]. Taking advantage of the $\mathrm{SO}(4) \sim \mathrm{SO}(3)_1 \times \mathrm{SO}(3)_2$ symmetry of the solution, which is identified with the R-symmetry group of the dual $\mathcal{N}=4$ S-fold CFT, it proves convenient to describe the five-sphere S^5 as a product of two two-spheres $\mathrm{S}^2_{i=1,2}$ fibered over an interval I. Each of these two-spheres displays an SO(3) symmetry. We choose coordinates such that the two-spheres are parameterised by standard polar and azimutal angles (θ_i, φ_i) with $\theta_i \in [0,\pi]$ and $\varphi_i \in [0, 2\pi]$, and the interval is parameterised by an angle $\alpha \in [0, \frac{\pi}{2}]$. In terms of the electric coordinates y^i, these angles are defined as

$$
\begin{aligned}
y^1 &= \cos\alpha\, \cos\theta_1\,, & y^2 &= \sin\alpha\, \sin\theta_2\, \sin\varphi_2\,,\\
y^3 &= \cos\alpha\, \sin\theta_1\, \cos\varphi_1\,, & y^4 &= \sin\alpha\, \sin\theta_2\, \cos\varphi_2\,,\\
y^5 &= \cos\alpha\, \sin\theta_1\, \sin\varphi_1\,. &&
\end{aligned}
\tag{5.72}
$$

Then, the internal $\mathrm{S}^1 \times \mathrm{S}^5$ metric in (5.66) is given by

$$
ds_6^2 = d\eta^2 + d\alpha^2 + \frac{\cos^2\alpha}{2+\cos(2\alpha)} ds^2_{\mathrm{S}^2_1} + \frac{\sin^2\alpha}{2-\cos(2\alpha)} ds^2_{\mathrm{S}^2_2}\,, \tag{5.73}
$$

with

$$
ds^2_{\mathrm{S}^2_i} = d\theta_i^2 + \sin^2\theta_i\, d{\varphi_i}^2 \qquad , \qquad \mathrm{vol}_i = \sin\theta_i\, d\theta_i \wedge d\varphi_i\,, \tag{5.74}
$$

and the non-singular warping factor reads

$$
\Delta^{-4} = 4 - \cos^2(2\alpha)\,. \tag{5.75}
$$

The η-independent two-form potentials in (5.68) read

$$
\mathfrak{b}_1 = -2\sqrt{2}\,\frac{\cos^3\alpha}{2+\cos(2\alpha)}\,\mathrm{vol}_1 \quad , \quad \mathfrak{b}_2 = -2\sqrt{2}\,\frac{\sin^3\alpha}{2-\cos(2\alpha)}\,\mathrm{vol}_2\,, \tag{5.76}
$$

whereas the axion-dilaton matrix is

$$
\mathfrak{m}_{\alpha\beta} = \begin{pmatrix} \sqrt{\frac{2+\cos(2\alpha)}{2-\cos(2\alpha)}} & 0 \\ 0 & \sqrt{\frac{2-\cos(2\alpha)}{2+\cos(2\alpha)}} \end{pmatrix}. \tag{5.77}
$$

The self-dual five-form is given by

$$
\widetilde{F}_5 = \Delta^4\, \sin^2(2\alpha)\,(1+\star)\left[-\tfrac{3}{2}\,\mathrm{vol}_5 + \sin(2\alpha)\, d\eta\wedge\mathrm{vol}_1\wedge\mathrm{vol}_2\right]\,, \tag{5.78}
$$

where $\mathrm{vol}_5 = d\alpha\wedge\mathrm{vol}_1\wedge\mathrm{vol}_2$. Lastly, the AdS_4 radius in the external part of the metric (5.66) is set to $L^2 = 1$.

5.4.2 $\mathcal{N}=2$ and $\mathrm{SU}(2)\times\mathrm{U}(1)$ symmetric S-fold

This S-fold was put forward in [2]. The $\mathrm{SU}(2)\times\mathrm{U}(1)$ symmetry of the solution becomes manifest when describing the five-sphere S^5 as a three-sphere S^3 fibered over a two-sphere S^2. We choose standard polar $\theta \in [0, \pi]$ and azimutal $\phi \in [0, 2\pi]$ angles to describe the two-sphere, as well as three angular coordinates $\alpha \in [0, 2\pi]$, $\beta \in [0, \pi]$ and $\gamma \in [0, 4\pi]$ to describe the three-sphere. In terms of the electric coordinates y^i, these angles are defined using the changes of coordinates

$$y^1 = \frac{w^1+w^3}{\sqrt{2}}\,,\quad y^2 = \frac{w^1-w^3}{\sqrt{2}}\,,\quad y^3 = \frac{w^2-w^3}{\sqrt{2}}\,,\quad y^4 = \frac{w^2+w^3}{\sqrt{2}}\,,\quad y^5 = w^5\,. \tag{5.79}$$

$$\begin{aligned} &w^1 = \cos\theta\,\sin\tfrac{\beta}{2}\,\cos(\alpha-\gamma)\,, && w^2 = \cos\theta\,\sin\tfrac{\beta}{2}\,\sin(\alpha-\gamma)\,,\\ &w^3 = \cos\theta\,\cos\tfrac{\beta}{2}\,\cos(\alpha+\gamma)\,, && w^4 = \cos\theta\,\cos\tfrac{\beta}{2}\,\sin(\alpha+\gamma)\,,\\ &w^5 = \sin\theta\,\sin\phi\,. \end{aligned} \tag{5.80}$$

The three-sphere is better described using a set $\sigma_{1,2,3}$ of SU(2) left-invariant one-forms

$$\begin{aligned} \sigma_1 &= \tfrac{1}{2}\left(-\sin\alpha\,d\beta + \cos\alpha\sin\beta\,d\gamma\right)\,,\\ \sigma_2 &= \tfrac{1}{2}\left(\cos\alpha\,d\beta + \sin\alpha\sin\beta\,d\gamma\right)\,,\\ \sigma_3 &= \tfrac{1}{2}\left(d\alpha + \cos\beta\,d\gamma\right)\,. \end{aligned} \tag{5.81}$$

In terms of the above one-forms, the internal $\mathrm{S}^1\times\mathrm{S}^5$ metric in (5.66) is given by

$$ds_6^2 = \tfrac{1}{2}\left(d\eta^2 + ds^2_{\mathrm{S}^2} + \cos^2\theta\left[8\,\Delta^4\left(\sigma_1^2+\sigma_2^2\right)+\sigma_3^2\right]\right)\,, \tag{5.82}$$

with $ds^2_{\mathrm{S}^2} = d\theta^2 + \sin^2\theta\,d\phi^2$, and the warping factor reads

$$\Delta^{-4} = 6 - 2\cos(2\theta)\,. \tag{5.83}$$

The U(1) factor of the S-fold symmetry group is then realised as rotations in the (σ_1,σ_2)-plane generated by translations along the coordinate α. This is identified with the R-symmetry group in the dual $\mathcal{N}=2$ S-fold CFT. The η-independent two-form potentials in (5.68) are given by

$$\mathfrak{b}_1 + i\,\mathfrak{b}_2 = \frac{\cos\theta}{\sqrt{2}}e^{i\frac{\pi}{4}-i\phi}\left[(d\theta - \tfrac{i}{2}\sin(2\theta)\,d\phi)\wedge\sigma_3 - 4\,\Delta^4\,\sin(2\theta)\,\sigma_1\wedge\sigma_2\right]\,, \tag{5.84}$$

and the axion-dilaton matrix reads

$$\mathfrak{m}_{\alpha\beta} = \tfrac{1}{2}\,\Delta^2\begin{pmatrix} 5-\cos(2\theta)+2\sin^2\theta\sin(2\phi) & 2\sin^2\theta\cos(2\phi)\\ 2\sin^2\theta\cos(2\phi) & 5-\cos(2\theta)-2\sin^2\theta\sin(2\phi)\end{pmatrix}\,. \tag{5.85}$$

The self-dual five-form flux takes the form

$$\widetilde{F}_5 = 4\,\Delta^4\,\cos^3\theta\,(1+\star)\left[-3\,\mathrm{vol}_5 + \sin\theta\,d\eta\wedge\mathrm{Re}\left[e^{-2i\phi}(d\theta - \tfrac{i}{2}\sin(2\theta)\,d\phi)\right]\wedge\mathrm{vol}_3\right]\,, \tag{5.86}$$

where $\mathrm{vol}_5 = \mathrm{vol}_2\wedge\mathrm{vol}_3$ with $\mathrm{vol}_2 = \sin\theta\,d\theta\wedge d\phi$ and $\mathrm{vol}_3 = -\sigma_1\wedge\sigma_2\wedge\sigma_3$. The radius of the external AdS_4 space-time turns out to be $L^2=1$ thus coinciding with the one of the $\mathcal{N}=4$ S-fold.

5.4.3 $\mathcal{N}=1$ and SU(3) symmetric S-fold

The $\mathcal{N}=1$ and SU(3) symmetric S-fold with internal geometry $\mathcal{M}=\mathrm{S}^5$ was presented in [1]. Its generalisation to Sasaki–Einstein internal manifolds was discussed in [72] (see also [73] for a local characterisation of this type of solutions). We will take advantage of the SU(2)-structure of S^5 when viewed as a Sasaki–Einstein manifold, and discuss S^5 as a circle S^1 fibered over $\mathbb{CP}^2$. The SU(2) structure on S^5 is encoded in a real one-form $\boldsymbol{\eta}$, a real two-form $\mathbf{J}$ and a complex $(2,\,0)$ form $\boldsymbol{\Omega}$. These forms have the properties that

$$d\boldsymbol{\eta} = \mathbf{J} \quad , \quad d\mathbf{J} = 0 \quad \text{and} \quad d\boldsymbol{\Omega} = 3\, i\, \eta \wedge \boldsymbol{\Omega}\,. \tag{5.87}$$

Moreover, they satisfy the algebraic relations

$$\boldsymbol{\Omega} \wedge \bar{\boldsymbol{\Omega}} = \mathbf{J} \wedge \mathbf{J} \qquad \text{and} \qquad \mathbf{J} \wedge \boldsymbol{\Omega} = 0\,. \tag{5.88}$$

The coordinate on the S^1 is denoted $\phi \in [0,\,2\pi]$ and those on $\mathbb{CP}^2$ are

$$\theta \in [0, \pi] \quad , \quad \alpha \in [0, 2\pi] \quad , \quad \beta \in [0,\, \pi] \quad , \quad \gamma \in [0, 4\pi]\ . \tag{5.89}$$

In terms of the electric coordinates y^i, these angles are defined by

$$y^1 + i\, y^2 = \sin\theta\, \cos\tfrac{\beta}{2}\, \exp\left[i\, \left(\tfrac{\alpha+\gamma}{2} + \phi\right)\right]\,, \tag{5.90}$$

$$y^3 + i\, y^4 = \sin\theta\, \sin\tfrac{\beta}{2}\, \exp\left[i\, \left(\tfrac{\alpha-\gamma}{2} + \phi\right)\right]\,, \tag{5.91}$$

$$y^5 = -\, \cos\theta\, \sin\phi\,. \tag{5.92}$$

Introducing a basis of one-forms on S^5 of the form

$$\tau_0 = d\theta \ \ , \ \ \tau_1 = \sin\theta\, \sigma_1 \ \ , \ \ \tau_2 = \sin\theta\, \sigma_2 \ \ , \ \ \tau_3 = \tfrac{1}{2}\sin(2\theta)\, \sigma_3 \quad , \quad \boldsymbol{\eta} = d\phi \,+\, \sin^2\theta\, \sigma_3\,, \tag{5.93}$$

with $\sigma_{1,2,3}$ given in (5.81) and $\boldsymbol{\eta}$ being the real one-form of the $\mathrm{SU}(2)$-structure, the internal part of the metric in (5.66) reads

$$ds_6^2 = \frac{5\sqrt{5}}{18}\left(\frac{2}{3}\, d\eta^2 + ds^2_{\mathbb{CP}^2} + \frac{6}{5}\, \boldsymbol{\eta}^2\right) \qquad \text{with} \qquad ds^2_{\mathbb{CP}^2} = \sum_{a=0}^{3} {\tau_a}^2\,, \tag{5.94}$$

and the warping factor takes the constant value

$$\Delta = \frac{5}{3\sqrt{6}}\ . \tag{5.95}$$

The η-independent two-form potentials and axion-dilaton matrix in (5.68) are given by

$$\mathfrak{b}_1 - i\, \mathfrak{b}_2 = \frac{1}{\sqrt{6}}\, e^{i\frac{\pi}{4}}\, \boldsymbol{\Omega} \qquad \text{and} \qquad \mathfrak{m}_{\alpha\beta} = \mathbb{I}_{\alpha\beta}\ , \tag{5.96}$$

with $\boldsymbol{\Omega} = e^{3i\phi}\,(\tau_0 + i\,\tau_3) \wedge (\tau_1 + i\,\tau_2)$ being the complex $(2,0)$-form of the $\mathrm{SU}(2)$-structure. The two-form $\boldsymbol{\Omega}$ is charged under the $\mathrm{U}(1)_\phi$ isometry of the fibration $\mathrm{S}^5 \sim \mathbb{CP}^2 \rtimes \mathrm{S}^1$. Therefore, the complex two-form potential in (5.96) breaks this $\mathrm{U}(1)_\phi$ isometry of the metric (5.94). The self-dual five-form reads

$$\widetilde{F}_5 = -3\,(1 + \star)\, \mathrm{vol}_5\ , \tag{5.97}$$

with $\mathrm{vol}_5 = \mathrm{vol}_{\mathbb{CP}^2} \wedge \boldsymbol{\eta}$ and $\mathrm{vol}_{\mathbb{CP}^2} = -\tau_0 \wedge \tau_1 \wedge \tau_2 \wedge \tau_3$. The AdS_4 radius takes the value $L^2 = 5^{\frac{5}{2}} \cdot 3^{-3} \cdot 2^{-1}$.

5.4.4 $\mathcal{N} = 0$ and $\mathrm{SO}(6)$ symmetric S-fold

This S-fold first discussed in [1] is the simplest one. The internal S^5 is round and features its largest possible $\mathrm{SO}(6)$ symmetry, although no supersymmetry is preserved by the solution. The internal metric, constant warping factor and self-dual five-form flux are given by

$$ds_6^2 = \tfrac{1}{2\sqrt{2}}\, d\eta^2 + \tfrac{1}{\sqrt{2}}\, ds^2_{\mathrm{S}^5} \quad , \quad \Delta = \frac{1}{\sqrt{2}} \quad , \quad \widetilde{F}_5 = \mp 4\,(1+\star)\,\mathrm{vol}_5 \,, \tag{5.98}$$

with vol_5 being the volume form of the round S^5 of unit radius. The η-independent two-forms and axion-dilaton matrix in (5.68) read

$$\mathfrak{b}_i = 0 \qquad \text{and} \qquad \mathfrak{m}_{\alpha\beta} = \mathbb{I}_{\alpha\beta} \;. \tag{5.99}$$

Finally, the AdS_4 radius takes the value $L^2 = 3 \cdot 2^{-\frac{3}{2}}$. The analysis of scalar fluctuations performed in [1, 2] at the four-dimensional supergravity level showed that this non-supersymmetric S-fold features perturbatively instabilities: various scalar modes have a tachyonic mass $m^2L^2 = -3$ lying below the Breitenlohner–Freedman (BF) bound [51] for stability in AdS_4.

5.5 Janus solutions and their holographic dual

In this chapter we have worked out the uplift of our four-dimensional solutions to Type IIB supergravity in ten dimensions. Moreover, we have understood that the magnetic coupling "c" appearing in the 4d gaugings is related to the S-folding of the higher dimensional solutions. In this section, we will compare our solutions with solutions found in the literature. In particular, we will see that our S-folds can be understood as extremal limit of "Janus solutions".

Janus solutions are a type of domain-wall solutions interpolating between two $\mathrm{AdS}_5 \times S^5$ asymptotic regions with *different* and *finite* dilaton value. The first example of Janus solutions was built as a solution of Type IIB supergravity on $S^5 \times \mathbb{R} \times \mathrm{AdS}_4$ where S^5 is the round five-sphere and $\mathbb{R} \times \mathrm{AdS}_4$ is an AdS_4 slicing of AdS_5 [74]. Setting the three-form flux to zero, one finds solutions preserving $SO(6)$ symmetry, with an axio-dilaton varying along the $\mathbb{R}$ factor, and breaking all supersymmetry. The holographic interpretation of this solution was given in [75] and is an interface in SYM_4 allowing for a "jump" in the gauge coupling. Shortly after, a solution preserving $\mathcal{N} = 1$ and $SU(3)$ residual symmetry was build in an $SO(6)$-gauged 5d maximal supergravity [76]. This solution was then uplifted to ten dimensions in [77]. Our solutions, before any S-folding, can be understood as an extremal limit of Janus solutions where the dilaton interpolates between two diverging values at $\pm\infty$.

A more in depth study of SYM_4 interfaces was carried in [78] where the maximal symmetry group compatible with a certain amount of residual supersymmetry is determined. The most symmetric interfaces are of the form: $SO(4)$ if $\mathcal{N} = 4$ is preserved, $SU(2) \times U(1)$ if $\mathcal{N} = 2$ is preserved or $SU(3)$ if only $\mathcal{N} = 1$ is preserved (notice that there are no $\mathcal{N} = 3$ interfaces in this classification). This started the search for the dual of the $\mathcal{N} = 4$ interface. An exhaustive classification of Type IIB $\mathcal{N} = 4$ Janus solutions was carried in [79, 80] by directly solving the equations of motion of Type IIB supergravity on a manifold of the form

$$AdS_4 \times S^2 \times S^2 \times \Sigma \,. \tag{5.100}$$

The two S^2 are the round two-spheres, reflecting the $SO(4)$ R-symmetry of $\mathcal{N} = 4$, whereas Σ is a Riemann surface. One must also allow a warping of $AdS_4 \times S^2 \times S^2$ on Σ. The $\mathcal{N} = 4$ Janus solutions are parametrised by two harmonic functions $h_{1,2}$ on Σ which obey specific boundary conditions on $\partial\Sigma$. Interestingly, these boundary conditions allow for specific types of singularities on $\partial\Sigma$ reflecting the presence of D5 and NS5 branes in the geometry. Away from such singularities, the boundary conditions rescale the two S^2 such that the internal space has no boundary. On these solutions, the axio-dilaton is not constant and only diverges when approaching the singularities on $\partial\Sigma$.

From this construction it becomes clear that the $\mathcal{N}=4$ Janus solutions appear as the near-horizon limit of D3-branes intersecting NS5 and D5 branes. In its simplest

	0-1-2	3	4-5-6	7-8-9
D3	-	-	$\times$	$\times$
D5	-	$\times$	-	$\times$
NS5	-	$\times$	$\times$	-

TABLE 5.1: Hanany-Witten brane configuration on flat space. Branes are extended in the "-" directions and point-like in the "$\times$" directions.

form, this brane setup appears in the work of Hanany and Witten for a constant axio-dilaton background [81]. The generalisation for the varying axio-dilaton value was understood in [82]. Allowing for a varying *axio*-dilaton, and not just a varying dilaton, is crucial to preserve $\mathcal{N}=4$ supersymmetry. This construction provides CFT dual to the Janus solutions as the IR fixed point of specific quiver theories. Indeed, to each brane configuration, one can associate a specific quiver diagram, encoding how the D3-branes intersect with the D5 and NS5 branes. Then, this quiver diagram can be used to build the supergravity solutions, specifying the types of singularities and their positions on $\partial\Sigma$ [83].

Up to this point, no S-folding was done and, away from singularities, the axio-dilaton has a changing but finite value. To make the connection to the $\mathcal{N}=4$ solution presented here, we will first restrict to the case where Σ is an infinite strip. In this case, $\Sigma\sim\mathbb{R}\times[0,\,\frac{\pi}{2}]$ and, for concreteness, is spanned by two coordinates $(\eta,\,\alpha)$. Using the correspondence between the brane setup and the singularity structure of h_1 and h_2, prior to any S-folding, the Type IIB S-fold appears as a limit of the T[SU(N)] brane configuration [84]. The T[SU(N)] gravity dual has two singularities located at $\left(-\frac{1}{2}\ln\tan\left(\frac{\pi}{2N}\right),\,0\right)$ and $\left(\frac{1}{2}\ln\tan\left(\frac{\pi}{2N}\right),\,\frac{\pi}{2}\right)$ which represent a stack of NS5 brane and a stack of D5 brane localised at the boundaries of Σ. In the large N limit, these two stacks are sent to $\eta=\pm\infty$. Since near the branes, the dilaton diverges, this procedure leads to an extremal Janus solution. This allows us to recover the solution of section 5.4.1 prior to the S-folding procedure.

As we reviewed in the main text, this solution is now invariant under the combined action of the translation $\eta\to\eta+T(k)$ and the S-duality element J_k. This allows us to perform the S-folding, avoiding the dangerous limit where $\phi\to\infty$. The new solution is regular, smooth, and does not contain any branes. However, the S-folding procedure is not trivial. From the CFT point of view, it is equivalent to gauging the $U(N)^2$ global symmetry of the T[U(N)] theory with a CS terms at level k. Precise holographic comparison between the Type IIB action and the sphere partition function of T[U(N)] with both $U(N)$ symmetries gauged has been performed in [71] using localisation techniques with much success. Some more information can be recovered from the $E_{7(7)}$-ExFT formulation of the S-fold such as the spectrum of operators which is accessible through the KK spectrometry techniques of [85]. It would be very interesting to understand if the other S-folds we have presented in this work do have well defined CFT_3 duals and if tests of holography could be performed, in the spirit of [71].

Chapter 6

Flat deformations

In the previous chapters, various AdS$_4$ solutions have been found in the G $= [\mathrm{SO}(1,1)\times \mathrm{SO}(6)] \ltimes \mathbb{R}^{12}$ gauged maximal supergravity. These solutions were subsequently uplifted to S-fold backgrounds of type IIB supergravity. Except for the $\mathcal{N}=4$ and SO(4) symmetric S-fold, the S-fold solutions contain moduli representing flat directions in the 4d scalar potential. These deformations are parametrised by axion-like field, χ, present in the $\mathbb{Z}_2^3$ invariant sector of the theory. The aim of this chapter is to disclose the universal character of the axion-like deformations of the type IIB S-folds. In particular, we will show that we can build such axion-like deformations without the need of working in a specific invariant subsector of the full $\mathcal{N}=8$ theory. This is encoded in the first claim that we will prove in this chapter:

Claim 1. *(4d) For any vacuum $\mathcal{V}_0 \in E_{7(7)}/SU(8)$ of gauged maximal supergravity with gauge group $[SO(1,1)\times SO(6)]\ltimes \mathbb{R}^{12}$, there exists an r-dimensional family of solutions deforming $\mathcal{V}_0$. The dimension r is the rank of the residual symmetry group of $\mathcal{V}_0$. At a generic point on the moduli space, the residual gauge group is $U(1)^r$.*

Although we will focus on the dyonic gauging G, the proof of the claim also works for the gauge groups $[\mathbb{R}\times \mathrm{SO}(6)]\ltimes \mathbb{R}^{12}$ and $[\mathrm{SO}(2)\times \mathrm{SO}(6)]\ltimes \mathbb{R}^{12}$ (where it was applied in [86]).[1] We call the deformations of the seed solution $\mathcal{V}_0$ "*flat deformations*".

As an illustration of the claim, we establish the existence of two flat deformations $\chi_{1,2}$ of the $\mathcal{N}=4$ and $SO(4)$ symmetric S-fold presented in subsection 4.4.4 which lie outside the $\mathbb{Z}_2^3$-invariant sector studied in chapter 4. A generic choice of the two parameters $\chi_{1,2}$ produces non-supersymmetric S-folds with $U(1)^2$ symmetry. We show that these are perturbatively stable at the lower dimensional supergravity level. Interestingly, these families of solutions are non-compact at the 4d level. As we will see, the 4d interpretation of these flat deformations can be deceiving, and a 10d uplift is necessary to understand them fully.

To prove the claim 1, we will use the duality invariance of $\mathcal{N}=8$ supergravity under the $E_{7(7)}$ group. This will allow us to show that a deformed solution $\mathcal{V}_\chi \cdot \mathcal{V}_0 \in E_{7(7)}/SU(8)$, for $\mathcal{V}_\chi \in E_{7(7)}$, of the gauging G, specified by the embedding tensor Θ, is equivalent to a solution $\mathcal{V}_0$ for an embedding tensor:

$$\mathcal{V}_\chi \star \Theta = \Theta + \Theta^{CSS}\,, \tag{6.1}$$

where the embedding tensor Θ^{CSS} corresponds to the Cremmer-Scherk-Schwarz (CSS) gaugings. This gauging was initially constructed as a way to introduce masses on the T^7 compactification of 11d supergravities and appears from the compactification of 5d ungauged supergravities on a circle [87, 88] (see also [52]). We will review how these gaugings appear as duality twists. We will also show that, from a 5d perspective, we

[1] For these gauge groups, the S-folding procedure of the uplift rely on non-hyperbolic S-duality elements of $SL(2,\,\mathbb{Z})$.

can unify the description of the parameter c, gauging the $SO(1,1)$ group in G, and the χ-deformations.

Once the four-dimensional interpretation of the flat deformations is understood, we will proceed to uplift the flat deformations (still using the gSS ansatz) to type IIB solutions in 10 dimensions. This will show that the 4d families of solutions parametrised by the χ's are actually compact and that the flat deformations introduce a non-trivial χ-dependent fibration of S^5 over the S^1_η of the undeformed solution. This interpretation holds for all the vacua found in the $\mathbb{Z}_2^3$-invariant sector. The parameter χ specify a monodromy element $h(\chi) \subset SO(6)$[2] and the patterns of symmetry breakings are classified by the mapping tori $T_h(S^5)$. This construction will lead us to an equivalent claim for gravity theories:

Claim 2. *(10d) For any solution of a theory of gravity (i.e. with diffeomorphism invariance) on a manifold of the form $\mathcal{M} \times S^1$, and with a residual continuous symmetry group G_0 of rank r, there is a r-dimensional family of solutions classified by $T_h(\mathcal{M})$, $\forall\, h \in G_0$. At a generic point on the moduli space, the residual symmetry group is $U(1)^r$.*

Once we have clarified the generic 10d interpretation of the flat deformations, we will focus on their implications for the 10d $\mathcal{N}=4$ S-fold. We will study the perturbative stability of the χ deformed $\mathcal{N}=4$ solution, including higher Kaluza-Klein modes. Usually, computing higher KK modes requires to compute the eigenvalues of complicated differential operators. However, since our solution has a interpretation as the gSS ansatz in ExFT, one can use the KK spectroscopy techniques introduced in [85]. We will show that the χ deformations do not introduce any perturbative instabilities. Moreover, although the masses in the 4d truncation were not periodic in the χ deformations, the full tower of higher KK modes are periodic (as expected from our geometric construction).

We will next investigate the non-perturbative stability of the $\mathcal{N}=4$ χ-deformed family of solutions. We will show that they do not suffer the probe-brane instabilities described in [89]. Moreover, they also seem to be protected against bubble of nothing instabilities, $\frac{1}{N}$ corrections as well as semi-classical instabilities such as instanton instabilities. For these reasons, this family of solutions seem to be a first example of non-SUSY AdS solutions continuously connected to a supersymmetric solution. Finally, we will try to make some educated guess on the SCFT interpretation of the non-SUSY flat deformations.

This chapter is organised as follow. In section 6.1, we prove the claim 1. In section 6.2 we apply our findings to the $\mathcal{N}=4$ $SO(4)$ S-fold found in four dimensions. In section 6.3, we present the 5d origin of these flat deformations as duality twists. In section 6.4, we present the 10d origin of the flat deformations in terms of mapping tori. In section 6.5, we apply these results to the $\mathcal{N}=4$ $SO(4)$ S-fold in 10d and study its perturbative and non-perturbative stability. We also interpret these results in light of the swampland program and the AdS/CFT conjecture.

6.1 Flat deformations in 4d

Let us start by restating claim 1 and make it more precise. Starting from a vacuum solution of the G-gauged maximal supergravity represented by $\mathcal{V}_0 \in E_{7(7)}/SU(8)$, we will build axion-like deformations. We will assume some residual symmetry $\mathrm{G}_0 \subset$

[2] To be precise $h \in SU(4)$, the double cover of $SO(6)$, but this difference will not come into play until we discuss fermions. We will thus ignore it for most of our discussion.

SO(6) $\subset$ G at the vacuum and parametrise an element of its algebra $\mathfrak{g}_0$ by an antisymmetric constant matrix χ^{ij} . The axion-like deformations will correspond to the coset replacement

$$\mathcal{V}_0 \to \mathcal{V}_\chi \mathcal{V}_0 \qquad \text{with} \qquad \mathcal{V}_\chi = \exp\left(\tfrac{1}{2}\,\chi^{ij}\,t_{1ij8}\right) \in \mathrm{E}_{7(7)} \ . \tag{6.2}$$

What we have to show is that this new coset representative still describes a vacuum solution of the G maximal gauged supergravity (with the *same* value of the cosmological constant V_0). The new vacuum solution $\mathcal{V}_\chi\mathcal{V}_0$ no longer belongs to the G_0-invariant sector of the theory, but nevertheless parameterises an entire family of χ-dependent solutions with χ^{ij} corresponding to flat directions in the scalar potential. We will refer to the χ^{ij} parameters as axion-like flat deformations of the original S-fold solution $\mathcal{V}_0$. The residual symmetry group of $\mathcal{V}_\chi\,\mathcal{V}_0$ is denoted G_0^χ.

To prove the statement above we will take advantage of the $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ coset structure of the scalar manifold of maximal supergravity. The solution with non-zero axions $\mathcal{V}_\chi\mathcal{V}_0$ in the G-gauged maximal supergravity can be mapped into an axion-vanishing solution in a *different* theory with different gauge group $\widetilde{\mathrm{G}} \neq \mathrm{G}$ specified by an embedding tensor

$$\widetilde{\Theta}_M{}^\alpha = \mathcal{V}_\chi \star \Theta_M{}^\alpha = \Theta_M{}^\alpha + (\delta\Theta)_M{}^\alpha \ , \tag{6.3}$$

where the $\star$ denotes the action of the $\mathrm{E}_{7(7)}$ element $\mathcal{V}_\chi$ in (6.2) on the embedding tensor $\Theta_{\mathbb{M}}{}^\alpha \in \mathbf{912}$ of the original G-gauged theory.

To understand correctly the action of te specific element $\mathcal{V}_\chi$ in (6.2), it is useful to study, as in (4.11), the branching of specific $E_{7(7)}$ representations under the group

$$SL(6)\times SL(2)\times SO(1,1) \subset SL(8) \subset E_{7(7)} \tag{6.4}$$

which contains the compact part of the gauge group G. We rewrite the branching here:

$$\begin{aligned}
\mathbf{56} &\to \mathbf{28}\oplus\mathbf{28'} \\
&\to [(\mathbf{6},\,\mathbf{2})_{-1}\oplus(\mathbf{15},\,\mathbf{1})_{1}\oplus(\mathbf{1},\,\mathbf{1})_{-3}]\oplus[(\mathbf{6'},\,\mathbf{2})_{1}\oplus(\mathbf{15'},\,\mathbf{1})_{-1}\oplus(\mathbf{1},\,\mathbf{1})_{3}]\ ,
\end{aligned} \tag{6.5}$$

$$\begin{aligned}
\mathbf{133} &\to \mathbf{63}\oplus\mathbf{70} \\
&\to \left[(\mathbf{35},\mathbf{1})_0\oplus(\mathbf{6},\mathbf{2})_2\oplus(\mathbf{6'},\mathbf{2})_{-2}\oplus(\mathbf{1},\mathbf{3})_0\oplus(\mathbf{1},\mathbf{1})_0\right] \\
&\qquad \oplus\left[(\mathbf{15},\mathbf{1})_{-2}\oplus(\mathbf{20},\,\mathbf{2})_0\oplus(\mathbf{15'},\mathbf{1})_2\right]\ ,
\end{aligned} \tag{6.6}$$

$$\begin{aligned}
\mathbf{912} &\to \mathbf{36}\oplus\mathbf{36'}\oplus\mathbf{420}\oplus\mathbf{420'} \\
&\to [(\mathbf{21},\mathbf{1})_1\oplus(\mathbf{6},\mathbf{2})_{-1}\oplus(\mathbf{1},\mathbf{3})_{-3}]\oplus[(\mathbf{21'},\mathbf{1})_{-1}\oplus(\mathbf{6'},\mathbf{2})_1\oplus(\mathbf{1},\mathbf{3})_3] \\
&\quad \oplus[(\mathbf{35},\mathbf{1})_{-3}\oplus(\mathbf{84},\mathbf{2})_{-1}\oplus(\mathbf{6},\mathbf{2})_{-1}\oplus(\mathbf{105},\mathbf{1})_1\oplus(\mathbf{15},\mathbf{3})_1\oplus(\mathbf{15},\mathbf{1})_1\oplus(\mathbf{20},\mathbf{2})_3] \\
&\quad \oplus[(\mathbf{35},\mathbf{1})_3\oplus(\mathbf{84'},\mathbf{2})_1\oplus(\mathbf{6'},\mathbf{2})_1\oplus(\mathbf{105'},\mathbf{1})_{-1}\oplus(\mathbf{15'},\mathbf{3})_{-1}\oplus(\mathbf{15'},\mathbf{1})_{-1}\oplus(\mathbf{20},\mathbf{2})_{-3}]\ .
\end{aligned} \tag{6.7}$$

At the linear level, the element $\mathcal{V}_\chi$ is generated by $t_{1ij8} \in (\mathbf{15'},\,\mathbf{1})_2$. Then, an inspection of Table 6.1 shows that, upon acting on the original embedding tensor components in (θ and $\tilde{\theta}$), the action of $\mathcal{V}_\chi$ only produces a single term $\delta\Theta \in (\mathbf{35},\,\mathbf{1})_3 \subset \mathbf{420'}$ that originates from the electric piece $(\mathbf{21},\,\mathbf{1})_1 \subset \mathbf{36}$ in the $\mathbf{912}$ decomposition (6.7)[3] Equivalently, only a linear term appears since the $\mathrm{SO}(1,1)$ grading in the decomposition (6.7) solely allows for $SO(1,\,1)$-charges -3 , -1, 1 or $+3$, and not $+5$ or

[3]This implies that our argument holds in the limit where $c \to 0$ as well as for the magnetic gauging of an $\mathrm{SO}(2)$ factor instead of $\mathrm{SO}(1,1)$.

$\otimes$ **133**	rep $\in$ **912**	θ $(\mathbf{21},\mathbf{1})_1$	$\tilde{\theta}$ $(\mathbf{1},\mathbf{3})_3$	$\delta\Theta$ $(\mathbf{35},\mathbf{1})_3$
${t_a}^i$	$(\mathbf{6},\mathbf{2})_2$	$\times$	$\times$	$\times$
t_{1ij8}	$(\mathbf{15'},\mathbf{1})_2$	$(\mathbf{35},\mathbf{1})_3$	$\times$	$\times$
$\mathfrak{sl}(2)$	$(\mathbf{1},\mathbf{3})_0$	$\times$	$(\mathbf{1},\mathbf{3})_3$	$\times$
$\mathfrak{sl}(6)$	$(\mathbf{35},\mathbf{1})_0$	$(\mathbf{21},\mathbf{1})_1 \oplus (\mathbf{15},\mathbf{1})_1$	$\times$	$(\mathbf{35},\mathbf{1})_3$
$\mathfrak{so}(1,1)$	$(\mathbf{1},\mathbf{1})_0$	$(\mathbf{21},\mathbf{1})_1$	$(\mathbf{1},\mathbf{3})_3$	$(\mathbf{35},\mathbf{1})_3$
t_{aijk}	$(\mathbf{20},\mathbf{2})_0$	$(\mathbf{84'},\mathbf{2})_1$	$(\mathbf{20},\mathbf{2})_3$	$(\mathbf{20},\mathbf{2})_3$
${t_i}^a$	$(\mathbf{6'},\mathbf{2})_{-2}$	$(\mathbf{6},\mathbf{2})_{-1}$	$(\mathbf{6'},\mathbf{2})_1$	$(\mathbf{84'},\mathbf{2})_1 \oplus (\mathbf{6'},\mathbf{2})_1$
t_{ijkl}	$(\mathbf{15},\mathbf{1})_{-2}$	$(\mathbf{105'},\mathbf{1})_{-1}$	$(\mathbf{15},\mathbf{3})_1$	$(\mathbf{105},\mathbf{1})_1 \oplus (\mathbf{21},\mathbf{1})_1 \oplus (\mathbf{15},\mathbf{1})_1$

Table 6.1: Group-theoretical action of the **133** representation of $\mathrm{E}_{7(7)}$ on the **912** using the $\mathrm{SL}(6)\times\mathrm{SL}(2)\times\mathrm{SO}(1,1)$ basis. Only pieces belonging to the **912** are displayed.

higher which would be the ones produced beyond the linear level. As a result, $\delta\Theta$ is invariant under the action of $\mathcal{V}_\chi$. An explicit computation of $\delta\Theta$ in (6.3) yields

$$(\delta\Theta)_{\mathbb{M}}{}^{\alpha}\, t_\alpha \quad\to\quad \delta\Theta_{ij} = 2\,\chi_{[i|}{}^{k}\, t_{1k|j]8} \quad , \quad \delta\Theta^{ai} = -\,\epsilon^{ab}\,\chi_j{}^i\, t_b{}^j \quad , \quad \delta\Theta^{18} = \chi_j{}^i\, t_i{}^j \ , \tag{6.8}$$

where $\chi_i{}^j = \theta_{ik}\,\chi^{kj}$ with $\theta_{ij} = g\,\delta_{ij}$ in (4.10). This comes from the fact that the only terms in the product $\mathbf{56}\otimes\mathbf{133}$ contributing to the $(\mathbf{35},\mathbf{1})_3$ are precisely the $(\mathbf{15},\mathbf{1})_1\otimes(\mathbf{15'},\mathbf{1})_2$, the $(\mathbf{6'},\mathbf{2})_1\otimes(\mathbf{6},\mathbf{2})_2$ and the $(\mathbf{1},\mathbf{1})_3\otimes(\mathbf{35},\mathbf{1})_0$ (see Table 6.2). It is also worth noticing that $\delta\Theta$ in (6.8) verifies the quadratic constraints (3.74) so it defines a consistent gauging by itself.

It can now be proved that

$$V(\Theta,\mathcal{V}_\chi\mathcal{V}_{\mathrm{G}_0^\chi\text{-inv}}) = V(\widetilde{\Theta},\mathcal{V}_{\mathrm{G}_0^\chi\text{-inv}}) = V(\Theta,\mathcal{V}_{\mathrm{G}_0^\chi\text{-inv}}) \ , \tag{6.9}$$

where $\widetilde{\Theta}_M{}^\alpha$ is given in (6.3) in terms of the tensors $\Theta_M{}^\alpha$ and $(\delta\Theta)_M{}^\alpha$ in (4.15) and (6.8), respectively, and where $\mathcal{V}_{\mathrm{G}_0^\chi\text{-inv}}$ denotes the coset representative for the G_0^χ-invariant sector of maximal supergravity with $\mathrm{G}_0^\chi\subset\mathrm{G}_0\subset\mathrm{SO}(6)$. As already emphasised, the first equality in (6.9) follows from the $\mathrm{E}_{7(7)}$-covariant formulation of the maximal 4d gauged supergravities provided by the embedding tensor formalism. Before proving the second equality in (6.9), let us note that, since $\widetilde{\Theta} = \Theta + \delta\Theta$, and the scalar potential is quadratic in the embedding tensor, one has

$$V(\widetilde{\Theta},\mathcal{V}_{\mathrm{G}_0^\chi\text{-inv}}) = V(\Theta,\mathcal{V}_{\mathrm{G}_0^\chi\text{-inv}}) \,+\, 2\,B(\Theta,\delta\Theta\,;\,\mathcal{V}_{\mathrm{G}_0^\chi\text{-inv}}) \,+\, V(\delta\Theta,\mathcal{V}_{\mathrm{G}_0^\chi\text{-inv}}) \ , \tag{6.10}$$

where $B(\Theta,\delta\Theta\,;\,\mathcal{V}_{\mathrm{G}_0^\chi\text{-inv}})$ accounts for the cross terms in the scalar potential (3.92) which are bilinear in Θ and $\delta\Theta$.

To keep the discussion as general as possible, let us introduce a generic bilinear form

$$B(\Theta_1,\Theta_2\,;\,\mathcal{V}) = B(\Xi_1,\Xi_2) = \frac{1}{672}\,(\Xi_1)_{\mathbb{M}}{}^{\alpha}\,(\Xi_2)_{\mathbb{M}}{}^{\beta}\,(\delta_{\alpha\beta} + 7\,k_{\alpha\beta})\,, \tag{6.11}$$

where $\Xi_i = \mathcal{V} \star \Theta_i$. We will now argue that if $\Xi_2 \in (\mathbf{35},\,\mathbf{1})_3 \subset \mathbf{912}$ (equivalently Ξ_1) then only the projection of Ξ_1 (equivalently Ξ_2) onto the $(\mathbf{35},\,\mathbf{1})_3$ contributes to the bilinear form (6.11). This is obvious for the first contribution to the r.h.s of (6.11) as the $\delta_{\alpha\beta}$ selects only representations of Ξ_1 which are present in Ξ_2. The analysis of the second contribution to the r.h.s of (6.11) is more subtle as terms of the form $(\Xi_1)_{\mathbb{M}}{}^{\alpha}\,(\Xi_2)_{\mathbb{M}}{}^{\alpha^t}$ appear by virtue of the Killing-Cartan matrix in (3.93). Then, for any non-vanishing component of $\Xi_2 \in \mathbf{r}$ with $\mathbf{r}$ belonging to the decomposition of the $\mathbf{912}$ in (6.7) and originating from a pair $(\mathbf{r}_1,\,\mathbf{r}_2)$ of irreps with $\mathbf{r}_1$ belonging to the decomposition of the $\mathbf{56}$ in (6.5) and $\mathbf{r}_2$ belonging to the decomposition of the $\mathbf{133}$ in (6.6), only the non-vanishing components of Ξ_1 originating from a pair $(\mathbf{r}_1,\,\mathbf{r}_2')$ contribute to the second term in (6.11). In the case of $\Xi_2 \in (\mathbf{35},\,\mathbf{1})_3$ only the piece of Ξ_1 originating from the $(\mathbf{r}_1,\mathbf{r}_2') = \left((\mathbf{1},\mathbf{1})_3\,,\,(\mathbf{35},\mathbf{1})_0\right)$ and living in the $(\mathbf{35},\,\mathbf{1})_3$ contributes to the second term in (6.11) as can be seen by inspection of Table 6.2.

$\otimes$	$\mathbf{r}_2 \in \mathbf{133}$	$t_a{}^i$	t_{1ij8}	t_{ijkl}	$t_i{}^a$	$t_i{}^j$	t_{aijk}	$t_a{}^b$	$3\,t_a{}^a - t_i{}^i$
$\mathbf{r}_1 \in \mathbf{56}$		$(\mathbf{6},\,\mathbf{2})_2$	$(\mathbf{15}',\,\mathbf{1})_2$	$(\mathbf{15},\,\mathbf{1})_{-2}$	$(\mathbf{6}',\,\mathbf{2})_{-2}$	$(\mathbf{35},\,\mathbf{1})_0$	$(\mathbf{20},\,\mathbf{2})_0$	$(\mathbf{1},\,\mathbf{3})_0$	$(\mathbf{1},\,\mathbf{1})_0$
$\Theta^{[ab]}$	$(\mathbf{1},\,\mathbf{1})_3$	$\times$	$\times$	$(\mathbf{15},\,\mathbf{1})_1$	$(\mathbf{6}',\,\mathbf{2})_1$	$(\mathbf{35},\,\mathbf{1})_3$	$(\mathbf{20},\,\mathbf{2})_3$	$(\mathbf{1},\,\mathbf{3})_3$	$\times$
$\Theta_{[ij]}$	$(\mathbf{15},\,\mathbf{1})_1$	$(\mathbf{20},\,\mathbf{2})_3$	$(\mathbf{35},\,\mathbf{1})_3$	$(\mathbf{15}',\,\mathbf{1})_{-1}$ $(\mathbf{105}',\,\mathbf{1})_{-1}$	$(\mathbf{84},\,\mathbf{2})_{-1}$ $(\mathbf{6},\,\mathbf{2})_{-1}$	$(\mathbf{15},\,\mathbf{1})_1$ $(\mathbf{21},\,\mathbf{1})_1$ $(\mathbf{105},\,\mathbf{1})_1$	$(\mathbf{6}',\,\mathbf{2})_1$ $(\mathbf{84}',\,\mathbf{2})_1$	$(\mathbf{15},\,\mathbf{3})_1$	$(\mathbf{15},\,\mathbf{1})_1$
$\Theta^{[ai]}$	$(\mathbf{6}',\,\mathbf{2})_1$	$(\mathbf{35},\,\mathbf{1})_3$ $(\mathbf{1},\,\mathbf{3})_3$	$(\mathbf{20},\,\mathbf{2})_3$	$(\mathbf{84},\,\mathbf{2})_{-1}$ $(\mathbf{6},\,\mathbf{2})_{-1}$	$(\mathbf{15}',\,\mathbf{1})_{-1}$ $(\mathbf{21}',\,\mathbf{1})_{-1}$	$(\mathbf{6}',\,\mathbf{2})_1$ $(\mathbf{84}',\,\mathbf{2})_1$	$(\mathbf{105},\,\mathbf{1})_1$ $(\mathbf{15},\,\mathbf{1})_1$ $(\mathbf{15},\,\mathbf{3})_1$	$(\mathbf{6}',\,\mathbf{2})_1$	$(\mathbf{6}',\,\mathbf{2})_1$
$\Theta_{[ai]}$	$(\mathbf{6},\,\mathbf{2})_{-1}$	$(\mathbf{15},\,\mathbf{1})_1$ $(\mathbf{21},\,\mathbf{1})_1$	$(\mathbf{6}',\,\mathbf{2})_1$ $(\mathbf{84}',\,\mathbf{2})_1$	$(\mathbf{20},\,\mathbf{2})_{-3}$	$(\mathbf{35},\,\mathbf{1})_{-3}$ $(\mathbf{1},\,\mathbf{3})_{-3}$	$(\mathbf{6},\,\mathbf{2})_{-1}$ $(\mathbf{84},\,\mathbf{2})_{-1}$	$(\mathbf{105}',\,\mathbf{1})_{-1}$ $(\mathbf{15}',\,\mathbf{1})_{-1}$ $(\mathbf{15}',\,\mathbf{3})_{-1}$	$(\mathbf{6},\,\mathbf{2})_{-1}$	$(\mathbf{6},\,\mathbf{2})_{-1}$
$\Theta^{[ij]}$	$(\mathbf{15}',\,\mathbf{1})_1$	$(\mathbf{84}',\,\mathbf{2})_1$ $(\mathbf{6}',\,\mathbf{2})_1$	$(\mathbf{105},\,\mathbf{1})_1$ $(\mathbf{15},\,\mathbf{1})_1$	$(\mathbf{35},\,\mathbf{1})_{-3}$	$(\mathbf{20},\,\mathbf{2})_{-3}$	$(\mathbf{15}',\,\mathbf{1})_{-1}$ $(\mathbf{21}',\,\mathbf{1})_{-1}$ $(\mathbf{105}',\,\mathbf{1})_{-1}$	$(\mathbf{6},\,\mathbf{2})_{-1}$ $(\mathbf{84},\,\mathbf{2})_{-1}$	$(\mathbf{15}',\,\mathbf{3})_{-1}$	$(\mathbf{15}',\,\mathbf{1})_{-1}$
$\Theta_{[ab]}$	$(\mathbf{1},\,\mathbf{1})_{-3}$	$(\mathbf{6},\,\mathbf{2})_{-1}$	$(\mathbf{15}',\,\mathbf{1})_{-1}$	$\times$	$\times$	$(\mathbf{35},\,\mathbf{1})_{-3}$	$(\mathbf{20},\,\mathbf{2})_{-3}$	$(\mathbf{1},\,\mathbf{3})_{-3}$	$\times$

TABLE 6.2: Contributions to the $\mathbf{912}$ originating from the tensor product $\mathbf{56} \times \mathbf{133}$. We have highlighted the three possible sources of $\delta\Xi \in (\mathbf{35},\mathbf{1})_3$. Only the blue one contributes to the second term in the bilinear $B(\delta\Xi,\delta\Xi)$ in (6.11).

Parametrising the two embedding tensors $\Xi_{1,2}$ in terms of two 6×6 matrices $\xi_{1,2} \in (\mathbf{35},\,\mathbf{1})_3$ as

$$
\begin{aligned}
(\Xi_{1,2})_{ij} &= (\xi_{1,2})_i{}^k\, t_{1kj8} - (\xi_{1,2})_j{}^k\, t_{1ki8}\ , \\
(\Xi_{1,2})^{ai} &= -\epsilon^{ab}\,(\xi_{1,2})_j{}^i\, t_b{}^j\ , \\
(\Xi_{1,2})^{18} &= (\xi_{1,2})_i{}^j\, t_j{}^i\ ,
\end{aligned}
\tag{6.12}
$$

an explicit computation shows that

$$
B(\Xi_1,\Xi_2) = \frac{1}{192}\,\mathrm{Tr}\left[(\xi_1 + \xi_1{}^t)(\xi_2 + \xi_2{}^t)\right]\ . \tag{6.13}
$$

As a result, whenever ξ_1 or ξ_2 are $\mathfrak{so}(6)$-valued (*i.e.* $\xi_{1,2} + {\xi_{1,2}}^t = 0$), the bilinear (6.13) vanishes identically.

The analysis above has been performed in terms of scalar-dependent Ξ-tensors living in the $(\mathbf{35},\,\mathbf{1})_3$ representation. However, flat deformations were introduced in (6.8) in terms of Θ-tensors living in the $(\mathbf{35},\,\mathbf{1})_3$ representation. Therefore, it remains to be shown that having $\Theta \in (\mathbf{35},\,\mathbf{1})_3$ implies $\Xi \in (\mathbf{35},\,\mathbf{1})_3$. Using the solvable parameterisation of the coset space $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ according to which scalars are associated with non-compact generators carrying non-negative SO(1,1) charge, we see from Table 6.1 that the scalars acting non-trivialy on $\Theta \in (\mathbf{35},\,\mathbf{1})_3$ separate in three families:

1. Scalars associated with generators of $\mathrm{SL}(6)$ transforming in the $(\mathbf{35},\,\mathbf{1})_0$.
2. A single scalar σ associated with the generator of $\mathrm{SO}(1,1)$

$$t_{\mathrm{SO}(1,1)} = \sqrt{3}\,(3\,({t_1}^1 + {t_8}^8) - ({t_2}^2 + {t_3}^3 + {t_4}^4 + {t_5}^5 + {t_6}^6 + {t_7}^7))\ , \tag{6.14}$$

   transforming in the $(\mathbf{1},\,\mathbf{1})_0$.
3. Scalars associated with generators of $\mathfrak{e}_{7(7)}$ transforming in the $(\mathbf{20},\mathbf{2})_0$. Note that only scalars in the $(\mathbf{20},\mathbf{2})_0$ are of relevance as they could generate an unwanted piece $\Xi \in (\mathbf{20},\mathbf{2})_{+3}$.

However, we computed that

$$\Xi(\Theta\,,\,\mathcal{V}) = e^{-3\sigma}\,\Theta \in (\mathbf{35},\,\mathbf{1})_3\ , \tag{6.15}$$

provided $\Theta \in (\mathbf{35},\,\mathbf{1})_3$ and

$$[\,\mathcal{V}\,,\,{\chi_i}^j\,{t_j}^i\,] = 0\ , \tag{6.16}$$

with ${\chi_i}^j\,{t_j}^i \in \mathfrak{g}_0 \subset \mathfrak{so}(6)$. A way of understanding (6.15), provided (6.16) holds, is by alternatively thinking about the relation $\Xi = \mathcal{V} \star \Theta$ as the coset representative $\mathcal{V}$ being acted upon by the embedding tensor Θ rather than the other way around. The condition (6.16) severely restricts the scalar dependence of the coset representative $\mathcal{V}$ so that $\delta_{{\chi_i}^j}\mathcal{V} = 0$. In particular $\mathcal{V} = \mathcal{V}_{\mathrm{G}_0^\chi\text{-inv}}$ satisfies (6.16).

Applying the above results to the case of $\Theta_1 = \Theta \in (\mathbf{21},\mathbf{1})_1 \oplus (\mathbf{1},\mathbf{3})_3$ in (4.6) and the flat deformation $\Theta_2 = \delta\Theta \in (\mathbf{35},\,\mathbf{1})_3$ in (6.8) induced by $\mathfrak{so}(6)$-valued axions $\chi^{ij} = \delta^{k[i}\,{\chi_k}^{j]}$, one has that

$$\begin{array}{rcl} B(\Theta,\delta\Theta\,;\,\mathcal{V}_{\mathrm{G}_0^\chi\text{-inv}}) & = & 0\ , \\ B(\delta\Theta,\delta\Theta\,;\,\mathcal{V}_{\mathrm{G}_0^\chi\text{-inv}}) & = & V(\delta\Theta,\mathcal{V}_{\mathrm{G}_0^\chi\text{-inv}}) \quad = \quad 0\ . \end{array} \tag{6.17}$$

Therefore, (6.10) reduces to

$$V(\widetilde{\Theta},\mathcal{V}_{\mathrm{G}_0^\chi\text{-inv}}) = V(\Theta,\mathcal{V}_{\mathrm{G}_0^\chi\text{-inv}})\ , \tag{6.18}$$

proving the second equality in (6.9). This implies that there always exist flat directions of the scalar potential at the vacuum $\mathcal{V}_0$ parameterised by the axions χ^{ij} that are *not* captured by the G_0-invariant sector of the theory. Finally, gauge inequivalent solutions of this type are parametrised by

$$\mathfrak{g}_0/G_0 \sim \mathbb{R}^r/\Gamma \tag{6.19}$$

where r is the rank of the gauge group and Γ is a discrete subgroup of G_0 to be computed. This finishes the proof of the Claim 1. This result is particularly useful to produce new solutions starting from a seed solution $\mathcal{V}_0$ because it does not rely on building a specific truncation of the full $\mathcal{N}=8$ supergravity. As such it allows us to describe solutions that were out of computational reach before. Moreover, this procedure will, in general, break the residual symmetry group G_0 down to its Cartan subgroup. When the residual symmetry group includes a continuous R-symmetry group, this can be used to break some, or all, of the residual supersymmetry of the solution as we will see in $\mathcal{N}=4$ $SO(4)$ example.

6.2 Deforming the $\mathcal{N}=4$ and $\mathrm{SO}(4)$ symmetric S-fold in 4d

A direct consequence of the claim 1 is the existence of two axion-like flat deformations[4] χ_α $(\alpha=1,2)$ of the original $\mathcal{N}=4$ S-fold with $\mathrm{G}_0=\mathrm{SO}(4)$ symmetry, which control the pattern of symmetry breaking down to its $\mathrm{G}_0^{\chi}=\mathrm{U}(1)^2$ Cartan subgroup. These flat deformations lie outside the $\mathbb{Z}_2^3$-invariant sector of the theory investigated in 4.4 and they specify a matrix χ^{ij} of the form

$$\chi^{ij}=12\sqrt{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \chi_1 & \chi_2\\ 0 & 0 & 0 & 0 & \chi_2 & \chi_1\\ 0 & 0 & -\chi_1 & -\chi_2 & 0 & 0\\ 0 & 0 & -\chi_2 & -\chi_1 & 0 & 0 \end{pmatrix}\in\mathfrak{so}(2)^2\subset\mathfrak{so}(4)\ . \tag{6.20}$$

An explicit computation of the full scalar and vector mass spectra at the corresponding AdS$_4$ vacua yields the following results. The normalised spectrum (masses and multiplicities) of scalar fields is given by

$$\begin{aligned} m^2L^2 \;=\;& 10\ (\times 1)\ ,\ 4\ (\times 2)\ ,\ -2\ (\times 3)\ ,\ 0\ (\times 32)\ ,\ \chi_\alpha^2\ (\times 2)\ ,\\ & (\chi_1\pm\chi_2)^2\ (\times 2)\ ,\ \tfrac{1}{4}\,(\chi_1\pm\chi_2)^2\ (\times 4)\ ,\ 1+\chi_\alpha^2\pm\sqrt{9+4\,\chi_\alpha^2}\ (\times 2)\ ,\\ & 1+\tfrac{1}{4}\,(\chi_1+\chi_2)^2\pm\sqrt{9+(\chi_1+\chi_2)^2}\ (\times 2)\ ,\\ & 1+\tfrac{1}{4}\,(\chi_1-\chi_2)^2\pm\sqrt{9+(\chi_1-\chi_2)^2}\ (\times 2)\ , \end{aligned} \tag{6.21}$$

in terms of the AdS$_4$ radius $L^2=-3/V_0=g^{-2}c$. The normalised spectrum (masses and multiplicities) of vector fields reads

$$\begin{aligned} m^2L^2 \;=\;& 0\ (\times 2)\ ,\ 2\ (\times 3)\ ,\ 6\ (\times 3)\ ,\ 2+\chi_\alpha^2\ (\times 2)\ ,\ 2+\tfrac{1}{4}(\chi_1\pm\chi_2)^2\ (\times 4)\ ,\\ & 3+\tfrac{1}{4}\,(\chi_1+\chi_2)^2\pm\sqrt{9+(\chi_1+\chi_2)^2}\ (\times 2)\ ,\\ & 3+\tfrac{1}{4}\,(\chi_1-\chi_2)^2\pm\sqrt{9+(\chi_1-\chi_2)^2}\ (\times 2)\ , \end{aligned} \tag{6.22}$$

and contains two massless vectors at generic values of χ_α. Lastly, the computation of the eight normalised gravitino masses yields[5]

$$m^2L^2 \;=\; \tfrac{5}{2}+\tfrac{1}{4}\chi_\alpha^2\pm\tfrac{1}{2}\sqrt{9+\chi_\alpha^2}\ (\times 2)\ . \tag{6.23}$$

[4]The index α in this section should not be confused with the $\mathrm{E}_{7(7)}$ adjoint index in the previous sections.

[5]In our conventions a massless gravitino associated with a preserved supersymmetry has $m^2L^2=1$.

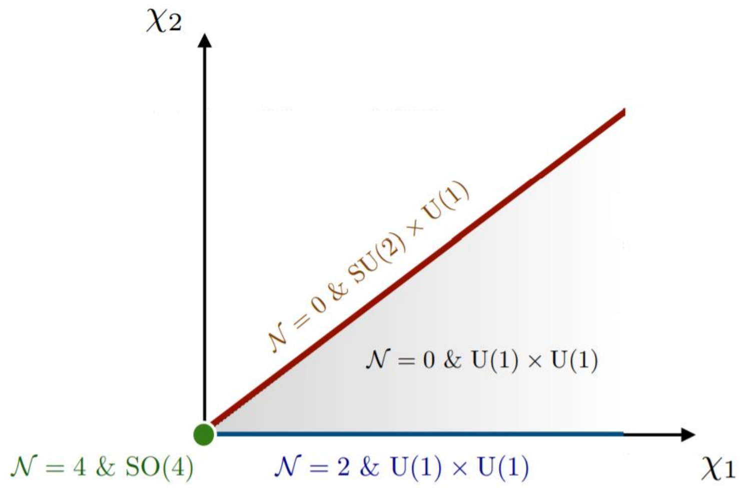

FIGURE 6.1: Two-dimensional parameter space (χ_1, χ_2) of 4d S-fold solutions induced by axion-like flat deformations of the $\mathcal{N}=4$ and SO(4) symmetric S-fold (green dot). The blue and red lines correspond to special choices of parameters $\chi_{1,2}$ and produce supersymmetry (blue, $\chi_2=0$) and residual symmetry (red, $\chi_1=\chi_2$) enhancements. These two special lines define the boundary of the parameter space.

By inspection of (6.21)-(6.23) we identify four different classes of flat deformations of the $\mathcal{N}=4$ and SO(4) symmetric S-fold (see Figure 6.1):

- At generic values of $\chi_{1,2}$ one finds non-supersymmetric S-folds with a $\mathrm{U}(1)^2$ symmetry which is interpreted as a flavour symmetry in the dual S-fold CFT's.
- Setting $\chi_2=0$ (equivalently $\chi_1=0$) produces a one-parameter family of $\mathcal{N}=2$ supersymmetric S-folds with $\mathrm{U}(1)\times\mathrm{U}(1)$ symmetry. These holographically describe a subclass of the $\mathcal{N}=2$ S-fold CFT's with a U(1) flavour symmetry investigated in [56].
- Setting $\chi_1=\pm\chi_2$ gives rise to a one-parameter family of non-supersymmetric S-folds with $\mathrm{SU}(2)\times\mathrm{U}(1)$ symmetry. This implies a flavour symmetry enhancement of the form $\mathrm{U}(1)\times\mathrm{U}(1)\to\mathrm{SU}(2)\times\mathrm{U}(1)$ in the dual S-fold CFT's.
- Setting $\chi_1=\chi_2=0$ gives back the original (undeformed) $\mathcal{N}=4$ supersymmetric S-folds with SO(4) symmetry.

Note that χ_1 and χ_2 enter (6.21)-(6.23) symmetrically, as expected, and that the scalar mass spectrum in (6.21) does not display any instability associated with a normalised mass mode violating the Breitenlohner–Freedman (BF) bound $m^2L^2\geq -9/4$ for perturbative stability in AdS_4 [51]. Therefore, this family of non-supersymmetric S-folds discussed above turn out to be pertubatively stable at the lower-dimensional supergravity level. This computation however does not protect us from having modes with a mass below the BF bound in the higher KK modes of the theory. We will come back to this issue when studying this moduli space in ten dimensions.

As a last remark, it is possible to do equivalent computations for the other families of S-fold we have discussed here.

- For the $\mathcal{N}=0$ family, the $\mathbb{Z}_2^3$ truncation of the theory already captured the three axionic deformations expected from the rank of the residual symmetry group $SO(6)$.

- For the $\mathcal{N}=1$ family, the $\mathbb{Z}_2^3$ truncation of the theory captures the two axionic deformations expected from the rank of the residual symmetry group $SU(3)$.
- For the $\mathcal{N}=2$ family of solutions, only one of the two deformations expected from the rank of $SU(2)\times U(1)$ is present in the $\mathbb{Z}_2^3$ model. The missing axionic deformation breaks the $\mathcal{N}=2$ supersymmetry down to $\mathcal{N}=0$ supersymmetry.
- For the φ family of solution that connects the $\mathcal{N}=2$ solution and the $\mathcal{N}=4$ solution with residual symmetry group $U(1)^2$, one can also build 2 axionic deformations. A generic point on this moduli has in general $\mathcal{N}=0$ and $U(1)^2$ residual symmetry.

From these examples, it becomes clear that it is generally possible to break supersymmetry by turning on the axions dual to the R-symmetry group (if such a group is continuous i.e., $\mathcal{N}>1$).

6.3 5d origin and CSS gaugings

The results in the previous section followed from the very specific flat deformations introduced by the axions χ^{ij} which, as already emphasised, can be alternatively understood in terms of the $\delta\Theta$ deformation tensor in (6.8). The attentive reader might have recognised in (6.8) a structure similar to a CSS gauging [90]. This type of gaugings appears when compactifying 5d supergravity down to 4d. We will now elaborate more on this point.

Let us start by recalling that there is a formally $\mathrm{E}_{6(6)}$-covariant formulation of the maximal 5d gauged supergravities provided by the embedding tensor formalism [91]. The bosonic sector of the maximal $\mathcal{N}=8$ supergravity multiplet in five dimensions consists of the metric field $g_{\mu\nu}$, a set of $\mathbf{27}'$ vector fields and $\mathbf{27}$ two-form tensor fields, and 42 scalar fields parametrising the coset space $\mathcal{M}_{\text{scal}}=\mathrm{E}_{6(6)}/\mathrm{USp}(8)$. The embedding tensor is subject to a set of linear or representation constraints restricting it to the $\mathbf{351}$ representation. In addition, it must also obey a set of quadratic constraints in order to specify a consistent gauging of the theory. To establish a 4d $\leftrightarrow$ 5d connection we will perform a group-theoretical decomposition of the $\mathrm{E}_{7(7)}$ representations in $\mathbf{56}$, $\mathbf{133}$ and $\mathbf{912}$ under the maximal subgroup $\mathrm{E}_{6(6)}\times\mathrm{SO}(1,1)\subset\mathrm{E}_{7(7)}$. This yields

$$\mathbf{56}\rightarrow\mathbf{1}_{-3}\oplus\mathbf{27}'_{-1}\oplus\mathbf{27}_{+1}\oplus\mathbf{1}_{+3}\ ,\tag{6.24}$$

$$\mathbf{133}\rightarrow\mathbf{78}_{0}\oplus\mathbf{1}_{0}\oplus\mathbf{27}_{-2}\oplus\mathbf{27}'_{+2}\ ,\tag{6.25}$$

$$\mathbf{912}\rightarrow\mathbf{78}_{-3}\oplus\mathbf{27}'_{-1}\oplus\mathbf{351}'_{-1}\oplus\mathbf{351}_{+1}\oplus\mathbf{27}_{+1}\oplus\mathbf{78}_{+3}\ .\tag{6.26}$$

In (6.26), we observe the embedding tensor $\mathbf{351}_{+1}$ of the 5d theory descends from the embedding tensor $\mathbf{912}$ of the 4d theory. However, comparing $SO(1,1)$ charges, in order to understand the embedding tensor $\tilde{\Theta}$ in (6.3), we will further need the $\mathbf{78}_{+3}$ in (6.26). This becomes clear when looking at the group-theoretical decompositions of the said representations under $\mathrm{E}_{6(6)}\times\mathrm{SO}(1,1)\rightarrow\mathrm{SL}(6)\times\mathrm{SL}(2)\times\mathrm{SO}(1,1)$, namely,

$$\mathbf{351}_{+1}\rightarrow\underbrace{(\mathbf{21},\,\mathbf{1})_{+1}}_{SO(6)\text{ gauging}}\oplus(\mathbf{84}',\,\mathbf{2})_{+1}\oplus(\mathbf{105},\,\mathbf{1})_{+1}\oplus(\mathbf{15},\,\mathbf{3})_{+1}\oplus(\mathbf{6}',\,\mathbf{2})_{+1}\ ,\tag{6.27}$$

$$\mathbf{78}_{+3}\rightarrow\underbrace{(\mathbf{35},\,\mathbf{1})_{+3}}_{\text{flat deformation}}\oplus(\mathbf{20},\,\mathbf{2})_{+3}\oplus\underbrace{(\mathbf{1},\,\mathbf{3})_{+3}}_{SO(1,1)\text{ gauging}}\ .\tag{6.28}$$

Importantly, while the $\mathbf{351}_{+1}$ captures the electric $(\mathbf{21},\mathbf{1})_{+1}$ piece in the 4d embedding tensor $\widetilde{\Theta}$ induced by g, it does not capture the magnetic $(\mathbf{1},\mathbf{3})_{+3}$ and $(\mathbf{35},\mathbf{1})_{+3}$ pieces induced by c and $\chi_i{}^j$. These two pieces are instead contained in the $\mathbf{78}_{+3}$, because they have matching SO(1,1) charge. A direct consequence is that the 4d gauging $\widetilde{\Theta}$ involving the $(\mathbf{1},\,\mathbf{3})_{+3}\subset\mathbf{78}_{+3}$ (c-terms) and $(\mathbf{35},\,\mathbf{1})_{+3}\subset\mathbf{78}_{+3}$ (χ-terms) cannot be directly uplifted to an embedding tensor deformation in 5d. These terms are instead generated dynamically by introducing an explicit dependence on the S^1 coordinate (in the form of a duality twist [92]) in the reduction process from 5d to 4d.

A general duality twist takes the form [90, 88]

$$\phi(x^\mu,\,\eta)=e^{M\,\eta}\star\phi(x^\mu)\ , \tag{6.29}$$

where η is the coordinate along the S^1, $M\in\mathbf{78}=\mathfrak{e}_{6(6)}$ and ϕ is any field in the 5d theory. This type of duality twists has been studied in the context of 5d *ungauged* supergravity [93]. Within this context, the dependence on the η coordinate factorises out in the reduction process as a consequence of M being chosen in the *global* duality group $\mathrm{E}_{6(6)}$ of the 5d theory. Moreover, choosing M in the maximal compact subalgebra $\mathfrak{usp}(8)\subset\mathfrak{e}_{6(6)}$ makes the scalar potential vanish identically in the reduced 4d theory. Our scenario, however, differs from the one just discussed: the theory to begin with is the 5d SO(6)-gauged supergravity. The global duality group $\mathrm{E}_{6(6)}$ is broken to a *local* SO(6) and a *global* SL(2), and the axion-like flat deformation $M=\chi_i{}^j\in\mathfrak{g}_0\subset\mathfrak{so}(6)$ leaves invariant the embedding tensor of the SO(6) gauging. Then, in our case, the flat deformations $\chi_i{}^j$ are expected to describe apparently trivial twists leaving the putative 5d backgrounds *locally* invariant.

The simultaneous study of axion-like $\chi_i{}^j$ and electromagnetic c deformations requires to investigate a general twist in the $\mathbf{78}_{+3}$. From a 4d perspective, and by virtue of (6.28), the most general contraction $(\delta\Theta)_M{}^\alpha\,t_\alpha\in\mathbf{78}_{+3}$ takes the form

$$\begin{aligned}
\delta\Theta_{ij} &= \chi_i{}^k\,t_{1kj8}-\chi_j{}^k\,t_{1ki8}+\chi^{almn}\,t_a{}^k\,\epsilon_{kijlmn}\ ,\\
\delta\Theta^{ai} &= -\epsilon^{ab}\,\chi_j{}^i\,t_b{}^j-\epsilon^{ab}\,\chi_b{}^c\,t_c{}^i-3\,\epsilon^{ab}\,\chi^{cijk}\,t_{bjkc}\ ,\\
\delta\Theta^{18} &= \chi_i{}^j\,t_j{}^i+\chi_a{}^b\,t_b{}^a+\chi^{aijk}\,t_{aijk}\ ,
\end{aligned} \tag{6.30}$$

in terms of three tensors $\chi_i{}^j\in(\mathbf{35},\mathbf{1})_{+3}$, $\chi_a{}^b\in(\mathbf{1},\mathbf{3})_{+3}$ and $\chi^{aijk}=\chi^{a[ijk]}\in(\mathbf{20},\,\mathbf{2})_{+3}$. The embedding tensor $\delta\Theta$ in (6.30) satisfies the quadratic constraint (3.74) and therefore defines a consistent gauging of the maximal supergravity in 4d. Since $\delta\Theta\in\mathbf{78}_{+3}$ carries the highest SO(1,1) charge in the decomposition (6.26) and, when using the solvable parameterisation of $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$, the coset representative $\mathcal{V}$ in (4.17) solely involves $\mathrm{E}_{7(7)}$ generators with non-negative SO(1,1) charge (Cartan generators and positive roots), it follows that $\delta\Xi\in\mathbf{78}_{+3}$. Then, as for $\delta\Theta$, the scalar-dependent $\delta\Xi$ tensor can be expressed in terms of three scalar-dependent tensors $\delta\Xi_i{}^j\in(\mathbf{35},\mathbf{1})_{+3}$, $\delta\Xi'_a{}^b\in(\mathbf{1},\mathbf{3})_{+3}$ and $\delta\Xi^{aijk}=\delta\Xi^{a[ijk]}\in(\mathbf{20},\,\mathbf{2})_{+3}$.

An explicit computation of the scalar potential (3.92) gives

$$\begin{aligned}
V=\frac{1}{192}\Big\{&\mathrm{Tr}\Big[\Big(\delta\Xi+(\delta\Xi)^t\Big)^2\Big]+\mathrm{Tr}\Big[\Big(\delta\Xi'+(\delta\Xi')^t\Big)^2\Big]\\
&+3\sum_{a\,i\,j\,k}\left(\delta\Xi^{aijk}-\tfrac{1}{3!}\,\epsilon_{ab}\,\epsilon_{ijklmn}\,\delta\Xi^{blmn}\right)^2\Big\}\ ,
\end{aligned} \tag{6.31}$$

which is positive definite. From the potential in (6.31) we see that having an $\mathfrak{so}(6)$-valued $\delta\Xi_i{}^j$ implies $\delta\Xi + (\delta\Xi)^t = 0$ and thus a flat CSS deformation with $V = 0$. This is only possible if $\chi_i{}^j = -\chi_j{}^i$. On the contrary, having an $\mathfrak{so}(1,1)$-valued $\delta\Xi'_a{}^b$ yields $\delta\Xi' + (\delta\Xi')^t \neq 0$ and therefore $V \neq 0$. This explains why the axions χ^{ij} specifying a compact $\mathfrak{so}(6)$ twist produce flat deformations whereas the magnetic parameter c specifying a non-compact $\mathfrak{so}(1,1)$ twist does not. In addition, there is the contribution to the scalar potential (6.31) coming from $\delta\Xi^{aijk} \in (\mathbf{20}, \mathbf{2})_{+3}$. The 20 linear combinations satisfying $\delta\Xi^{aijk} - \frac{1}{3!}\,\epsilon_{ab}\,\epsilon_{ijklmn}\,\delta\Xi^{blmn} = 0$ belong to $\mathfrak{usp}(8) \in \mathfrak{e}_{6(6)}$ and, in combination with $\delta\Xi_i{}^j \in \mathfrak{so}(6)$ and $\delta\Xi'_a{}^b \in \mathfrak{so}(2)$, they specify the most general flat CSS gauging yielding $V = 0$.

To conclude, group-theoretical arguments put forward in [3], that we will study in the next chapter, suggest that axion-like deformations should be related to one-form deformations of $\mathcal{N} = 4$ SYM on $\mathbb{R}^{1,2} \times \mathrm{S}^1$ [94]. Such one-forms are often discarded within the context of Janus solutions by a gauge-fixing argument without much regard for large gauge transformations. Working out the explicit 5d oxidation of the AdS$_4$ vacua with $\chi^{ij} \neq 0$ would be the next step towards testing these ideas. In addition, it would also be interesting to further investigate the interplay between embedding tensor deformations ($\mathbf{351}_{+1}$) and $\mathfrak{e}_{6(6)}$ duality twists ($\mathbf{78}_{+3}$) in S^1 compactifications of other 5d gauged supergravities. Also to understand the physical meaning (if any) of the rest of representations appearing in the group-theoretical decomposition of the **912** in (6.26).[6] We leave these and other related questions for future work.

6.4 Ten-dimensional uplift and mapping tori

To study the higher dimensional origin of the flat deformations, let us start with presenting here the ten-dimensional uplift of the (χ_1, χ_2)-family of deformations of the $\mathcal{N} = 4$ S-fold [95]. For $\chi = 0$, the internal S^5 was expressed as two S^2's, parametrised by angular variables $(\theta_i,\, \varphi_i)$ fibred over an interval. The S^1 factor of the compactification was parametrised by a coordinate η. When uplifting the χ-deformed solution, choosing an appropriate Cartan subalgebra of $\mathfrak{so}(4)$ to parameterise the constant deformations (χ_1, χ_2), one finds the same ten-dimensional S-fold solution as in section 5.4.1 with one crucial difference: the one-forms $d\varphi_i$ along the azimutal angles of the two-spheres S^2_i in (5.73)-(5.74) get replaced by new one-forms

$$d\varphi_i \to d\varphi_i + \chi_i\, d\eta\,. \tag{6.32}$$

This change of one-form basis can be *locally* reabsorbed in a change of coordinates

$$\varphi'_i = \varphi_i + \chi_i\, \eta\ . \tag{6.33}$$

Since all the type IIB fields in the S-fold solution of section 5.4.1 are independent of φ_i, the new backgrounds with $\chi_i \neq 0$ obtained by the minimal replacement $d\varphi_i \to d\varphi'_i$ automatically solve the equations of motion. At this stage, the reader might be tempted to conclude that the solutions with $\chi_i = 0$ and $\chi_i \neq 0$ are simply the same solution but in different coordinate patches. However, due to the periodicities $\varphi_i \sim \varphi_i + 2\pi$ and $\eta \sim \eta + T$, there is no *global* diffeomorphism connecting the two solutions unless $\chi_i = n_i \frac{2\pi}{T}$ with $n_i \in \mathbb{Z}$. This can be understood also from the fact that φ'_i is not a globally well-defined coordinate on $\mathrm{S}^5 \times \mathrm{S}^1$. Indeed, the variable φ'_i gets shifted by the quantity $\chi_i\, T$ when making a loop around S^1. And this quantity becomes a multiple of 2π only when $\chi_i = n_i \frac{2\pi}{T}$ with $n_i \in \mathbb{Z}$. Only in this case, φ'_i

[6] It is tempting to speculate about the $\mathbf{27}'_{-1}$ and the $\mathcal{A}_\eta{}^{1,\ldots,27}$ components of the 5d vector fields.

is a globally well-defined new azimutal angle. The $U(1)^2$ symmetries of the 4d theory can now be understood as the translation along the two φ_i azimutal angles.

For the other Type IIB S-folds $\mathcal{N} = 0$, 1 and 2, in a well-chosen basis, the axionic deformations can be understood in a similar manner.

- For the $\mathcal{N} = 0$ solution, one starts by selecting the three angles θ_i generating rotations in the three orthogonal planes $\mathbb{R}^2 \in \mathbb{R}^6$. Embedding the round S^5 in $\mathbb{R}^6$, the axionic deformations are equivalent to the replacement $\mathrm{d}\theta_i \to \mathrm{d}\theta_i + \chi_i\, \mathrm{d}\eta$.

- For the $\mathcal{N} = 1$ solution, introducing the deformations $\chi_{1,2,3}$ consist in making the replacements

$$\mathrm{d}\alpha \to \mathrm{d}\alpha + (\chi_1 - 2\,\chi_2 + \chi_3)\,\mathrm{d}\eta \tag{6.34}$$

$$\mathrm{d}\gamma \to \mathrm{d}\gamma + (\chi_1 - 3\chi_3)\,\mathrm{d}\eta \tag{6.35}$$

$$\mathrm{d}\phi \to \mathrm{d}\phi + \chi_2\,\mathrm{d}\eta \tag{6.36}$$

  This replacement only gives a solution when $\chi_1 + \chi_2 + \chi_3 = 0$ because the two-forms depend explicitly $\alpha + \phi$ which get translated by a term proportional to $\sum \chi_i$ upon the associated local changes of variables.

- For the $\mathcal{N} = 2$ solution, there are two axionic deformations parametrised by an element of $\mathfrak{su}(2) \times \mathfrak{u}(1)_R$. The $\mathfrak{su}(2)$ deformation replaces

$$\mathrm{d}\gamma \to \mathrm{d}\gamma + \chi_1\,\mathrm{d}\eta\,, \tag{6.37}$$

  and breaks $SU(2)$ to $U(1)$. The other deformation in $\mathfrak{u}(1)_R$ replaces

$$\mathrm{d}\alpha \to \mathrm{d}\alpha + \chi_2\,\mathrm{d}\eta\,. \tag{6.38}$$

  An explicit computation of the mass spectrum of the gravitons for these deformations shows that, while the χ_1 transformation, associated with the $SU(2)$ flavour symmetry, does not break supersymmetry at the 4d level, the χ_2 deformation associated with the $U(1)_R$ symmetry does break the residual supersymmetry.

We will now generalise the results obtained in specific examples and show that they are just applications of Claim 2. This claim gives a more general and systematic way of introducing flat deformations χ for a seed solution of any diffeomorphism-invariant theory involving a factorised geometry of the form $\mathcal{M} \times \mathrm{S}^1$ [3]. Let us denote such a seed solution Φ_0 which encodes the metric and the various p-form fluxes solving the equations of motion of the theory. Let us also assume that Φ_0 is invariant under the action of a Lie group G_0 of rank r. Then, we claim that it is always possible to construct an r-dimensional family of solutions generically breaking G_0 down to its maximal Cartan subgroup $\mathrm{U}(1)^r$. To do so, we first choose an element $h \in \mathrm{G}_0$ which, without loss of generality, can be chosen in a Cartan subgroup of G_0. From the element h, we build the so-called *mapping torus*

$$T(\mathcal{M})_h = \frac{\mathcal{M} \times [0,\, T]}{(p,\, 0) \sim (h \cdot p,\, 2\pi)}\ , \tag{6.39}$$

where the equivalence relation holds for all the points $p \in \mathcal{M}$. The quotient introduced in (6.39) is just a way of encoding different "boundary conditions" (periodicities) for the angular coordinates φ_i on $\mathcal{M}$ which are associated with commuting isometries of the seed solution Φ_0. This amounts to introduce a non-trivial G_0 monodromy h when moving around the S^1 in the factorised geometry $\mathcal{M} \times \mathrm{S}^1$.

To construct the new solutions with non-zero flat deformations, $\chi_i \neq 0$, we first project the seed solution Φ_0 defined on $\mathcal{M} \times \mathrm{S}^1$ to a solution on the mapping torus $T(\mathcal{M})_h$. This is done by the canonical projection

$$\pi : \mathcal{M} \times \mathrm{S}^1 \to \mathcal{M} \times [0,\, T] \xrightarrow{\sim} T(\mathcal{M})_h \ . \tag{6.40}$$

If $h \neq \mathbb{I}$, this projection is *not* a global diffeomorphism. Still, the projection $\pi(\Phi_0)$ is well-defined and $\pi(\Phi_0)$ is smooth as the action of $h \in \mathrm{G}_0$ on Φ_0 is trivial by definition. Moreover, $\pi(\Phi_0)$ continues being a solution of the theory as the equations of motion are local. However the symmetry group of $T(\mathcal{M})_h$ gets reduced to those elements of G_0 commuting with h. Otherwise, their action is not globally well-defined.

Let us illustrate the above construction when selecting the $\mathcal{N} = 4$ and $\mathrm{SO}(4)$ symmetric S-fold as the seed solution Φ_0. In this case we choose $h = \exp(T\,\chi)$ with $\chi \in \mathfrak{u}(1)^2 \subset \mathfrak{so}(4)$. We also introduce coordinates φ_i' on the mapping torus $T(\mathcal{M})_h$ which get shifted by $\chi_i \in \mathfrak{u}(1)$ and are subject to the identifications

$$(\varphi_i',\, \eta) \sim (\varphi_i' + \chi_i\, T,\, \eta + T) \quad \text{and} \quad (\varphi_i',\, \eta) \sim (\varphi_i' + 2\pi,\, \eta) \ . \tag{6.41}$$

The projection from $\mathrm{S}^5 \times \mathrm{S}^1$ to the mapping torus $T(\mathrm{S}^5)_h$ is trivial: it is the replacement $\varphi_i \to \varphi_i'$. Using the φ_i' coordinates, the type IIB fields are χ_i-independent and the flat deformations are totally encoded in the choice of periodicities in (6.41). However, there is an alternative coordinate system in which the deformations are encoded in the type IIB fields and not in the monodromy h. Let us consider a diffeomorphism of the form

$$m_\chi : T(\mathcal{M})_h \to \mathcal{M} \times \mathrm{S}^1 : (p,\, \eta) \to (e^{\chi\eta} \cdot p,\, \eta) \ . \tag{6.42}$$

The map m_χ is indeed a well-defined diffeomorphism, even globally, due to the different choice of coordinates identification when moving around the S^1. This corresponds, locally, to a change of coordinates of the form

$$(\varphi_i',\, \eta) \to (\varphi_i + \chi_i\, \eta,\, \eta) \ , \tag{6.43}$$

where $\varphi_i \sim \varphi_i + 2\pi$ and $\eta \sim \eta + T$. Since m_χ in (6.42) is globally a well-defined diffeomorphism, this implies that solutions depending explicitly on the χ_i deformations and involving an internal manifold with *trivial* monodromy h are equivalent to solutions with no explicit dependence on the χ_i deformations but involving an internal manifold with *non-trivial* monodromy h. For the confused reader (we were too), this is summarised in Figure 6.2. Finally, this construction shows that only the conjugacy class of $h \in G_0$ is relevant in 10d because the following equivalence relations are the same:

$$\begin{aligned} [(p,\, 0) \sim (h \cdot p,\, 2\pi)] &= [(g \cdot p,\, 0) \sim (h \cdot g \cdot p,\, 2\pi)] \\ &= \left[(p,\, 0) \sim (g^{-1} \cdot h \cdot g \cdot p,\, 2\pi)\right] \ . \end{aligned} \tag{6.44}$$

This implies that the moduli space of solutions can be identified with the conjugacy classes of G_0. This is why we could choose $\chi \in \mathfrak{u}(1)^2 \subset \mathfrak{so}(4)$ without loss of generality.

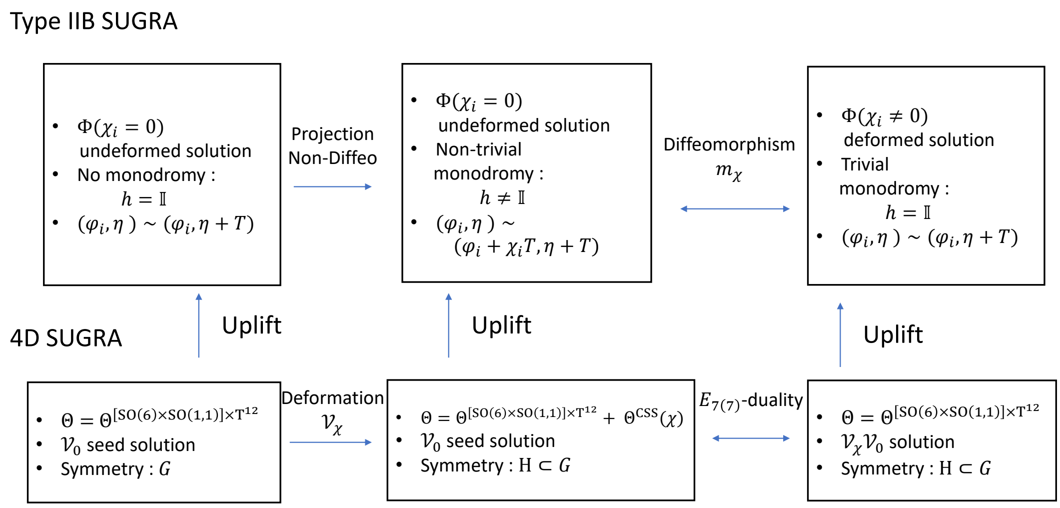

FIGURE 6.2: Schematic representation of the flat deformations from four- and ten-dimensional perspectives.

6.4.1 Relation to TsT

It is instructive to compare the deformation of the $\mathcal{N} = 4$ S-fold solution analysed here, with the deformation, discussed in [96], of the maximally supersymmetric $\text{AdS}_5 \times S^5$ Type IIB background, which generalises the Lunin-Maldacena construction [97]. The holographic dual to this solution is conjectured to be a non-supersymmetric marginal deformation of $\mathcal{N} = 4$ four-dimensional SYM theory. However, [98] suggested that conformal symmetry of this dual theory is absent, while [99, 100] hint at the existence of a tachyonic instability in the corresponding superstring background. In [96], the deformation parameters γ_I, $I = 1, 2, 3$, were the effect of shift transformations in the $\text{O}(3, 3)$ group acting on the three angular directions associated with translational isometries [101] along internal angular coordinates. These shift transformations were, however, preceded and followed by T-dualities ("factorised dualities") of the kind $R_I \to 1/R_I$ along all the three directions. Just as S^5 in the $\text{AdS}_5 \times S^5$ background, the internal manifold $\mathcal{I} \times S^2_1 \times S^2_2 \times S^1_\eta$ of the $\mathcal{N} = 4$ S-fold solution features three angular coordinates $\xi^I = \varphi_1, \varphi_2, \eta$ each associated with a translational symmetry of the internal metric, although, in the latter case, a constant translation along η is not a symmetry of the whole solution due to the $\text{SL}(2, \mathbb{R})_{\text{IIB}}$-twist.

As opposed to the construction of [96], the χ_i-deformation discussed here only results from a shift transformation in $\text{GL}(3, \mathbb{R}) \subset \text{O}(3, 3)$, with no T-dualities. This is effected by the $\text{GL}(3, \mathbb{R})$ matrix

$$A = \begin{pmatrix} 1 & 0 & \chi_1 \\ 0 & 1 & \chi_2 \\ 0 & 0 & 1 \end{pmatrix} , \tag{6.45}$$

which acts linearly on the I-component of all the fields. The components $g = (g_{IJ})$ of the internal metric along the angular directions ξ^I, for instance, transforms as follows:

$$g \to A^t \, g \, A \, . \tag{6.46}$$

Our χ_i deformations thus change the metric on the $S^5 \times S^1_\eta$ compactification, while leaving the fibration structure unchanged. This is analogous to complex structure deformations of $T^2 \sim S^1 \times S^1$, which can also locally be absorbed by diffeomorphisms which are, however, not globally well-defined. Indeed, our χ_i appear like the real part of complex structure deformations of the $\varphi_i \times S^1_\eta$ tori. A more precise definition is in terms of the mapping torus of S^5 [4]: the χ_i deformations imply that as we move around S^1_η, we end up in a different point on S^5. If $\chi_i \to \chi_i + 2\pi k_i/T$, $k_i \in \mathbb{Z}$, the deformation is in $\mathrm{GL}(3,\mathbb{Z})$ and the internal geometry is not affected. Invariance of the full spectrum, however, due to the presence of states with half-integer j_1, j_2, extends the periodicity of χ_i to $4\pi/T$, as will be discussed below.

6.5 More on the χ-deformation of the $\mathcal{N}=4$ solution

Via the AdS/CFT correspondence, our family of non-supersymmetric AdS_4 vacua of IIB string theory deforming the $\mathcal{N}=4$ solution suggests that the dual "J-fold" CFT_3 should belong to a non-supersymmetric conformal manifold. However, this is not the case if the non-supersymmetric AdS_4 vacua are unstable, as conjectured in [89]. These instabilities could arise due to some scalar fluctuation in the Kaluza-Klein spectrum violating the Breitenlohner-Freedman bound, or via a non-perturbative phenomenon. Let us now address these concerns for the (χ_1, χ_2) family of deformations of the $\mathcal{N}=4$ solution.

6.5.1 Perturbative stability and higher KK modes

First, we will prove that the Kaluza-Klein spectrum has no tachyons, i.e. the AdS_4 vacua are perturbatively stable. To do this, we use the technology developed in [85, 102] to compute the full Kaluza-Klein spectrum around the family of non-supersymmetric AdS_4 vacua we consider here.

It is easiest to express the Kaluza-Klein spectrum as a deformation of the spectrum of the $\mathcal{N}=4$ vacuum. The full $\mathcal{N}=4$ spectrum was computed in [103, 95]. Note that our S^1 radius differs from the convention of [95] such that $T_{\text{there}} = \frac{T_{\text{here}}}{2}$. The conformal dimension of the highest weight state of each supermultiplet is given by

$$\Delta = \frac{3}{2} + \frac{1}{2}\sqrt{9 + 2\ell(\ell+4) + 4\sum_{i=1,2}\ell_i(\ell_i+1) + 2\left(\frac{2n\pi}{T}\right)^2}\,, \tag{6.47}$$

where ℓ denotes the S^5 Kaluza-Klein level, n the S^1 level and ℓ_1, ℓ_2 the SO(4) spin of the highest weight state (in this case, the graviton). These $\mathcal{N}=4$ supermultiplets are counted by the generating function for their highest weight states:

$$\nu = \frac{1}{(1-q^2)(1-q\,u)(1-q\,v)}\frac{1+s}{1-s}\,, \tag{6.48}$$

where the exponent of q and s determine the Kaluza-Klein levels on the S^5, ℓ, and S^1, n, while the exponents of u and v count the $SU(2)\times SU(2)$ spins, ℓ_1 and ℓ_2. The effect of the $\chi_{1,2}$ deformations is to shift the conformal dimension of each field by replacing

$$\frac{2n\pi}{T} \longrightarrow \frac{2n\pi}{T} + (j_1+j_2)\chi_+ + (j_1-j_2)\chi_-\,, \tag{6.49}$$

in (6.47), where j_1, j_2 are the charges of the field under the $U(1)\times U(1)$ Cartan of SO(4) and we defined $\chi_\pm = \frac{1}{2}(\chi_1 \pm \chi_2)$. Note from (6.48) that, while j_1, j_2 are half-integers, $j_1 \pm j_2$ are always integers. Thus, we manifestly see that the full background has periodicity $\chi_\pm \to \chi_\pm + \frac{2\pi}{T}$, upon which the Kaluza-Klein spectrum gets mapped back to itself with an appropriate reshuffling of the fields amongst the S^1 level with $n \to n - (j_1 \pm j_2)$, just like in [95]. Notice that χ_1, χ_2 separately have period $4\pi/T$, which can only be seen from the spinors on the AdS$_4$ background which have half-integers charges under the $U(1) \times U(1)$ Cartan. This means that, although we have always written the $\mathcal{N} = 4$ solution to be $SO(4)$ invariant, it would have been more correct to write it as $Spin(4)$ invariant when including fermions in the mapping torus interpretation.

Even more importantly, we can see that the masses for all the fields are bounded from below by the masses of the fields of the four-dimensional supergravity at the $\mathcal{N} = 4$ vacuum, i.e. where $\ell = \ell_1 = \ell_2 = n = \chi_i = 0$. This in particular implies that all scalars have masses above the Breitenlohner-Freedman bound for any value of χ_i. Thus, the non-supersymmetric vacua are perturbatively stable.

One may also wonder whether the AdS$_4$ vacua are secretly supersymmetric in 10 dimensions, with some gravitinos amongst the higher Kaluza-Klein modes becoming light, akin to the "space invaders" scenario [104, 105, 95]. However, from (6.47), (6.49), we can easily see that such gravitinos can only restore supersymmetry when the combination $\frac{2n\pi}{T} + j_1\chi_1 + j_2\chi_2 = 0$. This can only occur when either $n = 0$ and $\chi_1 = \pm\chi_2$, corresponding to supersymmetry enhancement that already occurs in the four-dimensional supergravity, or $\chi_\pm = \frac{2\pi k_\pm}{T}$, for $k_\pm \in \mathbb{Z}$ when some gravitinos at S^1 level $n = -(j_1 + j_2)k_+ - (j_1 - j_2)k_-$ become massless. This latter condition is precisely when the background is mapped back to itself, so that for $0 < \chi_\pm < \frac{2\pi}{T}$, $\chi_1 \neq \pm\chi_2$, the AdS$_4$ vacua are not supersymmetric in the full Type IIB string theory.

In conclusion, our computation of the Kaluza-Klein spectrum (6.47), (6.49) reveals the $\frac{4\pi}{T}$ periodicity of the exactly marginal deformations parameterised by χ_i as shown in Figure 6.3. It also gives the anomalous dimensions of all operators of the CFT dual to supergravity modes along the non-supersymmetric conformal manifold and finally, it shows that the non-supersymmetric vacua are perturbatively stable.

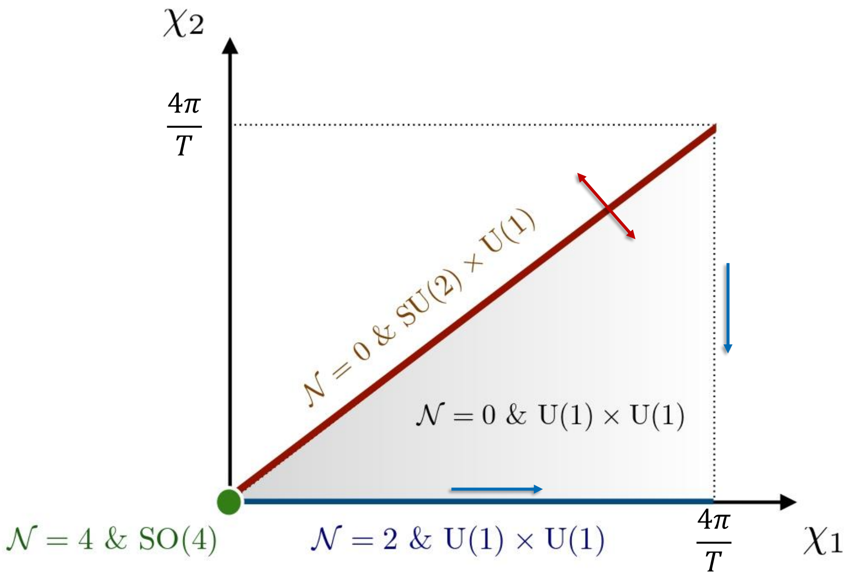

Figure 6.3: The moduli space of inequivalent flat deformations around the $\mathcal{N} = 4$ S-fold. The red arrow indicates a $\mathbb{Z}_2$ symmetry along the $\chi_1 = \chi_2$ axis. The blue arrows indicate how to identify the edges of the moduli space.

6.5.2 Non-perturbative stability and the Swampland

Next, we investigate the non-perturbative stability of the non-supersymmetric AdS_4 vacua. Since the AdS_4 vacua arise as near-horizon limits of certain brane configurations, one may worry that for the non-supersymmetric vacua the corresponding brane configurations become unstable [106]. We search for signs of such instabilities by considering single probe Dp-branes (and single probe NS5-branes) with rigid embeddings in our AdS_4 vacua. In particular, we check whether the branes are unstable due to a greater repulsive force of the fluxes coming from the WZ term than the attractive (i.e. towards the interior of the AdS spacetime) gravitational force due to the DBI term. Indeed, [89] conjectures that there should always be some branes that are unstable in this way, see also [107]. However, we find that single probe Dp-branes and NS5-branes without worldvolume flux remain stable when placed in the non-supersymmetric backgrounds.

The stability of these probe branes can be understood in the following way. Firstly, note that we can perform the diffeomorphism (6.33) to remove the χ_i deformation from the metric. However, now the coordinates respect the deformed periodicities

$$\begin{aligned} \varphi_i' &\to \varphi_i' + 2\pi \,, \\ \eta \to \eta + T \,, &\quad \varphi_i' \to \varphi_i' + \chi_i T \,. \end{aligned} \tag{6.50}$$

As a result, the only well-defined embeddings of branes wrapping η must also wrap φ_i'. In particular, let us denote by $\xi \sim \xi + T$ the relevant wrapped worldvolume coordinate on the brane. Then, the only well-defined embeddings are given by

$$\eta(\xi) = q\,\xi \,, \qquad \varphi_i'(\xi) = (p_i\, 2\pi + q\,\chi_i)\,\xi \,, \tag{6.51}$$

with q, $p_i \in \mathbb{Z}$. We see that as χ_i is turned on, a brane wrapping S^1_η must also wrap increasing amounts of φ_i', so that the DBI part of the action increases. At the same time, for p-branes, with $p \neq 5$, the WZ coupling is insensitive to wrapping along φ_i', unless the brane is completely internal. Therefore, these branes either become more stable as χ_i are turned on or they are completely internal branes, which cannot trigger non-perturbative instabilities in the usual way. Finally, an explicit computation for NS5- and D5-branes shows that they also remain stable as χ_i are turned on in the backgrounds our non-supersymmetric backgrounds.

Non-supersymmetric vacua may also decay due to bubbles of nothing [108], which requires a circle or sphere [109] to collapse. However, our internal space $S^5 \times S^1_\eta$ is topologically protected from such a collapse: the S^5 cannot collapse as it is supported by flux, whereas the S^1_η cannot collapse since the spinors do not have anti-periodic boundary conditions on it [108], but instead general periodicities along S^1_η, provided $(\chi_1, \chi_2) \neq (\frac{2\pi}{T}, 0)$, $(0, \frac{2\pi}{T})$. This means that a straightforward bubble of nothing cannot occur. Still, our vacua could decay semi-classically via more complicated bubbles of nothing containing defects, e.g. a D3-brane in S^5 similar to [110, 111] or an O7-plane in S^1 [112]. However, because the volume form of the compactification is independent of the χ_i deformations, our non-supersymmetric AdS_4 vacua are likely to be stable against the instanton decay constructed in [111], which is *delocalised* on the compactification space. On the other hand, constructing the *localised* instanton solutions is extremely technically challenging. Moreover, the mechanism of [111] treats a shrinking dilaton as equivalent to a shrinking S^1. Aside from the validity of this equivalence, a similar shrinking dilaton would be problematic for our S-fold vacua, where the dilaton is not well-defined due to the $SL(2,\mathbb{Z})$ monodromy along S^1_η.

So far, we have proven that our AdS_4 vacua are perturbatively stable and have provided evidence that they may also be stable against semi-classical decay. However, one may worry that while our AdS_4 geometries are solutions of IIB supergravity, the higher-derivative corrections of IIB string theory will spoil our solutions. In the dual CFT, this would imply that some $\frac{1}{N}$ corrections lift the conformal manifold. However, the deformations χ_i can always be locally absorbed by the coordinate redefinition (6.33), even if it is not globally well-defined. Therefore, all local diffeomorphism-invariant quantities are independent of the χ_i. In particular, this means that each term of the higher-derivative corrections of string theory, involving powers of the curvature tensor or the fluxes, are also independent of $\chi_{1,2}$. Thus, our non-supersymmetric AdS_4 vacua are equally valid solutions of IIB string theory as the $\mathcal{N}=4$ vacuum. Moreover, the χ_i deformations actually correspond to parity-odd (pseudo) scalars in the maximal supergravity [4], so the potential $1/N$ tadpole destabilisation of [113] cannot take place for our backgrounds.

There could still be some string corrections, e.g. from branes wrapping the compactification, which are sensitive to χ_i and which could thus spoil our solutions. For example, Dp-instantons could wrap some $(p+1)$-cycle of the compactification, and depend on χ_i. However, our solutions are also protected against such instanton corrections, since the compactification $S^5\times S^1_\eta$ only has non-trivial 1-, 5- and 6-cycles. Therefore, we can only have D5-instantons wrapped on the full $S^5\times S^1_\eta$. But since the volume form is independent of χ_i, these instantons give no corrections to our solutions. Nonetheless, one could expect some other extended state to do so, corresponding to some $\frac{1}{N}$ correction in the dual CFT.

6.5.3 The holographic interpretation

As was discussed in section 5.5, the SCFT dual to the $\mathcal{N}=4$ $SO(4)$ background emerges as the effective IR description of a $3d$ $\mathrm{T}[\mathrm{U}(N)]$ theory [84] in which the diagonal subgroup of the $\mathrm{U}(N)\times\mathrm{U}(N)$ flavour group has been gauged using an $\mathcal{N}=4$ vector multiplet [71]. In addition, a Chern-Simons term at level k must be introduced which is defined by the $J_k=-\mathcal{S}\,\mathcal{T}^k\in\mathrm{SL}(2,\mathbb{Z})_{\mathrm{IIB}}$ monodromy along the S^1_η. The effective $\mathcal{N}=4$ superpotential $W_{\text{eff}}=(2\pi/k)\,\mathrm{Tr}(\mu_H\,\mu_C)$ [114] is marginal in the IR and, in [56], a shift

$$W_{\text{eff}}\to W_{\text{eff}}+\lambda\,\mathrm{Tr}(\mu_H\,\mu_C)\quad\text{with}\quad\lambda\in\mathbb{C}\tag{6.52}$$

was proposed as an exactly marginal deformation preserving $\mathcal{N}=2$. The scalar superconformal primary operators μ_H and μ_C are respectively described by the moment maps of the Higgs and Coulomb branch of $\mathrm{T}[\mathrm{U}(N)]$. Each of the μ_H and μ_C fields is a component of a vector in the adjoint representation of the corresponding SU(2) subgroup of the SO(4) R-symmetry group (to be denoted by $\mathrm{SU}(2)_H$ and $\mathrm{SU}(2)_C$, respectively). We can therefore associate with μ_H the quantum numbers $j_1=1$, $j_2=0$ and with μ_C the values $j_1=0$, $j_2=1$, having identified j_1, j_2 with the eigenvalues of the Cartan generators of $\mathrm{SU}(2)_H$ and $\mathrm{SU}(2)_C$, respectively. Note that χ_1 (χ_2) only breaks $\mathrm{SU}(2)_H$ ($\mathrm{SU}(2)_C$) to its $\mathrm{U}(1)_H$ ($\mathrm{U}(1)_C$) subgroup. The combination $(\chi_1-\chi_2)/2$ of these two parameters, for $\chi_1=-\chi_2$, should already be encoded in the λ parameter of the $\mathcal{N}=2$ exactly marginal deformation proposed in [56]. The second real parameter in $\lambda\in\mathbb{C}$ should encode the φ deformation connecting the $\mathcal{N}=4$ and $\mathcal{N}=2$ families of solutions present in 4d.

We suggest that the orthogonal combination $(\chi_1+\chi_2)/2$, be encoded in the conjectured exactly marginal deformation of the $3d$ Lagrangian:$\partial_\alpha\mathcal{O}\,\partial^\alpha\bar{\mathcal{O}}$, where $\mathcal{O}\equiv$

$\mathrm{Tr}(\mu_H\,\bar{\mu}_C)$ is an operator with $j_1=1$, $j_2=-1$ and ∂_α denote the partial derivatives with respect to the (real) scalar fields. As opposed to $\mathrm{Tr}(\mu_H\,\mu_C)$, the above term does not originate from a holomorphic deformation of the superpotential and thus would break all supersymmetries. The exact marginality of the operator $\partial_\alpha\mathcal{O}\,\partial^\alpha\bar{\mathcal{O}}$ is here conjectured in light of the holographic evidence we put forward. Note that the resulting $\mathcal{N}=0$ theory would be parity symmetric in both the Higgs and the Coulomb sector. By performing, for instance, a parity transformation in the Coulomb sector which changes sign to the complex structure of the hyper-Kähler manifold (described as a complex Kähler space), $\mu_C\to\bar{\mu}_C$, and $\mathcal{O}$ would be exchanged with the exactly marginal operator $\mathrm{Tr}(\mu_H\,\mu_C)$ in the superpotential proposed in [56]. The same transformation would correspond in the bulk to a parity $\varphi_2\to-\varphi_2$ in S_2^2 and, correspondingly, to $\chi_2\to-\chi_2$. It is therefore the simultaneous presence of the deformations $\mathcal{O}$, $\bar{\mathcal{O}}$ and $\mathrm{Tr}(\mu_H\,\mu_C)$ in the Lagrangian which breaks supersymmetry. Also, the $\chi_1\leftrightarrow\chi_2$ symmetry of the supergravity backgrounds amounts to an exchange symmetry between the Higgs and Coulomb branches in the dual non-supersymmetric CFT's.

6.5.4 Final remarks and generalisations

In this section, we provided the first holographic evidence for the existence of a non-supersymmetric conformal manifold. We did this by constructing a 2-parameter family of non-supersymmetric S-fold AdS_4 vacua of IIB string theory and proving that they are perturbatively stable. Moreover, we excluded several potential non-perturbative instability mechanisms, and showed that our solutions are even protected against some higher-derivative corrections.

Our findings here can be generalised and applied to other settings. For example, an analogous non-supersymmetric 2-parameter family of S-fold AdS_4 vacua can be obtained by performing similar axionic deformations to the $U(1)$ R-symmetry and $SU(2)$ flavour symmetry of the $\mathcal{N}=2$ $SU(2)\times U(1)$ AdS_4 S-fold vacuum of IIB string theory [2]. This moduli space has a one dimensional locus of $\mathcal{N}=0$ deformations of the $\mathcal{N}=2$ $SU(2)\times U(1)$ vacuum, also contains the supersymmetric deformation studied in [95] and should be connected to our conformal manifold since there is an exactly marginal deformation, parametrised by φ, connecting the $\mathcal{N}=2$ and $\mathcal{N}=4$ vacua [56]. We explicitly verified that this second 2-parameter family is also perturbatively stable and has the same protection against non-perturbative mechanisms as was shown by our brane-jet computation and topological arguments. Moreover, the axionic deformations can again be reabsorbed by local coordinate redefinitions that fail to be globally well-defined [4], yielding the same space-invaders scenario as here which leads to a T^2 moduli space. This also protects this 2-parameter family of AdS_4 vacua against higher-derivative corrections. Moreover, this same argument can be applied to the recently-constructed moduli space of $\mathcal{N}=1$ CFT_3's [86], which would suggest that also this $\mathcal{N}=1$ moduli space is protected against some higher-derivative corrections of string theory. The methods laid out here should also apply to a related class of S-folds where S^5 is replaced by a Sasaki-Einstein manifold.

The fate of our family of non-supersymmetric AdS_4 vacua deserves further investigation. The brane-web whose near-horizon limit corresponds to the AdS_4 vacua could still suffer from some other instability mechanism. For example, it could feature some tachyon in its fluctuation spectrum, see e.g. [115, 116] for recent discussions. However, because we do not know the brane-web that would give rise to the AdS_4 vacua, it is currently unclear which probe branes to use for this computation. Still, the existence of a continuous limit to the $\chi_i=0$ supersymmetric case could help in taming such potential instabilities. Also, some non-perturbative string corrections could lift the moduli space. Finally, the CFT_3 interpretation of the χ_i deformations deserves further exploration.

Chapter 7

Holographic RG flows

In the previous chapter, we studied, holographically, exactly marginal deformations, i.e. moduli space of vacua. In this chapter, we are going to discuss non-marginal deformations of CFTs which trigger RG-flow. The holographic dual of these RG-flows are Domain Wall (DW) solutions interpolating between two different asymptotic behaviour in the deep UV or in the deep IR. In this chapter we will study such domain wall solutions using their descriptions as solution of 4d $[SO(1,\,1)\times SO(6)]\ltimes\mathbb{R}^{12}$ gauged maximal supergravity.

This model admits two kinds of asymptotic behaviours. The first one is simply an AdS_4 vacua that can be either the IR or the UV fixed point of the flow. There is another type of UV asymptotic behaviour which is the D3-brane solution. This D3-brane solution appears in the limit where $c \to 0$ and it is a 4d DW which uplifts to the usual $\text{AdS}_5 \times S^5$ solution of Type IIB supergravity. We will show that, perturbing this solution in power of c correspond to perturbing it by adding anisotropic deformations. Moreover, this expansion in powers of c will actually depend only on $\frac{c}{(gz)^2}$ meaning that the $c \to 0$ limit is equivalent to the $z \to \infty$ which is the UV limit. As such the D3-brane is a well defined asymptotic behaviour of the flow in the UV

Depending on their asymptotic behaviour, two types of holographic RG flows appear in the $[SO(1,\,1)\times SO(6)]\ltimes\mathbb{R}^{12}$ gauged maximal supergravity:

i) SYM_4 to CFT_3 flows connect various AdS_4 vacua (dual to J-fold CFT_3's) in the IR to c-deformation of the D3-brane behaviour in the UV (dual to an anisotropic deformation of SYM_4).

ii) CFT_3 to CFT_3 flows connecting two AdS_4 vacua of the theory. These two asymptotic regions uplift to S-fold solutions reviewed in chapter 5.

A summary of these new supersymmetric domain-walls is also displayed in Figure 7.1 where the notation $\mathcal{N}\,\&\,\tilde{\text{G}}_0$ labels the various AdS_4 vacua. The RG flow we built here are numerical solutions of the equations of motion starting from an IR fixed point with vanishing axion fields (i.e. maximal symmetry group). However, the UV fixed point has in general non-vanishing axions. In particular, although the CFT_3 to CFT_3 flow depicted in the diagram corresponds to the case with vanishing axions in the UV, i.e. with $\text{SU}(3)$ residual symmetry, there also exists CFT_3 to CFT_3 flows that reach $\mathcal{N}=1$ AdS_4 vacua in the UV with non-vanishing axions so that a smaller $\text{SU}(2)\times\text{U}(1)\subset\text{SU}(3)$ symmetry is realised in the UV.

The RG flows of type *i)* will involve a set of sub-leading corrections in the parameter c inducing anisotropy in the UV. Type IIB examples of such an anisotropic behaviour of SYM_4 have previously been obtained in terms of a charge density of dissolved D7-branes [117] or backreacted geometries corresponding to the intersection of D3- and (smeared) D5-branes along $2+1$ dimensions [118, 119]. The type IIB background in [118, 119] involves non-trivial RR fluxes F_3 and $\widetilde{F}_5$ and involves a

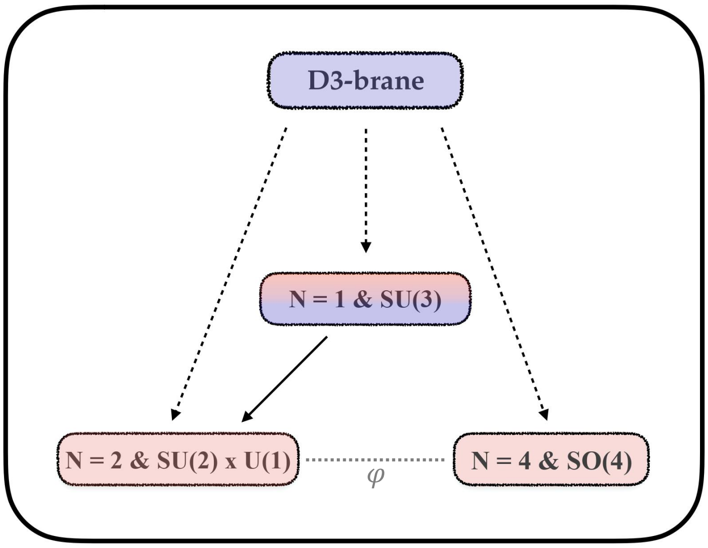

FIGURE 7.1: Network of domain-walls connecting the D3-brane behaviour (anisotropic SYM_4) and the known supersymmetric AdS_4 solutions (J-fold CFT_3's) of the dyonically-gauged $[\,\text{SO}(1,1)\times\text{SO}(6)\,]\ltimes\mathbb{R}^{12}$ maximal supergravity. Domain-walls connecting AdS_4 solutions are denoted by solid lines. The marginal φ-family of solution is denoted by the grey dashed line.

stack of N_c D3-branes and N_f flavour D5-branes. Here we obtain anisotropy in a purely closed string setup *without sources* – sourceless Bianchi identities are satisfied in ten dimensions as the effective four-dimensional supergravity enjoys maximal supersymmetry – by implementing the locally geometric $\text{SL}(2)_{\text{IIB}}$ twist that generates the S-fold background in the IR. As a by-product of the mechanism the results preserve $\text{SL}(2)_{\text{IIB}}$ covariance.

The chapter is organised as follows. In section 7.1, we will review the construction of DW in $\mathcal{N}=1$ supergravity and their holographic interpretation. In section 7.2, we will study the 4d description of the D3-brane solution at $c=0$. We will then study how the c parameter deforms this solution and triggers a flow. We will also study how flat deformations can be interpreted in this context. In section 7.3, we will construct the flow presented in Figure 7.1. In section 7.4, we perform the uplift of the UV behavior of the RG-flows. We will conclude with some remarks and interpretations of our results.

7.1 SUSY-domain walls $\mathcal{N}=1$ and holographic RG-flows

We start by reviewing Domain-Wall (DW) solutions in supergravity. Domain-walls are solution of the supergravity equations of motion corresponding to the ansatz

$$ds^2_{\text{DW}_4} = e^{2A(z)}\,\eta_{\alpha\beta}\,dx^\alpha dx^\beta + dz^2 \qquad \text{with} \qquad \eta_{\alpha\beta} = \text{diag}(-1,1,1)\ , \tag{7.1}$$

$$\mathcal{V}(x^\alpha,\, z) = \mathcal{V}(z) \tag{7.2}$$

where $z\in(-\infty,\infty)$ is the coordinate transverse to the domain-wall and $A(z)$ is the scale factor. In this ansatz, vectors and fermions have been consistently set to zero. By plugging this ansatz in the equations of motion of the Einstein-scalar Lagrangian, on get a set of second order equations of motions describing how the scale factor and the scalar vev's can change along the z direction.

This set of equations can be greatly simplified in two ways. The first one is simply to consistently truncate away certain scalar degrees of freedom. For example, restricting to the $\mathbb{Z}_2^3$-invariant sector of section 4.3 gives us a $\mathcal{N}=1$ model with "only" 14 real scalar fields. The second simplification only appear for $\mathcal{N}=1$ supergravities. Since they can be expressed in terms of a superpotential (see (4.28) for our model), the second order equations of motion we obtained from the DW-ansatz can be greatly simplified. This simplification yields the decoupled first order equations:

$$\partial_z A = \mp |\mathcal{W}| \qquad \text{and} \qquad \partial_z \Sigma^I = \pm K^{IJ}\, \partial_{\Sigma^J} |\mathcal{W}| \ , \tag{7.3}$$

$$\text{where} \qquad \mathcal{W} = e^{\frac{K}{2}} W = m_{3/2} \tag{7.4}$$

and where Σ^J are the real field build out of the complex scalar z^α and we will refer to them as *flow equations*. Solving the flow equations implies solving the full set of equations of motion. In particular, one can first solve for Σ^J and then integrate the solution for the scale factor $A(z)$ to obtain domain-wall solution. However, for generic superpotential, these solutions are usually found numerically. The flow equations ares equivalent to requiring the vanishing of the supersymmetry variations of fermions (gravitino and chiralini) in the $\mathcal{N}=1$ supergravity model. The BPS flow, solving the flow equations, will preserve at least $\mathcal{N}=1$ supersymmetry, which is why they are also called the BPS equations. It is important to note that there are also DW solutions of the full equations of motion which are not BPS and do not solve (7.3).

The simplest solutions to the BPS equations (7.3) are supersymmetric AdS_4 vacua. These solutions have constant scalars and thus satisfy (7.3) provided

$$\partial_{\Sigma^I} |\mathcal{W}| = 0 \qquad \text{and} \qquad A(z) = \mp |\mathcal{W}_0|\, z + C \ , \tag{7.5}$$

where $|\mathcal{W}_0|^{-2} = L^2 = -3/V_0$ corresponds to values evaluated at the AdS_4 vacuum and C is an arbitrary constant that can be reabsorbed by a rescaling of the coordinates x^α in (7.1). These types of solutions are the start- and endpoint of DW which makes DW the right ansatz to interpolate between a region where $A(z) \sim z/L$ when $z \to \infty$ and $A(z) \sim z/L'$ when $z \to -\infty$, i.e. between two CFT_3's.

The $\mathrm{AdS}_4/\mathrm{CFT}_3$ holographic dictionary then states that, at an AdS_4 vacuum, scalars fields with a normalised mass $m^2L^2 < 0$ correspond to relevant operators, $m^2L^2 = 0$ to marginal operators and $m^2L^2 > 0$ to irrelevant operators in the dual field theory. Each scalar field comes along with two modes with conformal dimensions $\Delta_\pm$, where Δ_+ (Δ_-) is the larger (smaller) root of

$$\Delta(\Delta - 3) = m^2 L^2 \ . \tag{7.6}$$

The conformal dimension of the dual operator is then identified with Δ_+. The question about which of the two modes is selected by supersymmetry is answered by the diagonalisation of the matrix

$$\Delta^I{}_J \equiv L\, K^{IM}\, \partial_{M,J} |\mathcal{W}| \ . \tag{7.7}$$

From a field theory perspective, Δ_- is interpreted as a source for the operator whereas Δ_+ is interpreted as a vacuum expectation value (VEV) for the operator. There is an ambiguity with this interpretation whenever the masses lie within the window $-\frac{9}{4} < m^2L^2 < -\frac{5}{4}$, for which an alternative quantisation of the scalar field is possible that interchanges the source and the VEV [120].

Summary of the $\mathcal{N} = 1$ model

For the $\mathcal{N} = 1$ model we will consider, see section 4.3 for its construction, the real scalar fields $\Sigma^I = \{y_i, \chi_i\}$ for $i = 1, \dots, 7$ parametrize the complex scalar fields as

$$z_i = -\chi_i + i\, y_i \qquad y_i > 0\,. \tag{7.8}$$

The superpotential W is

$$\begin{aligned} W = 2\,g\,[z_1 z_5 z_6 + z_2 z_4 z_6 + z_3 z_4 z_5 + (z_1 z_4 + z_2 z_5 + z_3 z_6)\, z_7] \\ + 2\,g\,c\,(1 - z_4 z_5 z_6 z_7)\,. \end{aligned} \tag{7.9}$$

Finally, the kinetic matrix for the real fields reads

$$K_{IJ} = \text{diag}\left(\frac{1}{4y_1^2}, \frac{1}{4y_1^2}, \dots, \frac{1}{4y_7^2}, \frac{1}{4y_7^2}\right)\,. \tag{7.10}$$

7.2 The D3-brane anisotropic deformations

In this section we will obtain (semi-)analytic non-AdS$_4$ solutions of the BPS flow equations (7.3): first in the purely electric case with $c = 0$, and then turning on the electromagnetic deformation c. We will see that the expansion in powers of c, around $c = 0$, can be reinterpreted as a UV expansion in powers of $\frac{1}{z^2}$ around $z = \infty$.

7.2.1 Analytic flow at $c = 0$

when the gauging in the maximal theory is purely electric, namely, $c = 0$, there is a simple solution of the BPS equations given by

$$z_{1,2,3} = -\chi^{(0)}_{1,2,3} + i\,\frac{(g\,z)^2}{8} \quad , \quad z_4 = z_5 = z_6 = z_7 = i\,e^{-\frac{1}{2}\Phi_0} \quad \text{and} \quad e^A = (g\,z)^3\,, \tag{7.11}$$

subject to the constraint[1]

$$\sum_{i=1}^{3} \text{Re} z_i \;=\; -\sum_{i=1}^{3} \chi_i^{(0)} \;=\; 0\,, \tag{7.12}$$

and with Φ_0 being an arbitrary constant. The four-dimensional solution (7.11) with arbitrary (constant) values of the axions $\chi^{(0)}_{1,2,3}$ has an uplift to a ten-dimensional background of type IIB supergravity that is locally equivalent to the D3-brane solution as we will show in section 7.4.

The $\mathcal{N} = 1$ scalar potential (3.28) evaluated at the solution (7.11) yields

$$g^{-2}\,V(z) = -\frac{24}{(gz)^2}\,, \tag{7.13}$$

whereas the $\mathcal{N} = 1$ gravitino mass (7.4) reads

$$g^{-2}\,m_{3/2}^2 = \frac{9}{(gz)^2} + \frac{64}{(gz)^6}\left(\sum_{i=1}^{3}\chi_i^{(0)}\right)^2\,, \tag{7.14}$$

[1] The axions $\chi^{(0)}_{1,2,3}$ must be constant by virtue of the BPS equations (7.3) when setting $c = 0$.

thus being independent of the arbitrary parameter Φ_0 in (7.11). Lastly, the constraint (7.12) further eliminates the dependence of (7.14) on the axion fields $\chi^{(0)}_{1,2,3}$.

The amount of four-dimensional supersymmetry preserved by a solution can be assessed by direct evaluation of the eight gravitino masses, namely, the eigenvalues of $|A_1|^2 = A_1 A_1^\dagger$, where $A_1(z_i) = A_{AB}(z_i)$ is the scalar-dependent gravitino mass matrix in the maximal theory 7.4. Substituting the analytic BPS solution (7.11) into the expression for $A_1(z_i)$ one finds a set of (normalised) eigenvalues given by

$$g^{-2}\, m^2_{3/2} \;=\; g^{-2}\,\text{Eigen}\left(|A_1|^2\right) \;=\; \frac{9}{(gz)^2} + \frac{64}{(gz)^6}\left(\pm\chi^{(0)}_1 \pm \chi^{(0)}_2 \pm \chi^{(0)}_3\right)^2 \;, \qquad (7.15)$$

where the $\pm$ signs are not correlated. Note that the $(+,+,+)$ and $(-,-,-)$ eigenvalues in (7.15) precisely reproduce the $\mathcal{N}=1$ gravitino mass (7.14) belonging to the $\mathbb{Z}_2^3$-invariant sector of the maximal supergravity by virtue of the constraint (7.12). However, such an algebraic constraint does *not* eliminate the dependence of the six remaining gravitino masses in (7.15) on the axions $\chi^{(0)}_{1,2,3}$. And we have explicitly verified that the analytic flow in (7.11) with $\chi^{(0)}_{1,2,3} \neq 0$ is BPS only with respect to two gravitino masses (superpotentials), thus reducing the amount of supersymmetry of the solution by a factor of $1/4$.

It is worth mentioning that the condition (7.12) is required by the BPS equations (7.3) but *not* by the second-order equations of motion.

7.2.2 Semi-analytic flows at $c \neq 0$

Let us now focus on the BPS equations (7.3) when the gauging in the maximal theory is of dyonic type, namely, $c \neq 0$. In this case there is no simple analytic solution of the BPS equations. However, we will be interested in perturbing the analytic solution (7.11) and solve the flow equations (7.3) order by order in powers of the deformation parameter c. We will refer to the resulting power series solution as the *deformed* D3-brane solution.

At zeroth order the analytic solution in (7.11) is recovered, which depends on the arbitrary parameters $(\chi^{(0)}_{1,2,3}\,,\,\Phi_0)$ subject to the constraint

$$\sum_{i=1}^{3} \chi^{(0)}_{1,2,3} = 0 \;. \qquad (7.16)$$

Following the discussion below (7.15), we will set $\chi^{(0)}_{1,2,3} = 0$ in the zeroth order solution so that the largest possible amount of supersymmetry is preserved at this order and the ten-dimensional $\text{AdS}_5 \times \text{S}^5$ geometry of the D3-brane is globally recovered.

First order corrections and universality

At first order in the deformation parameter c, an uneventful integration of the BPS equations in (7.3) yields

$$\begin{aligned}
z_{1,2,3} &= c\,\chi^{(1)}_{1,2,3}(z) + i\,\frac{(g\,z)^2}{8}\left[\,1 + c\,y^{(1)}_{1,2,3}(z)\,\right] ,\\
z_{4,5,6,7} &= c\,\chi^{(1)}_{4,5,6,7}(z) + i\,e^{-\frac{1}{2}\Phi_0}\left[\,1 + c\,y^{(1)}_{4,5,6,7}(z)\,\right] ,\\
e^{A} &= (g\,z)^3\left[\,1 + c\,j(z)\,\right] ,
\end{aligned} \qquad (7.17)$$

in terms of a set of z-dependent functions

$$
\begin{aligned}
\chi^{(1)}_{1,2,3}(z) &= \tfrac{1}{3}\sinh\Phi_0 - \rho_{1,2,3} - \frac{\lambda_4}{(gz)^4} \ , \\
y^{(1)}_{1,2,3}(z) &= \frac{\lambda_1}{gz} + \frac{\kappa_{1,2,3}}{(gz)^4} \ , \\
\chi^{(1)}_{4,5,6,7}(z) &= e^{-\frac{1}{2}\Phi_0}\cosh\Phi_0\,\frac{4}{(gz)^2} - \frac{\rho_{4,5,6,7}}{(gz)^2} - \frac{\lambda_6}{(gz)^6} \ , \\
y^{(1)}_{4,5,6,7}(z) &= \lambda_0 + \frac{\kappa_{4,5,6,7}}{(gz)^4} \ , \\
j(z) &= \tilde{\lambda}_0 + \frac{3}{2}\frac{\lambda_1}{gz} \ ,
\end{aligned}
\tag{7.18}
$$

which in turn depend on a set of integration constants

$$
(\,\rho_{1,\dots,7}\,,\,\kappa_{1,\dots,7}\,) \qquad \text{and} \qquad (\,\tilde{\lambda}_0\,,\,\lambda_{0,1,4,6}\,)\ . \tag{7.19}
$$

These are subject to the following constraints

$$
\sum_{i=1}^{3}\rho_i = 0 \quad , \quad \sum_{i=4}^{7}\rho_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{3}\kappa_i = 0 \quad , \quad \sum_{i=4}^{7}\kappa_i = 0\ . \tag{7.20}
$$

It is worth noticing that the c-deformed solution in (7.17)-(7.18) does *not* reduce to the purely electric ($c = 0$) solution in (7.11)-(7.12) upon adjustment of the integration constants (7.19). The obstruction to have $\chi^{(1)}_{4,5,6,7}(z) = 0$ and $\chi^{(1)}_{1,2,3}(z) = 0$ (the latter whenever $\Phi_0 \neq 0$) is caused by the first two constraints in (7.20). This in turn implies a generic breaking of the $\mathrm{SO}(6)$ symmetry of (7.11) when $\chi^{(0)}_{1,2,3} = 0$ down to an $\mathrm{SU}(3) \subset \mathrm{SO}(6)$ subgroup. Moreover, while the parameters $(\tilde{\lambda}_0\,,\,\lambda_{0,1,4,6})$ preserve such an $\mathrm{SU}(3)$ symmetry, the parameters $(\rho_{1,\dots,7}\,,\,\kappa_{1,\dots,7})$ do not, thus causing a further breaking of flavour symmetries in the dual field theory.

In the following we will analyse in more detail the case where all the integration constants are set to zero, both flavour breaking and $\mathrm{SU}(3)$-preserving constants in (7.19). Then the solution (7.17)-(7.18) acquires a *universal* form (to first order in the parameter c) given by

$$
\begin{aligned}
z_{1,2,3} &= \tfrac{1}{3}\,c\,\sinh\Phi_0 \,+\, i\,\frac{(g\,z)^2}{8} \ , \\
z_{4,5,6,7} &= 4\,e^{-\frac{1}{2}\Phi_0}\cosh\Phi_0\,\frac{c}{(gz)^2} \,+\, i\,e^{-\frac{1}{2}\Phi_0} \ , \\
e^{A} &= (g\,z)^3 \ ,
\end{aligned}
\tag{7.21}
$$

which necessarily induces a deviation from (7.11) that is linear in the parameter c and sub-leading around $(gz) \to \infty$ (UV). However, (7.21) does not capture corrections in $\mathrm{Im}z_{1,2,3}$, $\mathrm{Im}z_{4,5,6,7}$ or the scale factor e^{A}. To get those one must go to higher-orders in c. Finally, it also follows from (7.21) that

$$
\sum_{i=1}^{3}\mathrm{Re}z_i = c\,\sinh\Phi_0 \ , \tag{7.22}
$$

in contrast to the relation (7.12) obtained at $c = 0$.

Higher-order universal corrections

The power series procedure can be iterated to solve the BPS equations (7.3) to any desired order in the deformation parameter c. Setting all the integration constants that appear to zero, the general structure of the *universal* nth-order solution is

$$\begin{aligned}
z_{1,2,3} &= \tfrac{1}{3}\, c \sinh \Phi_0 \left(1 - 384 \cosh^2 \Phi_0\, \mu^2 \log(gz) + \sum_{n=2}^{\infty} \sum_{m=0}^{n-1} \tilde{f}_{n,m}(\Phi_0)\, \mu^{2n} \log^m(gz) \right) \\
&+ i \frac{(g\,z)^2}{8} \left(1 + 32 \cosh^2 \Phi_0\, \mu^2 + \sum_{n=2}^{\infty} \sum_{m=0}^{n-1} f_{n,m}(\Phi_0)\, \mu^{2n} \log^m(gz) \right) , \\
z_{4,5,6,7} &= 4\, e^{-\frac{1}{2}\Phi_0} \cosh \Phi_0\, \mu \left(1 + 64 \left(1 - 3\cosh(2\Phi_0) \right) \mu^2 \log(gz) \right. \\
&\qquad \left. + \sum_{n=2}^{\infty} \sum_{m=0}^{n-1} \tilde{g}_{n,m}(\Phi_0)\, \mu^{2n} \log^m(gz) \right) \\
&+ i\, e^{-\frac{1}{2}\Phi_0} \left(1 - 8 \left(\cosh^2 \Phi_0 - 2\sinh(2\,\Phi_0) \right) \mu^2 + \sum_{n=2}^{\infty} \sum_{m=0}^{n-1} g_{n,m}(\Phi_0)\, \mu^{2n} \log^m(gz) \right) , \\
e^A &= (g\,z)^3 \left(1 + 16 \cosh^2 \Phi_0\, \mu^2 + \sum_{n=2}^{\infty} \sum_{m=0}^{n-1} j_{n,m}(\Phi_0)\, \mu^{2n} \log^m(gz) \right) .
\end{aligned} \tag{7.23}$$

Note that (7.23) actually becomes an expansion in powers of the quantity

$$\mu \equiv \frac{c}{(gz)^2} \,, \tag{7.24}$$

around $\mu = 0$. Therefore, the original expansion in powers of c around $c = 0$ can be re-interpreted as an expansion in powers of $c/(gz)^2$ around the UV ($z \to \infty$).

Although we are not displaying the μ^{2n} higher-order corrections with $n \geq 2$ (it turns out that some of them vanish identically), we have explicitly computed the solution up to $n = 6$. At quadratic order, two corrections of the form $\mu^2 \log(gz)$ involving non-vanishing functions of Φ_0 appear in the axions $\text{Re}z_{1,2,3}$ and $\text{Re}z_{4,5,6,7}$. At quartic order, three corrections of the form $\mu^4 \log(gz)$ involving three different non-vanishing functions $f_{2,1}(\Phi_0)$, $g_{2,1}(\Phi_0)$ and $j_{2,1}(\Phi_0)$ appear in $\text{Im}z_{1,2,3}$, $\text{Im}z_{4,5,6,7}$ and the scale factor e^A, respectively. These are the relevant orders at which the logarithms enter the universal solution to (7.3) when $c \neq 0$.

In what follows we will consider a truncation of the universal solution (7.23) to cubic order in the deformation parameter c, namely,

$$\begin{aligned}
z_{1,2,3} &= \tfrac{1}{3}\, c \sinh \Phi_0 \left(1 - 384 \cosh^2 \Phi_0 \frac{c^2}{(g\,z)^4} \log(gz) \right) \\
&+ i \frac{(g\,z)^2}{8} \left(1 + 32 \cosh^2 \Phi_0 \frac{c^2}{(g\,z)^4} \right) ,
\end{aligned} \tag{7.25}$$

$$
\begin{aligned}
z_{4,5,6,7} &= 4\, e^{-\frac{1}{2}\Phi_0} \cosh\Phi_0 \, \frac{c}{(gz)^2} \left(1 + 64 \left(1 - 3\cosh(2\Phi_0)\right) \frac{c^2}{(g\,z)^4} \log(gz)\right) \\
&+ \; i\, e^{-\frac{1}{2}\Phi_0} \left(1 - 8 \left(\cosh^2\Phi_0 - 2\sinh(2\,\Phi_0)\right) \frac{c^2}{(g\,z)^4}\right) \;, \\
e^A &= (g\,z)^3 \left(1 + 16 \cosh^2\Phi_0 \, \frac{c^2}{(g\,z)^4}\right) \;.
\end{aligned} \tag{7.26}
$$

This order suffices to capture the first relevant terms in each of the scalar fields as well as in the scale factor for the deformed D3-brane solution. From (7.25) one has that

$$
\sum_{i=1}^{3} \text{Re} z_i = c \sinh\Phi_0 \left(1 - 384 \cosh^2\Phi_0 \, \frac{c^2}{(g\,z)^4} \log(gz)\right) \;, \tag{7.27}
$$

which picks up a dependence on the coordinate z in contrast to the relation (7.22) obtained at linear order in $c = 0$.

7.3 4d RG-flows

In this section we numerically construct BPS domain-wall solutions that interpolate between the supersymmetric AdS_4 vacua of Section 7.2 in the IR ($z \to -\infty$) and the anisotropic D3-brane solution of Section 7.2 with $c \neq 0$ in the UV ($z \to \infty$). We will also present an example of a domain-wall that interpolates between the AdS_4 vacuum with $\mathcal{N} = 2\,\&\,\text{SU}(2) \times \text{U}(1)$ symmetry in the IR and the AdS_4 vacuum with $\mathcal{N} = 1\,\&\,\text{SU}(3)$ symmetry in the UV. All these domain-walls in supergravity correspond to holographic RG flows in the field theory side.

Let us discuss the system of first-order and non-linear differential equations in (7.3). The set of equations for the scalars can be solved independently of the one for the scale factor, which can be readily integrated once the profiles for the scalars are known. This means that we must set one boundary condition per (real) scalar field. Moreover, it proves more efficient to numerically shoot from the IR and flow up to the UV.

Perturbing around an AdS_4 configuration dual to the J-fold CFT_3 in the deep IR with scalar VEVs $\Sigma^{(0)I}$ translates into a choice of boundary conditions of the form

$$
\Sigma^I = \Sigma^{(0)I} \left(1 + \lambda^I{}_J \, e^{-\Delta_J \frac{z}{L}}\right) \;, \tag{7.28}
$$

restricted to the set of modes with $\Delta_J < 0$ in the AdS_4 spectrum, as demanded by regularity of the flow in the deep IR ($z \to -\infty$). After imposing (7.28), the BPS equations will determine the set of permitted $\lambda^I{}_J$. However, generic values of the permitted $\lambda^I{}_J$ will end up in a singular flow for which some scalars diverge at a finite radial distance. We will thus have to perform a scan of the $\lambda^I{}_J$ parameter space in order to determine the region yielding regular flows between the IR and the UV. In summary, for a given choice of parameters $\lambda^I{}_J$ in the IR boundary conditions (7.28), we will obtain two different possible types of UV behavior. For type i), the field will flow deformed D3-brane solution (7.17)-(7.18) around the deep UV. The choice of IR boundary conditions will translate into a specific choice of parameters $(\chi_i^{(0)}\,, \Phi_0\,, \rho_{1,2,3}\,, \lambda_0\,, \tilde{\lambda}_0)$, as well as of subleading ones $(\kappa_{1,\dots,7}\,, \rho_{4,\dots,7}\,, \lambda_1\,, \lambda_4\,, \lambda_6)$, in the first-order deformed D3-brane solution. For type ii), the BPS flow will stabilise around another vacuum of the theory leading to CFT_3 to CFT_3 flows. The type of flow will depend on the choice of IR boundary conditions.

7.3.1 SYM$_4$ to CFT$_3$ with $\mathcal{N}=1\,\&\,\mathrm{SU}(3)$

We will solve the BPS equations (7.3) numerically by perturbing around the $\mathcal{N}=1\,\&\,\mathrm{SU}(3)$ AdS$_4$ vacuum (4.35) in the IR ($z\to-\infty$). This will generate generic flows towards a non-conformal behaviour in the UV ($z\to\infty$).

IR boundary conditions

By virtue of (7.6), the set of normalised scalar masses in (4.38) implies a set of conformal dimensions $\Delta_\pm$ for the dual operators given by

$$\begin{array}{rclcccccc}
m^2L^2 & = & -\frac{20}{9}\,(\times 3) & , \; -2\,(\times 2) & , \; -\frac{8}{9}\,(\times 3) & ; \; 0\,(\times 2) & ; \; 4-\sqrt{6}\,(\times 2) & , \; 4+\sqrt{6}\,(\times 2) & , \\
\Delta_+ & = & \mathbf{\frac{5}{3}}\,(\times 3) & , \; \mathbf{2}\,(\times 2) & , \; \frac{8}{3} & ; \; 3 & ; \; \mathbf{1+\sqrt{6}}\,(\times 2) & , \; 2+\sqrt{6} & , \\
\Delta_- & = & \frac{4}{3} & , \; 1 & , \; \mathbf{\frac{1}{3}}\,(\times 3) & ; \; \mathbf{0}\,(\times 2) & ; \; 2-\sqrt{6} & , \; \mathbf{1-\sqrt{6}}\,(\times 2) & .
\end{array} \tag{7.29}$$

The highlighted conformal dimensions in (4.38) appear as eigenvalues of the matrix (7.7) and will play a role in fixing the boundary condition (7.28). Around the $\mathcal{N}=1\,\&\,\mathrm{SU}(3)$ solution, there are two irrelevant modes in the spectrum (7.29)

$$\Delta_- = 1-\sqrt{6}\;(\times 2)\,, \tag{7.30}$$

that are compatible with regularity of the flows in the IR ($z\to-\infty$). The linearised BPS equations then allow for two arbitrary real parameters $(\Lambda\,,\,\lambda)$ specifying the IR boundary conditions (7.28), which read

$$\begin{array}{rcll}
\mathrm{Im}z_{1,2,3} & = & c\,\frac{\sqrt{5}}{3}\left(1-\Lambda\,e^{-(1-\sqrt{6})\frac{z}{L}}\right) & , \\
\mathrm{Im}z_{4,5,6,7} & = & \sqrt{\frac{5}{6}}\left(1-\frac{1}{4}(3\,(2+\sqrt{6})\,\lambda+(\sqrt{6}-2)\,\Lambda)e^{-(1-\sqrt{6})\frac{z}{L}}\right) & , \\
\mathrm{Re}z_{1,2,3} & = & c\,\lambda\,e^{-(1-\sqrt{6})\frac{z}{L}} & , \\
\mathrm{Re}z_{4,5,6,7} & = & \frac{1}{\sqrt{6}}-\frac{1}{12}\left(3\left(\sqrt{6}+3\right)\lambda+5\left(\sqrt{6}-3\right)\Lambda\right)e^{-(1-\sqrt{6})\frac{z}{L}} & .
\end{array} \tag{7.31}$$

Note that, whenever non-vanishing, one of the parameters Λ or λ can be set at will by a shift on the coordinate z. We will set $\Lambda=-1$ without loss of generality[2], which translates into a one-dimensional parameter space to be scanned.

Behaviour of the flows

Fixing $\lambda=0$ implies $\mathrm{Re}z_{1,2,3}=0$ in the IR boundary conditions (7.31). In this case we obtain the numerical flow[3] depicted in Figure 7.2 that approaches the deformed D3-brane solution in the UV ($z\to\infty$). As previously discussed in Section 7.2.2, the UV behaviour of this flow is understood as a sub-leading correction in the electromagnetic deformation c of the D3-brane solution in (7.11) with

$$\mathrm{Re}z_{1,2,3}=0 \qquad \text{and} \qquad \Phi_0=0\,. \tag{7.32}$$

[2]Setting $\Lambda=0,1$ does not produce regular flows.

[3]All the figures presented in this work are produced by setting the initial value of the radial coordinate to $z_{\text{ini}}=\log(10^{-2})$. Note that this value can be set at will by virtue of (7.28).

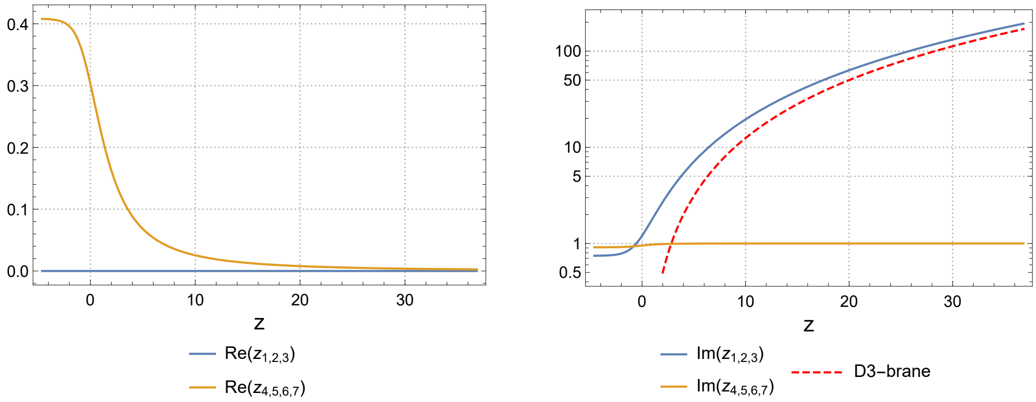

FIGURE 7.2: Holographic RG flow from $\mathcal{N}=4$ SYM$_4$ (UV, right) to $\mathcal{N}=1\,\&\,\mathrm{SU}(3)$ J-fold CFT$_3$ (IR, left) with $\Lambda=-1$ and $\lambda=0$.

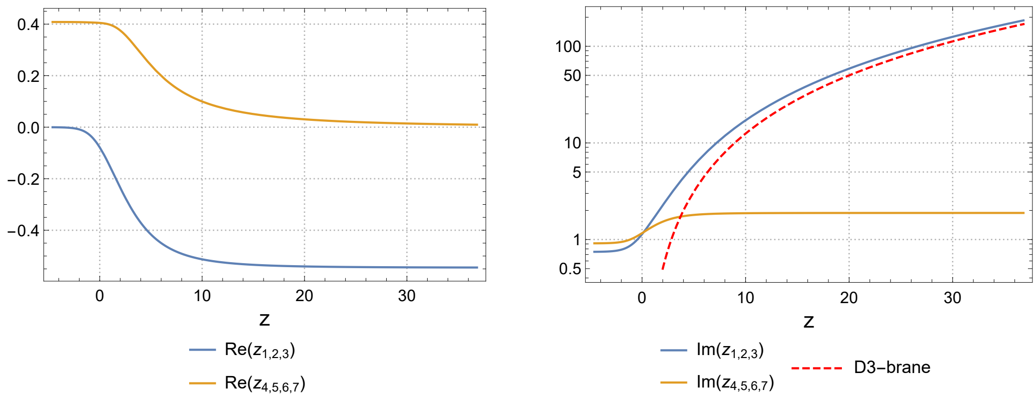

FIGURE 7.3: Holographic RG flow from $\mathcal{N}=4$ SYM$_4$ (UV, right) to $\mathcal{N}=1\,\&\,\mathrm{SU}(3)$ J-fold CFT$_3$ (IR, left) with $\Lambda=-1$ and $\lambda=-0.16$.

Activating the parameter λ makes the axions $\mathrm{Re}z_{1,2,3}$ run along the flow. In this case, the UV region is reached with

$$\sum_{i=1}^{3}\mathrm{Re}z_i \approx c\,\sinh\Phi_0 \quad , \quad \mathrm{Im}z_{4,5,6,7}\approx e^{-\frac{1}{2}\Phi_0} \quad \text{and} \quad \Phi_0\neq 0\;. \tag{7.33}$$

This agrees with (7.22) obtained at first-order in the deformation parameter c. One such generic flows is depicted in Figure 7.3.

Study of the parameter space

We have performed a numerical scanning of values for the parameter λ, and found regular flows only within the interval

$$\lambda\in[-0.171\,,\,0.171]\;. \tag{7.34}$$

Outside this range, singular flows occur with some scalar fields diverging at a finite radial distance.

Gravitino masses and supersymmetry

The $\mathcal{N} = 1\,\&\,\mathrm{SU}(3)$ AdS_4 vacuum in the IR realises an $\mathrm{SU}(3)$ flavour symmetry group in the dual J-fold CFT_3. Under this $\mathrm{SU}(3)$ symmetry the eight gravitini of the maximal theory decompose as

$$\begin{array}{ccc} \mathrm{SU}(8) & \supset & \mathrm{SU}(3) \\ \mathbf{8} & \rightarrow & \mathbf{1} + \mathbf{3} + \bar{\mathbf{3}} + \mathbf{1} \end{array} \tag{7.35}$$

For generic BPS flows with the parameter λ within the range (7.34), an explicit evaluation of the eight gravitino masses, namely the eigenvalues of $A_{IJ}\,A^{JK}$, shows that only one of them is compatible with the IR value of the $\mathcal{N} = 1$ gravitino mass (7.4) belonging to the $\mathbb{Z}_2^3$-invariant sector. More concretely, the eight gravitino masses of the maximal theory turn out to split as

$$\begin{array}{lcl} \lambda \neq 0 & : & 8 \rightarrow 1 + 3 + 3 + \boxed{1}\,, \\ \lambda = 0 & : & 8 \rightarrow 1 + 6 + \boxed{1}\,, \end{array} \tag{7.36}$$

when evaluated along the numerical flows. In (7.36) we have boxed the $\mathcal{N} = 1$ supersymmetry realised at the AdS_4 vacuum in the deep IR and highlighted (in blue) those gravitino masses with respect to which the numerical flows are BPS. Note that the IR boundary conditions in (7.31) are compatible with an $\mathrm{SO}(6)$ symmetry when $\lambda = 0$ (so that $\mathrm{Re}z_{1,2,3} = 0$), thus yielding the decomposition in (7.36).

7.3.2 $\mathbf{SYM_4}$ to $\mathbf{CFT_3}$ with $\mathcal{N} = 2\,\&\,\mathbf{SU}(2)$

Let us now solve the BPS equations (7.3) by perturbing around the $\mathcal{N} = 2\,\&\,\mathrm{SU}(2) \times \mathrm{U}(1)$ AdS_4 vacuum (4.41) in the IR ($z \rightarrow -\infty$). This will cause again the appearance of generic flows towards a non-conformal behaviour in the UV ($z \rightarrow \infty$).

IR boundary conditions

Upon solving (7.6), the set of normalised scalar masses in (4.43) implies a set of conformal dimensions $\Delta_{\pm}$ for the dual operators given by

$$\begin{array}{rclcccccccccc} m^2L^2 & = & -2\,(\times 4) & , & 3-\sqrt{17}\,(\times 2) & ; & 0\,(\times 2) & ; & 2\,(\times 4) & , & 3+\sqrt{17}\,(\times 2) & , \\ \Delta_{+} & = & \mathbf{2}\,(\times 2) & , & \tfrac{1}{2}(\mathbf{1}+\sqrt{\mathbf{17}})\,(\times 2) & ; & 3 & ; & \tfrac{1}{2}(\mathbf{3}+\sqrt{\mathbf{17}})\,(\times 2) & , & \tfrac{1}{2}(5+\sqrt{17}) & , \\ \Delta_{-} & = & \mathbf{1}\,(\times 2) & , & \tfrac{1}{2}(5-\sqrt{17}) & ; & \mathbf{0}\,(\times 2) & ; & \tfrac{1}{2}(\mathbf{3}-\sqrt{\mathbf{17}})\,(\times 2) & , & \tfrac{1}{2}(\mathbf{1}-\sqrt{\mathbf{17}})\,(\times 2) & . \end{array} \tag{7.37}$$

Around the $\mathcal{N} = 2\,\&\,\mathrm{SU}(2) \times \mathrm{U}(1)$ solution, there are four irrelevant modes in the spectrum (7.37)

$$\Delta_{-} = \tfrac{1}{2}\,(1-\sqrt{17})\ (\times 2) \qquad \text{and} \qquad \Delta_{-} = \tfrac{1}{2}\,(3-\sqrt{17})\ (\times 2)\,, \tag{7.38}$$

that are compatible with regularity of the flows in the IR ($z \rightarrow -\infty$). The linearised BPS equations then allow for four parameters $(\Lambda_1,\, \Lambda_2)$ and $(\lambda_1,\, \lambda_2)$ specifying the

IR boundary conditions (7.28) which read

$$
\begin{array}{rcll}
\text{Im}z_{1,3} & = & c\,\frac{1}{\sqrt{2}}\left(1-\frac{1+\sqrt{17}}{4}\,\Lambda_1\,e^{-\frac{1}{2}(1-\sqrt{17})\frac{z}{L}}\right) & , \\
\text{Im}z_{2} & = & c\left(1-\Lambda_1\,e^{-\frac{1}{2}(1-\sqrt{17})\frac{z}{L}}-\lambda_2\,e^{-\frac{1}{2}(3-\sqrt{17})\frac{z}{L}}\right) & , \\
\text{Im}z_{4,6} & = & 1+\frac{1+\sqrt{17}}{4}\,\Lambda_2\,e^{-\frac{1}{2}(1-\sqrt{17})\frac{z}{L}} & , \\
\text{Im}z_{5,7} & = & \frac{1}{2\sqrt{2}}\left(2+(\Lambda_2-\Lambda_1)\,e^{-\frac{1}{2}(1-\sqrt{17})\frac{z}{L}}+(\lambda_2-\lambda_1)\,e^{-\frac{1}{2}(3-\sqrt{17})\frac{z}{L}}\right) & , \\
\text{Re}z_{1,3} & = & -c\,\frac{1-\sqrt{17}}{4\sqrt{2}}\,\lambda_1\,e^{-\frac{1}{2}(3-\sqrt{17})\frac{z}{L}} & , \\
\text{Re}z_{2} & = & -c\left(\Lambda_2\,e^{-\frac{1}{2}(1-\sqrt{17})\frac{z}{L}}+\lambda_1\,e^{-\frac{1}{2}(3-\sqrt{17})\frac{z}{L}}\right) & , \\
\text{Re}z_{4,6} & = & -\frac{1-\sqrt{17}}{4}\,\lambda_2\,e^{-\frac{1}{2}(3-\sqrt{17})\frac{z}{L}} & , \\
\text{Re}z_{5,7} & = & \frac{1}{2\sqrt{2}}\left(2+(\Lambda_1+\Lambda_2)\,e^{-\frac{1}{2}(1-\sqrt{17})\frac{z}{L}}-(\lambda_1+\lambda_2)\,e^{-\frac{1}{2}(3-\sqrt{17})\frac{z}{L}}\right) & .
\end{array}
\tag{7.39}
$$

As before, and whenever non-vanishing, one of the parameters $\Lambda_{1,2}$ or $\lambda_{1,2}$ can be set at will by a shift on the coordinate z. We will set $\Lambda_1 = -1$ without loss of generality[4], which leaves us this time with a three-dimensional parameter space to be scanned.

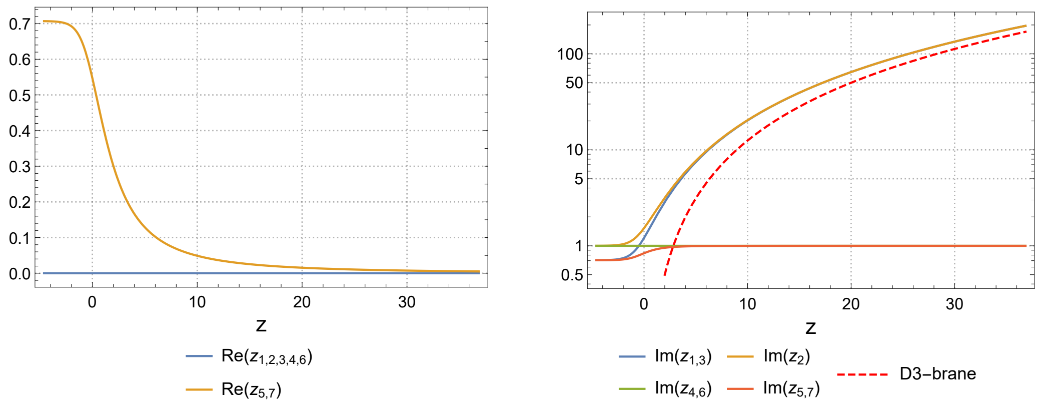

FIGURE 7.4: Holographic RG flow from $\mathcal{N}=4$ SYM$_4$ (UV) to $\mathcal{N}=2\,\&\,\text{SU}(2)$ J-fold CFT$_3$ (IR) with $\Lambda_1=-1$, $\Lambda_2=0$ and $(\lambda_1,\lambda_2)=(0,0)$.

Behaviour of the flows

Fixing $\Lambda_2 = 0$ and $\lambda_{1,2} = 0$ implies $\text{Re}z_{1,2,3} = 0$ and $\text{Re}z_{4,6} = 0$ in the IR boundary conditions (7.39). In this case we obtain the flow depicted in Figure 7.4. The UV ($z \to \infty$) behaviour of this flow is again understood as a sub-leading correction in the electromagnetic deformation c of the D3-brane solution in (7.11) with

$$\text{Re}z_{1,2,3} = 0 \qquad \text{and} \qquad \Phi_0 = 0\ . \tag{7.40}$$

Now we can explore the UV behaviour of the flows when activating the parameters Λ_2 and $\lambda_{1,2}$ in the boundary conditions (7.39). As it can be directly seen from there, the parameter λ_1 controls the pair-wise identified perturbation of $\text{Re}z_1 = \text{Re}z_3$

[4] Setting $\Lambda_1 = 0$ or $= 1$ does not produce regular flows.

whereas Λ_2 subsequently controls the perturbation of $\mathrm{Re}z_2$. Turning on just the parameter Λ_2 generates flows reaching the UV with

$$\sum_{i=1}^{3} \mathrm{Re}z_i \approx c\,\sinh\Phi_0 \quad , \qquad \mathrm{Im}z_{4,5,6,7} \approx e^{-\frac{1}{2}\Phi_0} \qquad \text{and} \qquad \Phi_0 \neq 0\;, \tag{7.41}$$

in agreement with (7.22). Turning on the remaining parameters $\lambda_{1,2}$ makes more scalars run along the flow. Flows of these types are presented in Figure 7.5.

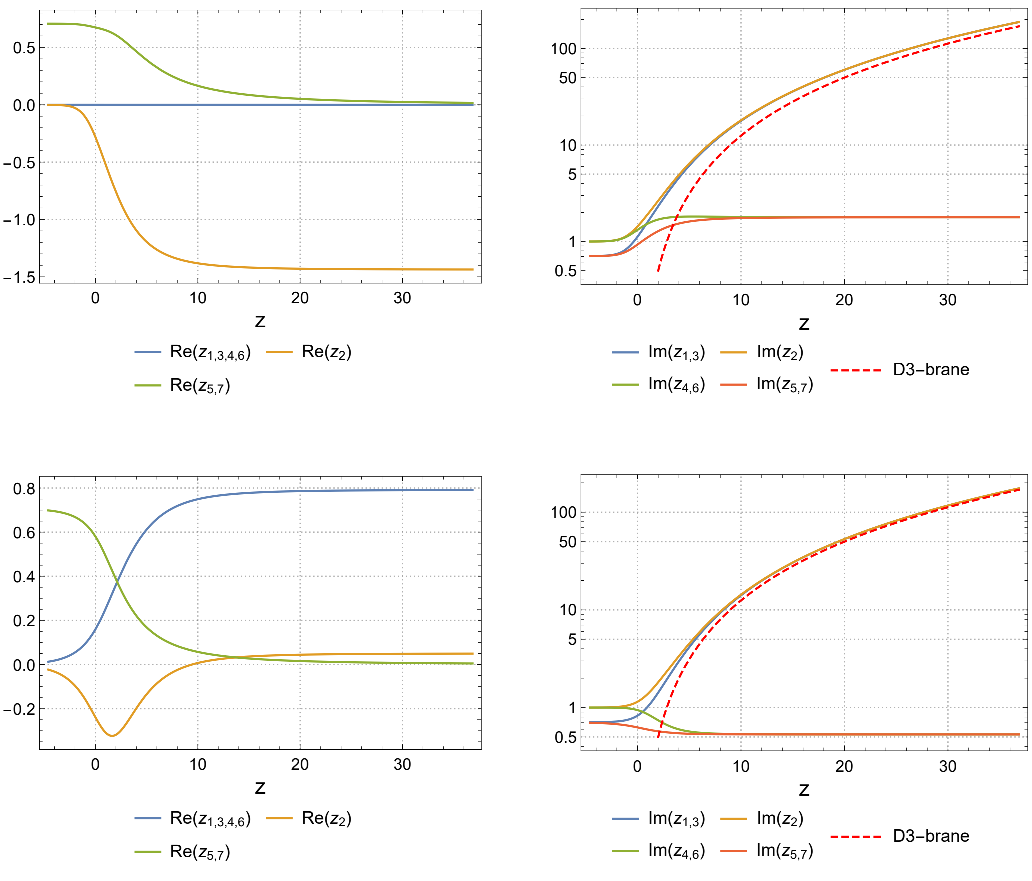

FIGURE 7.5: Holographic RG flow from $\mathcal{N}=4$ SYM$_4$ (UV) to $\mathcal{N}=2\,\&\,\mathrm{SU}(2)$ J-fold CFT$_3$ (IR). Top plots: $(\Lambda_1,\Lambda_2)=(-1,0.7)$ and $(\lambda_1,\lambda_2)=(0,0)$. Bottom plots: $(\Lambda_1,\Lambda_2)=(-1,0)$ and $(\lambda_1,\lambda_2)=(0.307,0.011)$.

Study of the parameter space

This time we must perform a numerical scan of flows in a three-dimensional parameter space $(\Lambda_2\,;\,\lambda_1\,,\,\lambda_2)$. Various sections of the parameter space can be taken which are depicted in Figure 7.6. The three parameters Λ_2 and $\lambda_{1,2}$ simply turn out to control the values of the axions when generically approaching the D3-brane solution in the UV.

Let us discuss in more detail some features of the parameter space depicted in Figure 7.6. The borders of the various sections delimit the region of the three-dimensional parameter space producing regular flows. Outside this region the flows have some scalar field diverging at a finite radial distance. Importantly, the region around the

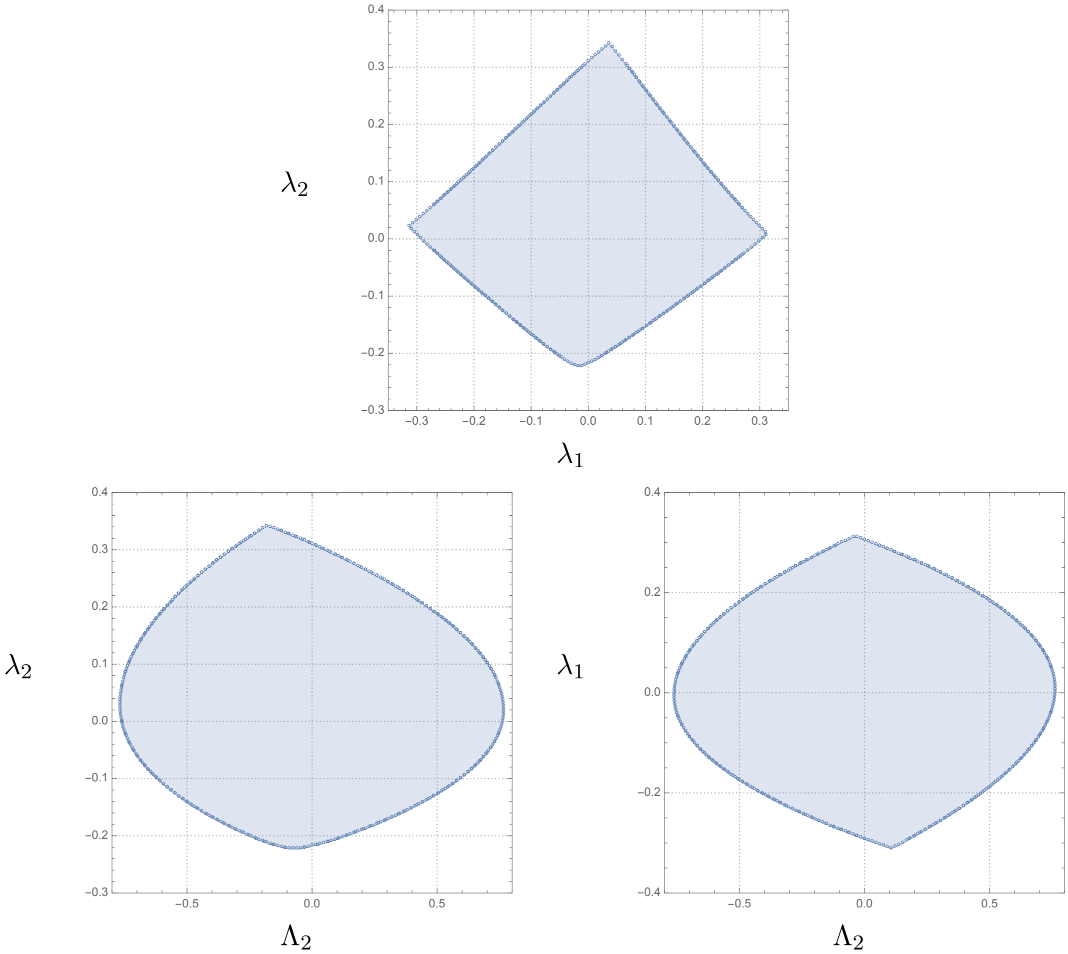

FIGURE 7.6: Three sections of the parameter space showing the region accommodating regular holographic RG flows involving the $\mathcal{N}=2\,\&\,\mathrm{SU}(2)$ J-fold CFT_3 in the IR: $\Lambda_2=0$ (top), $\lambda_1=0$ (bottom-left) and $\lambda_2=0$ (bottom-right).

upper corner in the (λ_1,λ_2)-projection is very special as it produces flows passing arbitrarily close to the $\mathcal{N}=1$ J-fold CFT_3's before continue flowing to SYM_4 in the deep UV. This limiting CFT_3 to CFT_3 holographic RG flows are presented separately in Section 7.3.4.

Gravitino masses and supersymmetry

For the $\mathcal{N}=1\,\&\,\mathrm{SU}(2)\times\mathrm{U}(1)$ AdS_4 vacuum in the IR the flavour symmetry group realised in the dual J-fold CFT_3 is $\mathrm{SU}(2)$. Under this $\mathrm{SU}(2)$ symmetry the eight gravitini of the maximal theory decompose this time as

$$\begin{array}{ccc} \mathrm{SU}(8) & \supset & \mathrm{SU}(2) \\ \mathbf{8} & \rightarrow & \mathbf{2}+\mathbf{2}+\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1} \end{array} \tag{7.42}$$

For generic BPS flows with parameters Λ_2 and $(\lambda_1\,,\,\lambda_2)$ in the regions shown in Figure 7.6, the evaluation of the eight eigenvalues of $A_{AB}\,A^{BC}$ shows that, as before, only one of them is generically compatible with the IR value of the $\mathcal{N}=1$ gravitino mass (7.4) belonging to the $\mathbb{Z}_2^3$-invariant sector. However, specific choices of the parameters $(\lambda_1\,,\,\lambda_2)$ this time yield different splittings of the eight gravitino masses

$$\begin{array}{lcl} \lambda_1^2+\lambda_2^2\neq 0 & : & 8\rightarrow 4+1+1+\boxed{1+1}\;, \\ \lambda_1^2+\lambda_2^2=0 & : & 8\rightarrow 4+2+\boxed{2}\;, \end{array} \tag{7.43}$$

when evaluated along the numerical flows. In (7.43) we have boxed the $\mathcal{N}=2$ supersymmetry realised at the AdS$_4$ vacuum in the deep IR and highlighted (in blue) those gravitino masses with respect to which the numerical flows are BPS.

7.3.3 SYM$_4$ to CFT$_3$ with $\mathcal{N}=4$

Lastly we will solve the BPS equations (7.3) by perturbing around the $\mathcal{N}=4\,\&\,\mathrm{SO}(4)$ AdS$_4$ vacuum (4.46) at $\varphi=0$ in the IR ($z\rightarrow-\infty$). This will trigger again the appearance of generic flows towards a non-conformal behaviour in the UV ($z\rightarrow\infty$).

IR boundary conditions

At $\varphi=0$, the spectrum of $\mathbb{Z}_2^3$-invariant normalised scalar masses is of the form

$$m^2L^2 \;=\; -2\,(\times 3) \quad , \quad 0\,(\times 6) \quad , \quad 4\,(\times 4) \quad , \quad 10\,(\times 1)\;. \tag{7.44}$$

Solving (7.6) for the set of normalised scalar masses in (7.44) yields a set of conformal dimensions $\Delta_{\pm}$ for the dual operators given by

$$\begin{array}{rclcccccccc} m^2L^2 & = & -2\,(\times 3) & ; & 0\,(\times 6) & ; & 4\,(\times 4) & , & 10\,(\times 1) & , \\ \Delta_+ & = & \mathbf{2}\,(\times 3) & ; & \mathbf{3}\,(\times 3) & ; & \mathbf{4}\,(\times 1) & , & 5 & , \\ \Delta_- & = & 1 & ; & \mathbf{0}\,(\times 3) & ; & -\mathbf{1}\,(\times 3) & , & -\mathbf{2}\,(\times 1) & . \end{array} \tag{7.45}$$

The highlighted conformal dimensions in (7.44) appear as eigenvalues of the matrix (7.7) and will play a role in fixing the boundary condition (7.28). Around the $\mathcal{N}=4\,\&\,\mathrm{SO}(4)$ solution, there are again four irrelevant modes in the spectrum (7.45)

$$\Delta_-=-2 \qquad \text{and} \qquad \Delta_-=-1\;(\times 3)\;, \tag{7.46}$$

that are compatible with regularity of the flow in the IR ($z\rightarrow-\infty$). The linearised BPS equations then allow for four parameters Λ and λ_i, with $i=1,2,3$, specifying the IR boundary conditions (7.28). These read

$$\begin{array}{rcll} \mathrm{Im}z_i & = & c\left(1-2\Lambda\,e^{2\frac{z}{L}}\right) & , \\ \mathrm{Im}z_{3+i} & = & \frac{1}{\sqrt{2}}\left(1-\Lambda\,e^{2\frac{z}{L}}-\frac{1}{4}\,(\lambda+\lambda_i)\,e^{\frac{z}{L}}\right) & , \\ \mathrm{Im}z_7 & = & \frac{1}{\sqrt{2}}\left(1-\Lambda\,e^{2\frac{z}{L}}-\frac{1}{2}\,\lambda\,e^{\frac{z}{L}}\right) & , \\ \mathrm{Re}z_i & = & \frac{1}{2}\,c\,\lambda_i\,e^{\frac{z}{L}} & , \\ \mathrm{Re}z_{3+i} & = & \frac{1}{\sqrt{2}}\left(1+\Lambda\,e^{2\frac{z}{L}}-\frac{1}{4}\,(\lambda+\lambda_i)e^{\frac{z}{L}}\right) & , \\ \mathrm{Re}z_7 & = & -\frac{1}{\sqrt{2}}\left(1+\Lambda\,e^{2\frac{z}{L}}-\frac{1}{2}\,\lambda\,e^{\frac{z}{L}}\right) & , \end{array} \tag{7.47}$$

with $\lambda\equiv\lambda_1+\lambda_2+\lambda_3$. Note that the parameters λ_i enter the IR boundary conditions (7.47) in a symmetric manner and fully specify $\mathrm{Re}z_i$. As before, one of the parameters, either Λ or λ_i, can be set at will by a shift on the coordinate z. We

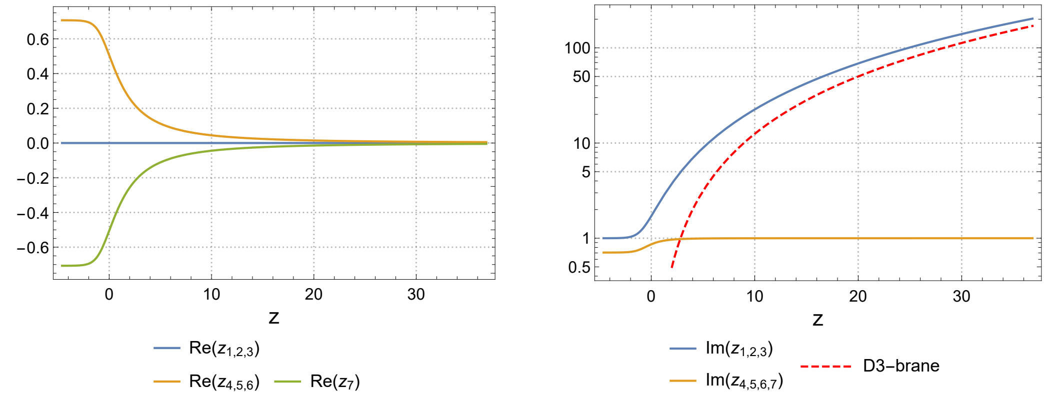

FIGURE 7.7: Holographic RG flow from $\mathcal{N}=4$ SYM$_4$ (UV) to $\mathcal{N}=4$ J-fold CFT$_3$ (IR) with $\Lambda=-1$ and $\lambda_i=0$.

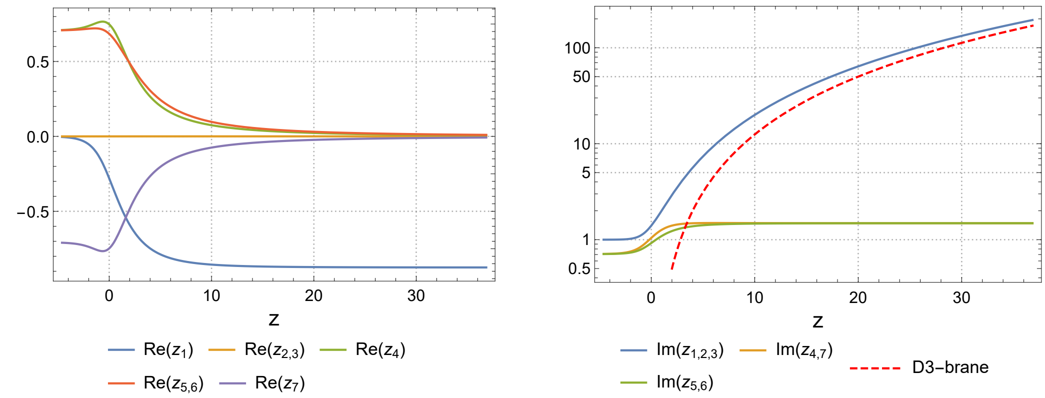

FIGURE 7.8: Holographic RG flow from $\mathcal{N}=4$ SYM$_4$ (UV) to $\mathcal{N}=4$ J-fold CFT$_3$ (IR) with $\Lambda=-1$ and $(\lambda_1,\lambda_2,\lambda_3)=(-0.8,0,0)$.

will set $\Lambda=-1$ without loss of generality[5], which leaves us also this time with a three-dimensional parameter space to scan.

Behaviour of the flows

The IR boundary conditions (7.47) are highly symmetric. Fixing $\lambda_i=0$ in (7.47) implies $\text{Re}z_{1,2,3}=0$. In this case we obtain the flow depicted in Figure 7.7. The UV ($z\to\infty$) behaviour of this flow is again understood as a sub-leading correction in the electromagnetic deformation c of the D3-brane solution in (7.11) with

$$\text{Re}z_{1,2,3}=0 \qquad \text{and} \qquad \Phi_0=0\ . \tag{7.48}$$

Turning on the parameters λ_i activates the asymptotic values of the axions $\text{Re}z_i$ in the UV. More concretely, the UV region is again reached as

$$\sum_{i=1}^{3}\text{Re}z_i\approx c\sinh\Phi_0 \quad , \quad \text{Im}z_{4,5,6,7}\approx e^{-\frac{1}{2}\Phi_0} \quad \text{and} \quad \Phi_0\neq 0\ . \tag{7.49}$$

[5]Setting $\Lambda=0,1$ does not produce regular flows.

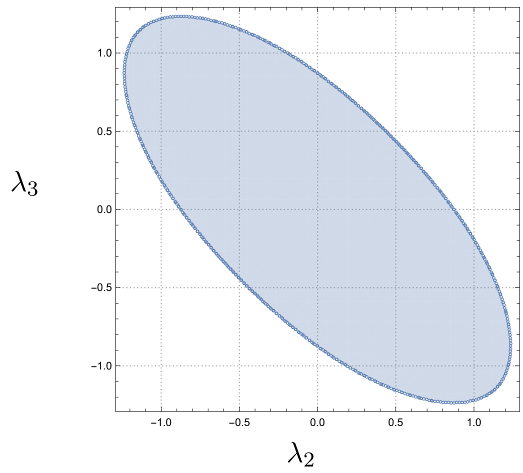

FIGURE 7.9: Section of the parameter space with $\lambda_1 = 0$ showing the region accommodating regular holographic RG flows involving the $\mathcal{N} = 4$ J-fold CFT$_3$ in the IR. Sections with $\lambda_2 = 0$ or $\lambda_3 = 0$ are equivalent due to the exchangeability of the parameters λ_i .

This agrees again with (7.22) obtained at first-order in the deformation parameter c. A generic flow with $\lambda_1 \neq 0$ and $\text{Re}z_1 \neq 0$ in the UV is depicted in Figure 7.8.

Study of the parameter space

The fact that $\lambda_{1,2,3}$ enter the IR boundary conditions (7.47) symmetrically renders the three parameters completely interchangeable as far as the induced flows are concerned. In Figure 7.9 the section of the parameter space allowing for regular holographic RG flows with $\lambda_1 = 0$ is depicted. Similar figures are obtained upon setting $\lambda_2 = 0$ or $\lambda_3 = 0$. Finally, within our numerical precision, we do not observe flows reaching the $\mathcal{N} = 1$ family of AdS$_4$ vacua (4.35) in the UV. The three parameters $\lambda_{1,2,3}$ simply turn out to control the values of the axions $\text{Re}z_{1,2,3}$ when approaching the D3-brane solution in the UV.

Gravitino masses and supersymmetry

The $\mathcal{N} = 4 \,\&\, \text{SO}(4)$ AdS$_4$ vacuum in the IR realises a trivial flavour symmetry group in the dual J-fold CFT$_3$. For generic BPS flows with parameters λ_i in the parameter space of Figure 7.9, the evaluation of the eight eigenvalues of $A_{AB}\, A^{BC}$ shows that, as in the previous cases, only one of them is generically compatible with the IR value of the $\mathcal{N} = 1$ gravitino mass (7.4) belonging to the $\mathbb{Z}_2^3$-invariant sector. Specific choices of the parameters λ_i yield again a different splitting of the eight gravitino masses

$$
\begin{array}{lcl}
\lambda \neq \lambda_i \quad \forall i & : & 8 \to 1 + 3 + \boxed{1 + 3}\,, \\
\lambda = \lambda_i \quad (\text{for one } \lambda_i) & : & 8 \to 4 + \boxed{2 + 2}\,, \\
\lambda = \lambda_i \quad (\text{for two } \lambda_i \text{ and } \lambda_j) & : & 8 \to 1 + 3 + \boxed{1 + 3}\,, \\
\lambda = \lambda_i \quad \forall i \quad \Leftrightarrow \quad \lambda_i = 0 \quad \forall i & : & 8 \to 4 + \boxed{4}\,,
\end{array}
\tag{7.50}
$$

with $\lambda \equiv \lambda_1 + \lambda_2 + \lambda_3$ when evaluated along the numerical flows. In (7.50) we have boxed the $\mathcal{N} = 4$ supersymmetry realised at the AdS$_4$ vacuum in the deep IR and highlighted (in blue) those gravitino masses with respect to which the numerical flows are BPS. Note that the IR boundary conditions in (7.47) are compatible with a symmetry that ranges from $\mathrm{SO}(3)$ $(\lambda \neq \lambda_i)$ to $\mathrm{SO}(4)$ $(\lambda_i = 0)$.

7.3.4 CFT$_3$ to CFT$_3$

We now present an example of CFT$_3$ to CFT$_3$ holographic RG flow that connects the J-fold CFT$_3$ with $\mathcal{N}=1\,\&\,\mathrm{SU}(3)$ symmetry in the UV to the J-fold CFT$_3$ with $\mathcal{N}=2\,\&\,\mathrm{SU}(2)$ symmetry in the IR (see Figure 7.1). This flow requires an extreme fine tuning of the IR boundary conditions in (7.39) and is depicted in Figure 7.10.

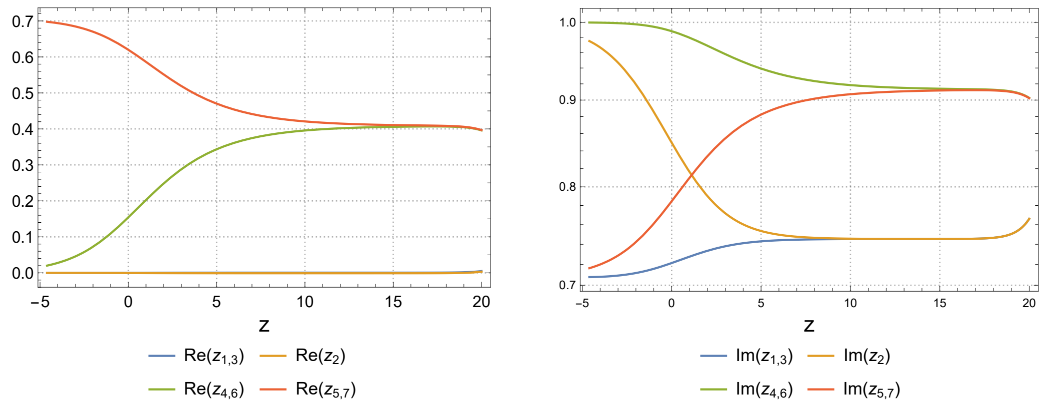

FIGURE 7.10: Holographic RG flow from $\mathcal{N}=1\,\&\,\mathrm{SU}(3)$ J-fold CFT$_3$ (UV, right) to $\mathcal{N}=2\,\&\,\mathrm{SU}(2)$ J-fold CFT$_3$ (IR, left) with $(\Lambda_1,\Lambda_2) = (-1,-0.1566939789)$ and $(\lambda_1,\lambda_2) = (0.0042958950\,,\,0.3421361222)$.

This extremely fine-tuned flow is actually a limiting case within a more general class of flows connecting the $\mathcal{N}=1\,\&\,\mathrm{SU}(2)\times\mathrm{U}(1)$ CFT$_3$'s in the UV to the $\mathcal{N}=2\,\&\,\mathrm{SU}(2)$ CFT$_3$ in the IR. Such flows are described by domain-walls connecting the associated AdS$_4$ vacua in (4.35) and (4.41) at $\chi=0$, and approach the UV with non-vanishing axions satisfying $\sum_i \chi_i = 0$ with

$$\mathrm{Re}z_1 = \mathrm{Re}z_3 = -\tfrac{1}{2}\mathrm{Re}z_2\ , \tag{7.51}$$

as shown in Figure 7.11. Therefore, the UV symmetry enhancement to $\mathrm{SU}(3)$ does not take place.

Finally, a more detailed study of CFT$_3$ to CFT$_3$ holographic RG flows including the axions $\mathrm{Re}z_{1,2,3}$ dual to exactly marginal deformations in the corresponding J-fold CFT$_3$'s could be studied in the future.

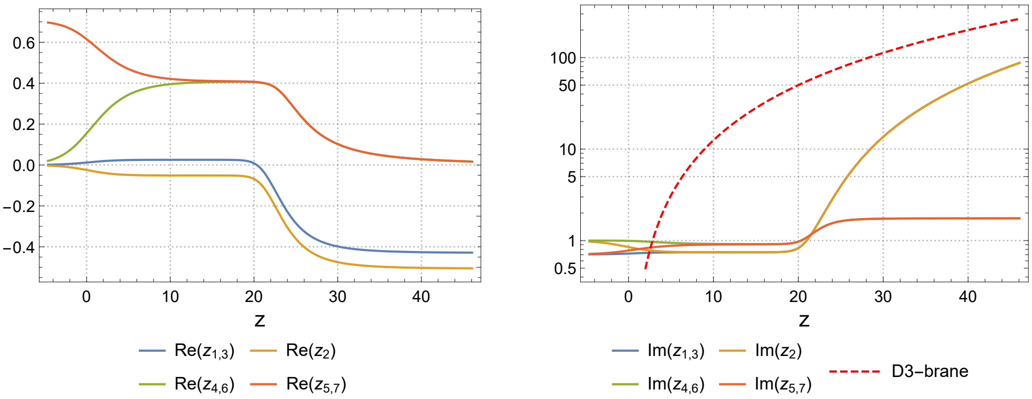

FIGURE 7.11: Holographic RG flow from $\mathcal{N} = 1\,\&\,\mathrm{SU}(2)\times\mathrm{U}(1)$ J-fold CFT$_3$ (UV, right) to $\mathcal{N} = 2\,\&\,\mathrm{SU}(2)$ J-fold CFT$_3$ (IR, left) with $(\Lambda_1, \Lambda_2) = (-1, 0)$ and $(\lambda_1, \lambda_2) = (0.0364510166\,,\, 0.3417265522)$.

7.4 10d RG Flows

In Sections 7.3.1, 7.3.2 and 7.3.3 we numerically constructed holographic RG flows across dimensions connecting (c-dependent deformations of) $\mathcal{N} = 4$ SYM$_4$ in the UV to different J-fold CFT$_3$'s in the IR. The latter with various amounts of supersymmetry and the largest possible flavour symmetry by turning off any axionic deformation: $\mathcal{N} = 1\,\&\,\mathrm{SU}(3)\,,\ \mathcal{N} = 2\,\&\,\mathrm{SU}(2)\times U(1)_R$ and $\mathcal{N} = 4\,\&\,SO(4)_R\,.$ These holographic RG-flow were build in 4d, and can be uplifted to Type IIB supergravity using the gSS ansatz which offer a more natural setting to study their holographic interpretation.

In ten dimensions, these RG flow present a geometry interpolating between the S-folds reviewed in section 5.4 in the IR and a the solution that asymptotes the c-dependent subleading correction of the $\mathrm{AdS}_5\times\mathrm{S}^5$ geometry. In this section, we will review the uplift of the $c = 0$ DW solution and show that it is a 4d description of the $\mathrm{AdS}_5\times\mathrm{S}^5$ solution. We will then comment on the role of the axionic deformations in this context. We will then uplift the deformed D3-brane solution in Section 7.2.2. To do so, we will restrict to an SU(3) invariant deformation of the D3-brane which captures the most relevant features of this flow in the UV.

$$\boxed{\mathrm{IR}} \qquad\qquad\qquad\qquad \boxed{\mathrm{UV}}$$

$$\mathrm{AdS}_4\times\mathrm{S}^1\times\mathrm{S}^5 \qquad \Longleftrightarrow \qquad \mathrm{AdS}_5\times\mathrm{S}^5 \quad (c\text{-deformed})$$

$$ds^2_{10} = \tfrac{1}{2}\,\Delta_{\mathrm{IR}}^{-1}\left(ds^2_{\mathrm{AdS}_4} + 2\,f(z_i)\,d\eta^2\right) + g^{-2}\,ds^2_{\mathrm{S}^5} \qquad ds^2_{10} = \tfrac{1}{2}\,\Delta_{\mathrm{UV}}^{-1}\left(ds^2_{\mathrm{DW}_4} + 2\,\Delta_{\mathrm{UV}}\,H(z)\,d\eta^2\right) + g^{-2}\,ds^2_{\mathrm{S}^5}$$

FIGURE 7.12: Type IIB geometry describing an holographic RG flows across dimensions from (c-dependent deformations of) $\mathcal{N} = 4$ SYM$_4$ in the UV to different J-fold CFT$_3$'s in the IR.

7.4.1 Uplifting the UV: D3-brane at $c = 0$

Let us first set the three axions $\chi^{(0)}_{1,2,3} = 0$ so that the largest amount of symmetry is preserved. The four-dimensional solution contains two arbitrary parameters (g, Φ_0)

and uplifts to a ten-dimensional type IIB background with a factorised internal geometry of the form $\mathrm{S}^1 \times \mathrm{S}^5$ in the limit of an infinite radius for S^1 so that $\mathrm{S}^1 \to \mathbb{R}$. The five-sphere is *round* and displays its largest possible $\mathrm{SO}(6)$ symmetry. The various ten-dimensional fields are given by

$$\begin{aligned} ds_{10}^2 &= \tfrac{1}{2}\,\Delta^{-1}\,ds_{\mathrm{DW}_4}^2 + \Delta^2\,d\eta^2 + g^{-2}\,d\mathring{s}_{\mathrm{S}^5}^2 \ , \\ \widetilde{F}_5 &= 4\,g\,(1+\star)\,\mathrm{vol}_5 \ , \\ m_{\alpha\beta} &= \begin{pmatrix} e^{-\Phi_0} & 0 \\ 0 & e^{\Phi_0} \end{pmatrix} \qquad \text{with} \qquad \Phi_0 = cst \ , \\ \mathbb{H}^{\alpha} &= 0 \ , \end{aligned} \tag{7.52}$$

where $ds_{\mathrm{DW}_4}^2$ is the domain-wall metric displayed in (7.1) and $\mathrm{vol}_5 = g^{-5}\,\mathring{\mathrm{vol}}_5$ is a rescaled volume form on the *round* five-sphere. The warping function $\Delta(z)$ takes the simple form

$$\Delta = \frac{(g\,z)^2}{8} = \mathrm{Im} z_{1,2,3} \ . \tag{7.53}$$

At first sight the DW_4 metric in (7.52) seems to break conformal invariance. However, the five-dimensional piece of the metric (7.52) spanned by the domain-wall and the coordinate η can be recast as an AdS_5 metric

$$\begin{aligned} \tfrac{1}{2}\,\Delta^{-1}\,ds_{\mathrm{DW}_4}^2 + \Delta^2\,d\eta^2 &= (gz)^4\left(4\,\eta_{\alpha\beta}\,dx^{\alpha}\,dx^{\beta} + 2^{-6}\,d\eta^2\right) + \frac{4}{(gz)^2}\,dz^2 \\ &= \frac{g^{-2}}{r^2}\left(-dt^2 + dx^2 + dy^2 + dw^2 + dr^2\right) \\ &= ds_{\mathrm{AdS}_5}^2 \ , \end{aligned} \tag{7.54}$$

upon a change of coordinates

$$t = 2\,x^0 \ \ , \ \ x = 2\,x^1 \ \ , \ \ y = 2\,x^2 \ \ , \ \ w = 2^{-3}\,\eta \ \ , \ \ r = \frac{g^{-1}}{(gz)^2} \ , \tag{7.55}$$

thus recovering the maximally supersymmetric $\mathrm{AdS}_5 \times \mathrm{S}^5$ near-horizon geometry of the D3-brane with $L_{\mathrm{AdS}_5} = g^{-1}$. This is nothing but the holographic dual of $\mathcal{N}=4$ SYM_4.

The axion deformation have the same interpretation as in the case of the $\mathcal{N}=0$ $SO(6)$ symmetric solutions we discussed in the previous chapter. One can parametrise the three commuting orthogonal rotations of S^5 by angles θ_i with $i = 1, \ldots, 3$. Turning on the axion correspond to the local change of coordinates[6]

$$\theta_i' = \theta_1 + g\,\chi_i^{(0)}\,\eta \,. \tag{7.56}$$

This is what was expected from our discussion on flat deformations. However, there is a subtlety due to the $c=0$ limit we have taken here. When we were working on the S^1_η circle, the coordinate change 7.56 was not well defined because of global effects, namely periodicities issues when going along the S^1_η circle. Such issues do not arise in the case where S^1_η decompactify to $\mathbb{R}_\eta$. However, one should still be careful when using the change of coordinates 7.56 as it is not a *small* gauge transformation either.

[6]These angles must be chosen judiciously to correspond to the $\mathfrak{u}(1)^3 \subset \mathfrak{so}(6)$ generated by the axions.

It does not vanish as $\eta \to \infty$. We do not, at the moment, have 10d interpretation of these axionic deformations as good as on $S^1 \times S^5$ case.

7.4.2 Uplifting the UV: deformed D3-brane at $c \neq 0$

In this section we investigate various aspects of the $\mathbb{Z}_2 \times \mathrm{SU}(3)$ invariant sector of the $[\mathrm{SO}(1,1) \times \mathrm{SO}(6)] \ltimes \mathbb{R}^{12}$ gauged supergravity which is obtained upon identifying the scalar fields in the $\mathbb{Z}_2^3$ invariant sector as

$$z_1 = z_2 = z_3 \equiv z_{1,2,3} \qquad \text{and} \qquad z_4 = z_5 = z_6 = z_7 \equiv z_{4,5,6,7} \ . \tag{7.57}$$

This two scalars model is of particular interest as it allows us to capture the most interesting features of the D3-brane deformation we discussed earlier. Perturbing the four-dimensional incarnation of the D3-brane solution in (7.11), and solving the BPS equations perturbatively in the parameter c, we found the universal solution in (7.23) which, as already emphasised, is compatible with (7.57). Using the notation of section 5.4.3, we now present the uplift of this sector.

For $Re(z_{1,2,3})$, the ten-dimensional metric takes the form

$$ds_{10}^2 = \tfrac{1}{2}\,\Delta^{-1}\left(ds^2_{\mathrm{DW}_4} + 2\,(gc)^{-2}\,\Delta\,H(z_i)\,d\eta^2\right) + g^{-2}\,F(z_i)\,\left[ds^2_{\mathbb{CP}_2} + F(z_i)^{-2}\,\boldsymbol{\eta}^2\right] \ , \tag{7.58}$$

in terms of a four-dimensional space-time given by $ds^2_{\mathrm{DW}_4}$ in (7.1) and an internal space $\mathrm{M}_6 = \mathrm{S}^1_\eta \rtimes \mathrm{S}^5$ with $\mathrm{S}^5 = \mathbb{CP}_2 \rtimes \mathrm{S}^1$. As a result of the type IIB uplift, the metric in (7.58) depends on two scalar-dependent functions

$$F(z_i) = |z_{4,5,6,7}|^{-1}\,\mathrm{Im}z_{4,5,6,7} \qquad \text{and} \qquad H(z_i) = F(z_i)^{-1}\,(\mathrm{Im}z_{1,2,3})^2 \ , \tag{7.59}$$

and the warping factor

$$\Delta = F(z_i)\,\mathrm{Im}z_{1,2,3} \ . \tag{7.60}$$

Introducing the axion $\mathrm{Re}(z_{1,2,3}) \neq 0$ is equivalent replacing $\mathrm{d}\phi \to \mathrm{d}\phi + \chi_{1,2,3}\,\mathrm{d}\eta$. The complex combination of the two-forms is independent of $\mathrm{Re}(z_{1,2,3})$ and reads

$$\mathfrak{b}^2 + i\,|z_{4,5,6,7}|^2\,\mathfrak{b}^1 = -i\,g^{-2}\,\mathrm{Re}z_{4,5,6,7}\ \boldsymbol{\Omega} \ . \tag{7.61}$$

Equivalently,

$$g^2\,\mathfrak{b}^1 = -|z_{4,5,6,7}|^{-2}\,\mathrm{Re}z_{4,5,6,7}\ \mathrm{Re}\boldsymbol{\Omega} \qquad , \qquad g^2\,\mathfrak{b}^2 = \mathrm{Re}z_{4,5,6,7}\ \mathrm{Im}\boldsymbol{\Omega} \ . \tag{7.62}$$

The associated three-form field strengths $\mathbb{H}^\alpha = d\mathbb{B}^\alpha = (H_3, F_3)$ are directly computed from (5.68), (5.65) and (7.61). They take the form[7]

$$\mathbb{H}^\alpha = A^\alpha{}_\beta\left(d\eta \wedge \mathfrak{b}^\gamma\,\theta_\gamma{}^\beta + d\mathfrak{b}^\beta\right) \ , \tag{7.63}$$

with $\mathfrak{b}^\alpha$ given in (7.62) and

$$\begin{aligned} g^2\,d\mathfrak{b}^1 &= -d\left(\frac{\mathrm{Re}z_{4,5,6,7}}{|z_{4,5,6,7}|^2}\right) \wedge \mathrm{Re}\boldsymbol{\Omega} + 3\,\frac{\mathrm{Re}z_{4,5,6,7}}{|z_{4,5,6,7}|^2}\,(\boldsymbol{\eta} - \mathrm{Re}z_{1,2,3}\,d\eta) \wedge \mathrm{Im}\boldsymbol{\Omega} \ , \\ g^2\,d\mathfrak{b}^2 &= d\mathrm{Re}z_{4,5,6,7} \wedge \mathrm{Im}\boldsymbol{\Omega} + 3\,\mathrm{Re}z_{4,5,6,7}\,(\boldsymbol{\eta} - \mathrm{Re}z_{1,2,3}\,d\eta) \wedge \mathrm{Re}\boldsymbol{\Omega} \ . \end{aligned} \tag{7.64}$$

[7] We have used the relation $\partial_\eta A = A\,\theta^t = A\,\theta$.

In (7.63) we have introduced the constant matrix

$$\theta_\gamma{}^\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \, , \tag{7.65}$$

and used the third of the SU(2)-structure relations in (5.87).
The axion-dilaton matrix is given by

$$m_{\alpha\beta} = (A^{-t})_\alpha{}^\gamma \, \mathfrak{m}_{\gamma\delta} \, (A^{-1})^\delta{}_\beta \qquad \text{with} \qquad \mathfrak{m}_{\gamma\delta} = \begin{pmatrix} |z_{4,5,6,7}|^2 & 0 \\ 0 & |z_{4,5,6,7}|^{-2} \end{pmatrix} \, . \tag{7.66}$$

Finally the self-dual five-form field strength is given by

$$\begin{aligned} \widetilde{F}_5 \;\; &= \;\; g\,(1+\star)\,\Big[\Big(4-6\,(1-F(z_i)^2)\Big)\,\text{vol}_{\mathbb{CP}_2}\wedge d\beta \\ &+ \;\; \Big(4\,\text{Re}z_{1,2,3}+(\text{Re}z_{4,5,6,7})^2\;(1-|z_{4,5,6,7}|^{-4})\Big)\,\text{vol}_{\mathbb{CP}_2}\wedge d\eta \\ &- \;\; d\text{Re}z_{1,2,3}\wedge d\eta\wedge\Big(\tau^1\wedge\tau^2\wedge\boldsymbol{A}_1+2\boldsymbol{J}\wedge d\beta\Big)\Big] \, , \end{aligned} \tag{7.67}$$

with the frame fields on $\mathbb{CP}_2$ given in (5.93). This concludes the type IIB uplift of the $\mathbb{Z}_2 \times \text{SU}(3)$ invariant sector of the $[\text{SO}(1,1)\times\text{SO}(6)]\ltimes\mathbb{R}^{12}$ maximal supergravity.

In order to recover the ten-dimensional geometries for the S-folds and the D3-brane, we will express the metric (7.58) as

$$\begin{aligned} ds_{10}^2 \;\; &= \;\; \tfrac{1}{2}\,F(z_i)^{-1}\,\Big[e^{B(\rho)}\,\eta_{\alpha\beta}\,dx^\alpha dx^\beta+(gc)^{-2}\,e^{C(\rho)}\,dw^2+d\rho^2\Big] \\ &+ \;\; g^{-2}\,\Big[F(z_i)\,ds^2_{\mathbb{CP}_2}+F(z_i)^{-1}\,\boldsymbol{\eta}^2\Big] \, , \end{aligned} \tag{7.68}$$

with $\alpha=0,1,2$, in terms of a rescaled coordinate $w=2^{-3}\,\eta$, a new radial coordinate ρ defined as

$$d\rho = (\text{Im}z_{1,2,3})^{-\frac{1}{2}}\,dz \quad \Rightarrow \quad \rho = \int (\text{Im}z_{1,2,3})^{-\frac{1}{2}}\,dz \, , \tag{7.69}$$

and the functions

$$e^B = (\text{Im}z_{1,2,3})^{-1}\,e^{2A} \qquad \text{and} \qquad e^C = 2^7\,(\text{Im}z_{1,2,3})^2 \, . \tag{7.70}$$

IR and $c=0$ limits

Let us first consider the deep IR region of the BPS domain-walls constructed in Section 7.3. When approaching this region the scalars get a constant value. Then $\rho=(\text{Im}z_{1,2,3})^{-\frac{1}{2}}\,z$ and $A=L^{-1}\,z$ so that

$$e^B \sim e^{2\,(\text{Im}z_{1,2,3})^{\frac{1}{2}}\,L^{-1}\rho} \qquad , \qquad e^C \sim \text{cst} \, , \tag{7.71}$$

and the metric (7.68) boils down to the $\text{AdS}_4\times\text{S}^1\times\text{S}^5$ metric for the $\mathcal{N}=1$ S-fold in Section 5.4.3. On the other hand, substituting the analytic D3-brane solution (7.11) obtained at $c=0$ [8] yields $F(z_i)=1$ together with

$$e^B = 8\,e^{\sqrt{2}\,(g\rho)} \quad , \quad e^C = 2\,e^{\sqrt{2}\,(g\rho)} \qquad \text{with} \qquad e^{g\rho} = (gz)^{2\sqrt{2}} \, . \tag{7.72}$$

[8] Note that the factor $(gc)^{-2}$ in the metric (7.68) becomes pathological in this limit and must be just removed. Also, the A-twist matrix in (5.65) must be taken to be the identity when $c=0$.

Then the expected $\mathrm{AdS}_5\times\mathrm{S}^5$ local geometry with a round S^5 metric $d\mathring{s}^2_{\mathrm{S}^5}=ds^2_{\mathbb{CP}_2}+\boldsymbol{\eta}^2$ is recovered. Note also that, since $\mathrm{Re}z_{1,2,3}$ take constant values in the D3-brane solution (7.11), one must turn on the axionic deformations to reach the UV solution.

Recovering the deformed D3-brane

Let us investigate the ten-dimensional type IIB uplift of the domain-walls constructed in Sections 7.3.1, 7.3.2 and 7.3.3 when approaching the UV ($z\to\infty$). We will holographically relate such a UV behaviour to having an anisotropic deformation of SYM_4.

It is instructive to first study the uplift of the UV in five dimensions. Setting $c\neq 0$ modifies the AdS_5 metric in (7.54) so that it acquires a dependence on the scalar fields

$$ds_5^2=\tfrac{1}{2}\,\Delta_{\mathrm{UV}}^{-1}\left(ds^2_{\mathrm{DW}_4}+2\,(gc)^{-2}\,\Delta_{\mathrm{UV}}\,H(z_i)\,d\eta^2\right)\ , \tag{7.73}$$

in terms of the functions $F(z_i)$ and $H(z_i)$ in (7.59) and the warping factor Δ in (7.60). Consistently, $F(z_i)=1$ and $H(z_i)=\Delta^2$ when $c=0$ so that the undeformed AdS_5 metric in (7.54) is recovered. Note that the deformed AdS_5 metric in (7.73) singles out the coordinate η which, in the deep IR, will span the S^1 factor of the $\mathrm{AdS}_4\times\mathrm{S}^1\times\mathrm{S}^5$ S-fold geometry [55, 1, 2]. This is the same coordinate along which the type IIB fields acquire a dependence on as a consequence of the $\mathrm{SL}(2)_{\mathrm{IIB}}$ monodromy in (5.65). However, the RG flows constructed in Section 7.3 occur along the radial direction gz. Therefore, the dependence of the type IIB fields $m_{\alpha\beta}$ and $\mathbb{B}^\alpha$ on the η coordinate induced by the S-folding (equivalently by the parameter c) is ultimately connected to having an anisotropic deformation of $\mathcal{N}=4$ SYM_4 (see, *e.g.* [117, 121]). Let us look at this issue in more detail from a ten-dimensional perspective.

In order to present the ten-dimensional metric (7.58) in a suitable form to be compared with previous results in the literature regarding holographic anisotropy in SYM_4 (see [118, 119]), it is convenient to first perform a redefinition of the radial coordinate as

$$g\,\zeta=(gz)^2\ , \tag{7.74}$$

and then introduce a set of metric functions

$$h=4\,\Delta_{\mathrm{UV}}^2\,e^{-4A}\quad,\quad e^{2m}=2\,e^{-2A}\,\Delta_{\mathrm{UV}}\,H\quad,\quad e^{2f}=2^4\,\Delta_{\mathrm{UV}}^2\,e^{-2A}\,(g\zeta)^3\ , \tag{7.75}$$

so that (7.58) conforms with

$$\begin{aligned} ds^2_{10} &= h^{-\frac{1}{2}}\left(\eta_{\alpha\beta}\,dx^\alpha dx^\beta+(gc)^{-2}\,e^{2m}\,d\eta^2\right)\\ &+ h^{\frac{1}{2}}\left[(g\zeta)^2\,e^{-2f}\,d\zeta^2+g^{-2}\,h^{-\frac{1}{2}}\,F\,ds^2_{\mathbb{CP}_2}+g^{-2}\,h^{-\frac{1}{2}}\,F^{-1}\,\boldsymbol{\eta}^2\right]\ . \end{aligned}\tag{7.76}$$

It then becomes transparent that the function e^{2m} in (7.76) will be responsible for anisotropy in the dual 4D field theory. Using (7.59) and the four-dimensional universal solution in (7.25) the three metric functions in (7.75) read

$$\begin{aligned} h &= \frac{1}{16\,(g\zeta)^4}\left(1 - 16\cosh^2\Phi_0\,\frac{c^2}{(g\zeta)^2} + \dots\right) , \\ e^{2m} &= 2\,e^{-2A}\,(\text{Im}z_{1,2,3})^3 \;=\; 2^{-8}\left(1 + 64\cosh^2\Phi_0\,\frac{c^2}{(g\zeta)^2} + \dots\right) , \\ e^{2f} &= \frac{(g\zeta)^2}{4}\left(1 + 16\,\cosh^2\Phi_0\,\frac{c^2}{(g\zeta)^2} + \dots\right) . \end{aligned} \tag{7.77}$$

As a result the anisotropy function e^{2m} acquires a non-trivial $c^2/(g\zeta)^2$ correction in the UV. Note that this correction does *not* occur at linear order in c, as can be checked by direct substitution of the four-dimensional solution in (7.21) instead of the one in (7.25)[9].[10]

Note that, upon dimensional reduction on the five-sphere S^5, the three-form field strengths $\mathbb{H}^\alpha$ in (7.63) give rise to an $\text{SL}(2)_{\text{IIB}}$ doublet of one-form field strengths

$$\mathcal{F}_{(1)}{}^{\alpha} = A^{\alpha}{}_{\beta}\,d\eta \wedge \underbrace{\mathfrak{b}^{\gamma}\,\theta_{\gamma}{}^{\beta}}_{\text{purely along }\text{S}^5} , \tag{7.80}$$

along the $d\eta$ direction in the (external) geometry (7.73). From the expressions of $\mathfrak{b}^\alpha$ in (7.62), and using the four-dimensional solution in (7.25), one finds

$$\begin{aligned} g^2\,\mathfrak{b}^1 &= -\left(4\,e^{\frac{1}{2}\Phi_0}\,\cosh\Phi_0\,\frac{c}{g\,\zeta} + \dots\right)\text{Re}\boldsymbol{\Omega} , \\ g^2\,\mathfrak{b}^2 &= \left(4\,e^{-\frac{1}{2}\Phi_0}\,\cosh\Phi_0\,\frac{c}{g\,\zeta} + \dots\right)\text{Im}\boldsymbol{\Omega} . \end{aligned} \tag{7.81}$$

Importantly, the dependence on the coordinate η through the twist matrix $A(\eta)$ will factorise out due to the contraction with the axion-dilaton matrix $m_{\alpha\beta}$ in the ten-dimensional kinetic term $m_{\alpha\beta}\,\mathbb{H}^\alpha \wedge \star\mathbb{H}^\beta$.

Similar one-form terms along the $d\eta$ direction in the (external) geometry (7.73) appear from the axion-dilaton matrix in (7.66) with

$$\mathfrak{m}_{\gamma\delta} = \begin{pmatrix} \mathfrak{m}_+ & 0 \\ 0 & \mathfrak{m}_- \end{pmatrix} \qquad \text{and} \qquad \mathfrak{m}_\pm = e^{\mp\Phi_0}\left(1 \pm 32\,\sinh(2\Phi_0)\,\frac{c^2}{(g\,\zeta)^2} + \dots\right) , \tag{7.82}$$

[9]This is also the case for the non-universal solution in (7.17)-(7.18) computed at linear order in c provided the identifications in (7.57) hold.

[10]Similarly, the functions occurring in front of the internal S^5 metric in (7.76) are given by

$$\begin{aligned} h^{-\frac{1}{2}}\,F &= 4\,(g\,\zeta)^2\left(1 - \frac{384}{49}\,\cosh^2\Phi_0\left(f_1(\Phi_0) + f_2(\Phi_0)\,\log(g\zeta)\right)\frac{c^4}{(g\zeta)^4} + \dots\right) , \\ h^{-\frac{1}{2}}\,F^{-1} &= 4\,(g\,\zeta)^2\left(1 + 16\,\cosh^2\Phi_0\,\frac{c^2}{(g\zeta)^2} + \dots\right) , \end{aligned} \tag{7.78}$$

with

$$f_1(\Phi_0) = 9 + \cosh(2\Phi_0) - 28\,\sinh(2\Phi_0) \qquad \text{and} \qquad f_2(\Phi_0) = 56\,\big(1 - 3\,\cosh(2\Phi_0)\big). \tag{7.79}$$

and also from $\widetilde{F}_5$ in (7.67). From the axion-dilaton one finds

$$\begin{aligned}\mathcal{F}_{(1)\alpha\beta} &= -(|z_{4,5,6,7}|^2 + |z_{4,5,6,7}|^{-2})\,(A^{-t}\,\theta\,A^{-1})_{\alpha\beta}\,d\eta \\ &= -2\,\cosh\Phi_0\left(1 - 64\,\sinh^2\Phi_0\,\frac{c^2}{(g\,\zeta)^2} + \dots\right)(A^{-t}\,\theta\,A^{-1})_{\alpha\beta}\,d\eta\ ,\end{aligned} \tag{7.83}$$

whereas from the five-form field strength one gets

$$\begin{aligned}\widetilde{\mathcal{F}}_{(1)} &= \Big[\tfrac{4}{3}\,c\,\sinh\Phi_0\left(1 - 192\,\cosh^2\Phi_0\,\frac{c^2}{(g\,\zeta)^2}\,\log(g\,\zeta) + \dots\right) \\ &\quad - \left(16\,\sinh(2\Phi_0)\,\cosh\Phi_0\,\frac{c^2}{(g\,\zeta)^2} + \dots\right)\Big]\,d\eta\wedge\text{vol}_{\mathbb{CP}_2}\ .\end{aligned} \tag{7.84}$$

We have verified the stability of the above results using the four-dimensional solution computed up to twelfth order in the parameter c.

Other type IIB constructions have been put forward to generate holographic anisotropy in SYM$_4$. For instance, [118, 119] employ backreacted geometries involving D3- and (smeared) D5-branes along $2+1$ dimensions. Our setup involves only closed strings (without source terms) and differs from the one in [118, 119] in that it generates anisotropy in a purely geometric manner by implementing the locally geometric $\text{SL}(2)_{\text{IIB}}$ twist $A(\eta)$ in (5.65). As a consequence of the mechanism here, the various one-form sources of anisotropy, namely, $(\mathcal{F}_{(1)}{}^{\alpha}\,,\,\mathcal{F}_{(1)\alpha\beta}\,,\,\widetilde{\mathcal{F}}_{(1)})$, organise themselves into $\text{SL}(2)_{\text{IIB}}$ multiplets. The $\text{SL}(2)_{\text{IIB}}$ covariance here could explain the difference between the ζ-powers appearing in (7.77)-(7.78) and in [119]. Also along these lines, it would be interesting to characterise the operators triggering SYM$_4$ anisotropy in an $\text{SL}(2)_{\text{IIB}}$ covariant setup of the type investigated here. To this end, it would be helpful to have the oxidation of the RG flows presented in this work to five dimensions.

Finally, let us remark that the UV flow does necessarily have a dependence in the axion of the form

$$\sum_{i=1}^{3}\text{Re}z_i \approx c\,\sinh\Phi_0\ , \tag{7.85}$$

in the deformed D3-brane solution controlling the UV behaviour of the RG flows in Section 7.3. This necessarily causes an axionic breaking of the SO(6) symmetry down to a subgroup of SU(3). In addition, we have also verified that the axionic symmetry breaking on the D3-brane reproduces the patterns observed in chapter 6.

7.5 Summary and concluding remarks

Using the effective four-dimensional $[\text{SO}(1,1)\times\text{SO}(6)]\ltimes\mathbb{R}^{12}$ gauged supergravity, all the S-fold backgrounds were shown to be connected with a deformation of the D3-brane solution, namely, a deformation of the $\text{AdS}_5\times\text{S}^5$ background, via a holographic RG-flow. In other words, all the conjectured S-fold SCFT's can be viewed as IR fixed points of a supersymmetric RG-flow that starts from a deformation of $\mathcal{N}=4$ super Yang–Mills in the UV. This UV regime was found to correspond to an *anisotropic* deformation of $\mathcal{N}=4$ SYM. This is in agreement with the five-dimensional supergravity fields developing a dependence on the η coordinate along S^1, which becomes a spatial coordinate of the deformed field theory living at the boundary.

The above RG-flows were also analysed from a ten-dimensional perspective aiming at shedding some new light on their holographic interpretation. In this manner, the anisotropic deformation of SYM was interpreted in a purely geometric manner and connected to the locally geometric SL(2) S-duality twist matrix $A(\eta)$ in (5.65) generating the S-fold background. Moreover, a set of five-dimensional one-form deformations $(\mathcal{F}_{(1)}{}^{\alpha}\,,\,\mathcal{F}_{(1)\alpha\beta}\,,\,\widetilde{\mathcal{F}}_{(1)})$ were identified as the sources of anisotropy, thus coming in SL(2) representations. A more detailed characterisation of the operators triggering such an anisotropy in an SL(2) covariant context is yet to be worked out. Lastly, holographic RG-flows between S-fold CFT's were also explicitly constructed in [122].

Chapter 8

Half-maximal deformations

In this chapter, we will produce a third type of deformation of our S-folds, using a purely 4d perspective. This deformation will be a deformation of the embedding tensor of the maximal supergravity we have studied so far. To do so, we will firstly truncate our maximal supergravity down to a $\mathcal{N}=4$ gauged supergravity. We will then deform the embedding tensor obtained this way in a way which is not compatible with an embedding in maximal supergravity. Let us start by motivate our choice of such deformation.

We will concentrate on the holographic study of $\mathcal{N}=2$ CFT_3's with a potential type IIB gravity dual of the form $\text{AdS}_4 \times \text{S}^2 \times \text{S}^2 \times \Sigma$. In order to be general but still keep some control, we will make some assumptions on the effective 4d supergravity that could describe such a model. Firstly, we will consider four-dimensional supergravity Lagrangians preserving half-maximal supersymmetry (16 supercharges). Secondly, in order to be able to recover the conformal manifold of $\mathcal{N}=2$ S-fold CFT_3's constructed in the context of maximal supergravity [56], we will couple the half-maximal supergravity multiplet to *six* vector multiplets so that the field content of the half-maximal supergravity forms a subset of the (unique) field content of maximal supergravity. Thirdly, in order to be compatible with a potential uplift to a type IIB solution of the form $\text{AdS}_4 \times \text{S}^2 \times \text{S}^2 \times \Sigma$, the gauging must contain an $\text{SO}(3) \times \text{SO}(3)$ factor. It becomes then natural to investigate gaugings of the group

$$\text{G}_{\text{half-max}} = \text{ISO}(3) \times \text{ISO}(3) \subset \text{SL}(2) \times \text{SO}(6,6) \ , \tag{8.1}$$

with $\text{ISO}(3) = \text{SO}(3) \ltimes \mathbb{R}^3$, as well as their possible group-theoretical embeddings (8.1) into the $\text{SL}(2) \times \text{SO}(6,6)$ duality group of half-maximal supergravity coupled to six vector multiplets. The most general such embeddings turns out to depend on *eight* parameters. However, a full analysis of the eight-parameter family of gaugings in (8.1) goes beyond the scope of this work. In this work we will even more simplify this setup by choosing the same group-theoretical embedding for the two $\text{ISO}(3)$ factors in (8.1). This reduces the number of embedding parameters down to four. We will study a specific three parameter family of gaugings (c, g, $\tilde{\varphi}$) which allows us to identify this $\mathcal{N}=4$ theory with a $\mathbb{Z}_2$ truncation of our $[SO(1,1) \times SO(6)] \ltimes \mathbb{R}^{12}$ maximal supergravity whenever $\tilde{\varphi}=1$.

Turning on these three embedding parameters produces new multi-parametric families of AdS_4 solutions, all of them with the same AdS_4 radius. Interestingly, for any value of the three embedding parameters, the AdS_4 solutions still feature two scalar moduli $(\varphi, \chi) \in \mathbb{R}$ associated with non-compact flat directions in the scalar potential. By adjusting the embedding parameters and the scalar moduli (φ, χ) in the AdS_4 solutions, these can preserve $\mathcal{N}=2$ (8 supercharges), $\mathcal{N}=3$ (12 supercharges) or $\mathcal{N}=4$ (16 supercharges) supersymmetry. Via the gauge/gravity duality, these AdS_4 solutions are conjectured to be dual to new classes of strongly-coupled $\mathcal{N}=2$,

$\mathcal{N}=3$ or $\mathcal{N}=4$ CFT$_3$'s provided an embedding in string theory (yet to be worked out) exists.

Theses results point at the existence of a web of $\mathcal{N}=2$ CFT$_3$'s containing a special “line" of $\mathcal{N}=3$ supersymmetry enhancement which, in turn, contains isolated “points" where supersymmetry gets enhanced to $\mathcal{N}=4$. We will characterise this web of $\mathcal{N}=2$ CFT$_3$'s by arranging their set of low lying operators into superconformal multiplet of $\mathfrak{osp}(2|4)$, and by further discussing the phenomenon of supermultiplet shortening that occurs at the special loci where supersymmetry gets enhanced to $\mathcal{N}=3$ or $\mathcal{N}=4$. Regarding the latter, two isolated points describing $\mathcal{N}=4$ CFT$_3$'s are identified. The first one describes the $\mathcal{N}=4$ S-fold CFT$_3$ of [71, 55] dual to the $\mathcal{N}=4$ AdS$_4$ solution of the maximal supergravity with gauge group[1]

$$\mathrm{G}_{\mathrm{max}} = [SO(1,1)\times SO(6)]\ltimes\mathbb{R}^{12}, \tag{8.2}$$

even if its realisation in $\mathcal{N}=4$ theory only shows a $\mathcal{N}=3$ residual supersymmetry. The second point describes a novel $\mathcal{N}=4$ CFT$_3$ – we will refer to it as the *exotic* $\mathcal{N}=4$ CFT$_3$ – dual to an $\mathcal{N}=4$ AdS$_4$ solution of a half-maximal supergravity with gauge group (8.1). This AdS$_4$ solution appeared originally in [123] where it was argued to be a non-geometric solution still admitting a locally geometric type IIB description. We now see that it is actually connected to the locally geometric $\mathcal{N}=4$ S-fold solution of type IIB supergravity [55], at least at the effective four-dimensional supergravity level. As a by-product, we will also present an additional set of generically non-supersymmetric marginal deformations of the exotic $\mathcal{N}=4$ CFT$_3$ which resemble (without being the same) the axion-like deformations of S-folds reviewed in chapter 6.

This chapter is organised as follows. In Section 8.1 we review a simple class of ISO(3) $\times$ ISO(3) gaugings of $\mathcal{N}=4$ supergravity which depends on a specific *deformation* parameter $\tilde{\varphi}$. We also show how this gaugings connects to maximal supergravity when $\tilde{\varphi}=1$. In Section 8.2 we construct the $\mathbb{Z}_2^2$-invariant sector of the theory and present a $(\varphi;\tilde{\varphi})$-family of $\mathcal{N}=2$ AdS$_4$ solutions that incorporates the φ modulus dual to one of the two marginal operators spanning the conformal manifold of $\mathcal{N}=2$ S-fold CFT$_3$'s. These AdS$_4$ solutions are holographically conjectured to describe a web of $\mathcal{N}=2$ CFT$_3$'s whose spectrum of low lying operators is arranged into superconformal multiplets of $\mathfrak{osp}(2|4)$. Supersymmetry as well as flavour symmetry enhancements are discussed together with the corresponding shortening of superconformal multiplets. In Section 8.3 we construct the U(1)$_\mathrm{R}$-invariant sector of the theory in order to also incorporate the modulus χ dual to the second marginal operator compatible with $\mathcal{N}=2$ supersymmetry in the dual CFT$_3$'s. We present a $(\varphi,\chi;\tilde{\varphi})$-family of $\mathcal{N}=2$ AdS$_4$ solutions that generalises the results of [56] to the context of half-maximal supergravity, and arrange the spectrum of low lying operators of the dual $\mathcal{N}=2$ CFT$_3$'s into superconformal multiplets of $\mathfrak{osp}(2|4)$. After a suitable treatment of vector fields and gauge redundancies in the U(1)$_\mathrm{R}$-invariant sector, the Zamolodchikov metric on the conformal manifold of such $\mathcal{N}=2$ CFT$_3$'s is shown to be

$$ds^2_{\mathrm{CM}} = \frac{1+2\varphi^2}{2(1+\varphi^2)^2}\left(d\varphi^2+(1+\varphi^2)\,d\chi^2\right)\ . \tag{8.3}$$

for arbitrary values of the half-maximal deformation parameter $\tilde{\varphi}$. We conclude in Section 8.4 with some implications and potential applications of the results presented

[1]Note that $\mathrm{G}_{\text{half-max}}\subset\mathrm{G}_{\mathrm{max}}$. This makes it possible to rediscover an AdS$_4$ solution of a maximal supergravity as an AdS$_4$ solution of a half-maximal supergravity provided certain relations between the couplings in the half-maximal supergravity Lagrangian hold, see (3.112).

in this work. Appendix B discusses more general gaugings of ISO(3) × ISO(3) in half-maximal supergravity.

8.1 ISO(3) × ISO(3) half-maximal supergravity

Our starting point is the four-dimensional maximal ($\mathcal{N} = 8$) supergravity with gauge group

$$\mathrm{G}_{\mathrm{max}} = [\mathrm{SO}(1,1) \times \mathrm{SO}(6)] \ltimes \mathbb{R}^{12} \subset \mathrm{E}_{7(7)} \ . \tag{8.4}$$

We are going to deform the corresponding embedding tensor of the four-dimensional maximal supergravity in a way that do not satisfy the extra constraints required to embed an half maximal supergravity in a maximal supergravity (3.74). To do so in an interesting way, and following the prescription in [37], we will mod out the $\mathrm{G}_{\mathrm{max}}$ -gauged maximal supergravity by a $\mathbb{Z}_2$ discrete group to produce a very specific half-maximal ($\mathcal{N} = 4$) supergravity with gauge group

$$\mathrm{G}_{\text{half-max}} = \mathrm{ISO}(3) \times \mathrm{ISO}(3) \subset \mathrm{SL}(2) \times \mathrm{SO}(6,6) \ . \tag{8.5}$$

Then, we will introduce a specific deformation of such a half-maximal supergravity – which we parameterise in terms of a continuous parameter $\tilde{\varphi} \in \mathbb{R}$ – to produce new gaugings of the group We investigate a specific class of gaugings of the group[2]

$$\mathrm{G} = \mathrm{ISO}(3)_1 \times \mathrm{ISO}(3)_2 \ , \tag{8.6}$$

which is embedded in the duality group as

$$\mathrm{G} \ \subset \ \mathrm{SO}(3,3)_1 \times \mathrm{SO}(3,3)_2 \ \subset \ \mathrm{SL}(2) \times \mathrm{SO}(6,6) \ , \tag{8.7}$$

where we have attached labels $_1$ and $_2$ in order to keep track of each independent ISO(3) and SO(3,3) factor in (8.6) and (8.7). General classes of gaugings of $\mathrm{G} \subset \mathrm{SO}(3,3)_1 \times \mathrm{SO}(3,3)_2$ in half-maximal supergravity [126, 127, 124] have been extensively investigated in the past, for example, with the aim of charting the landscape of flux compactifications [128, 123].

8.1.1 From $[\mathrm{SO}(1,1) \times \mathrm{SO}(6)] \ltimes \mathbb{R}^{12}$ to $\mathrm{ISO}(3) \times \mathrm{ISO}(3)$

Starting from the $\mathrm{G}_{\mathrm{max}} = [\mathrm{SO}(1,1) \times \mathrm{SO}(6)] \ltimes \mathbb{R}^{12}$ gauged maximal supergravity of [55] and modding it out by a discrete subgroup $\mathbb{Z}_2 \subset \mathrm{G}_{\mathrm{max}}$ [37], one is left with a very specific gauging

$$\mathrm{G} = \mathrm{ISO}(3)_1 \times \mathrm{ISO}(3)_2 \ , \tag{8.8}$$

of half-maximal supergravity. In order to describe the resulting half-maximal supergravity, it will prove convenient to first perform a light-cone splitting $M = (m, \bar{m})$ with respect to the SO(6,6) invariant metric

$$\eta_{MN} = \begin{pmatrix} 0 & \delta_{m\bar{n}} \\ \delta_{\bar{m}n} & 0 \end{pmatrix} \qquad \text{with} \qquad m = 1, \dots, 6 \quad , \quad \bar{n} = \bar{1}, \dots, \bar{6} \ , \tag{8.9}$$

[2] The gauge group $\mathrm{G} = \mathrm{ISO}(3)_1 \times \mathrm{ISO}(3)_2$ is a particular example within the larger class of $\mathrm{G} = \mathrm{CSO}(p,q,r) \times \mathrm{CSO}(p',q',r')$ gaugings investigated in [124, 125]. The case of ISO(3) corresponds to $(p,q,r) = (3,0,1)$ or $(p,q,r) = (0,3,1)$ (and equivalently for the primed factor).

and then a further splitting $m = (a, i)$ and $\bar{m} = (\bar{a}, \bar{i})$ with $a = 1, 3, 5$ and $i = 2, 4, 6$. In this manner, the original $\mathrm{SO}(6,6)$ fundamental index M has a decomposition

$$\begin{array}{ccccccc} \mathrm{SO}(6,6) & \supset & \mathrm{SO}(3,3)_1 & \times & \mathrm{SO}(3,3)_2 \\ M & \rightarrow & (\,a\,,\,\bar{a}\,) & \oplus & (\,i\,,\,\bar{i}\,) \end{array} \tag{8.10}$$

and the $\mathrm{ISO}(3)_{1,2}$ factors in (8.8) are embedded into $\mathrm{SO}(3,3)_{1,2} \sim \mathrm{SL}(4)_{1,2}$, respectively. An explicit computation of the resulting embedding tensor $f_{\alpha MNP}$ specifying the half-maximal supergravity yields (using conventions in [123])

$$\begin{array}{cccc} f_{+\bar{a}bc} = 2\,g & , & f_{-abc} = \pm\,2\,g\,c & , \\ f_{-i\bar{j}\bar{k}} = 2\,g & , & f_{+\bar{i}\bar{j}\bar{k}} = \pm\,2\,g\,c & , \end{array} \tag{8.11}$$

with all the other components vanishing. In what follows we are assuming a cyclic structure in all the embedding tensor components of the same type, *i.e.* $f_{+\bar{1}35} = f_{+\bar{3}51} = f_{+\bar{5}13} = 2\,g$, etc., so we are intentionally omitting epsilon symbols in (8.11) to lighten the notation. Lastly, since the embedding tensor in (8.11) is the result of *halving* the $\mathrm{G}_{\text{max}} = [\mathrm{SO}(1,1) \times \mathrm{SO}(6)] \ltimes \mathbb{R}^{12}$ gauging of maximal supergravity, it automatically satisfies the extra constraints in (3.112) for a half-maximal supergravity to be embeddable in a maximal supergravity.

8.1.2 Deforming $\mathrm{ISO}(3) \times \mathrm{ISO}(3)$ half-maximal supergravity

Following [125] (in the conventions of [123]), we will deform the $\mathrm{G} = \mathrm{ISO}(3)_1 \times \mathrm{ISO}(3)_2$ gauging specified by (8.11) while preserving the $\mathcal{N} = 4$ supersymmetry of half-maximal supergravity. We are doing so by activating two additional embedding tensor components

$$f_{+abc} \qquad \text{and} \qquad f_{-\bar{i}\bar{j}\bar{k}}\ . \tag{8.12}$$

As discussed in [125, 123], turning on these two components modifies how the gauge group $\mathrm{G} = \mathrm{ISO}(3)_1 \times \mathrm{ISO}(3)_2$ is embedded into the duality group, see (8.7). We will change this embedding in a parametrically controlled manner yielding a one-parameter generalisation of the gauging in (8.11).

Let us denote $\tilde{\varphi}$ the new parameter entering the embedding tensor, which now has components

$$\begin{array}{cccccc} f_{+\bar{a}bc} = \frac{2\sqrt{2}\,g}{\sqrt{1+\tilde{\varphi}^2}} & , & f_{-abc} = \pm 2\sqrt{2}\,g\,c\,\frac{\tilde{\varphi}}{\sqrt{1+\tilde{\varphi}^2}} & , & f_{+abc} = -2\sqrt{2}\,g\,c\,\frac{\tilde{\varphi}^2-1}{\tilde{\varphi}^2+1} & , \\ f_{-i\bar{j}\bar{k}} = \frac{2\sqrt{2}\,g}{\sqrt{1+\tilde{\varphi}^2}} & , & f_{+\bar{i}\bar{j}\bar{k}} = \pm 2\sqrt{2}\,g\,c\,\frac{\tilde{\varphi}}{\sqrt{1+\tilde{\varphi}^2}} & , & f_{-\bar{i}\bar{j}\bar{k}} = -2\sqrt{2}\,g\,c\,\frac{\tilde{\varphi}^2-1}{\tilde{\varphi}^2+1} & , \end{array} \tag{8.13}$$

and, as we will see in a moment, accommodates a rich structure of new AdS_4 vacua. The class of gaugings in (8.13) automatically solves the quadratic constrains in (3.111) required by half-maximal supersymmetry. However, an explicit computation of the additional quadratic constraints in (3.112) yields

$$f_{\alpha MNP}\, f_{\beta}{}^{MNP} = 0 \qquad \text{and} \qquad \epsilon^{\alpha\beta}\, f_{\alpha[MNP}\, f_{\beta QRS]}\Big|_{\text{SD}} \propto g^2 c\, \frac{\tilde{\varphi}^2 - 1}{(\tilde{\varphi}^2+1)^{\frac{3}{2}}}\ . \tag{8.14}$$

As a result, due to the violation of the constraint living in the $(\mathbf{1}, \mathbf{462'})$ irrep, the deformed theories do not admit an uplift to maximal supergravity unless $\tilde{\varphi}^2 = 1$. Note that, at $\tilde{\varphi}^2 = 1$, the embedding tensor in (8.13) consistently reduces to the one in (8.11) and the theory becomes a $\mathbb{Z}_2$-invariant subsector of the $\mathrm{G}_{\text{max}} = [\mathrm{SO}(1,1) \times \mathrm{SO}(6)] \ltimes \mathbb{R}^{12}$ maximal supergravity.

8.1.3 The gauge algebra of the deformed $\mathcal{N} = 4$ models

It is instructive to take a closer look at how the precise $\mathcal{N} = 4$ gauging specified by (8.13) is realised at an algebraic level in terms of the generators

$$T_{\alpha M} = (\, T_{\alpha a} \, , \, T_{\alpha \bar{a}} \ \ ; \ \ T_{\alpha i} \, , \, T_{\alpha \bar{i}} \,) \tag{8.15}$$

entering the commutation relations (3.110). We will show that each of the $\mathrm{ISO}(3)_{1,2}$ factors is at a different $\mathrm{SL}(2)$ angle in the spirit of [129]. Moreover, each $\mathrm{ISO}(3)_{1,2}$ factor by itself involves a non-trivial $\mathrm{SO}(3,3)_{1,2}$ angle (both unbar/bar generators are present) in the spirit of [125].

The algebraic realisation of the $\mathrm{G} = \mathrm{ISO}(3)_1 \times \mathrm{ISO}(3)_2$ gaugings specified by (8.13) involves a set of 12 independent generators. For example, choosing them to be $(T_{+a}, T_{+\bar{a}})$ and $(T_{-i}, T_{-\bar{i}})$, the antisymmetry of the brackets (3.110) further sets

$$T_{-a} = c\, \tilde{\varphi}\; T_{+\bar{a}} \quad , \quad T_{+\bar{i}} = c\, \tilde{\varphi}\; T_{-i} \quad , \quad T_{-\bar{a}} = T_{+i} = 0 \, , \tag{8.16}$$

and the independent generators satisfy non-trivial commutation relations of the form

$$\begin{array}{rcl} [\, T_{+a}, T_{+b} \,] & = & 2\sqrt{2}\, g \left(\dfrac{1}{\sqrt{1+\tilde{\varphi}^2}}\, \epsilon_{ab}{}^{c}\, T_{+c} + c\, \dfrac{1-\tilde{\varphi}^2}{1+\tilde{\varphi}^2}\, \epsilon_{ab}{}^{\bar{c}}\; T_{+\bar{c}} \right) \, , \\ \left[T_{+a}, T_{+\bar{b}} \right] & = & 2\sqrt{2}\, g\, \dfrac{1}{\sqrt{1+\tilde{\varphi}^2}}\, \epsilon_{a\bar{b}}{}^{\bar{c}}\, T_{+\bar{c}} \, , \\ \left[T_{+\bar{a}}, T_{+\bar{b}} \right] & = & 0 \, , \end{array} \tag{8.17}$$

for the $\mathrm{ISO}(3)_1$ factor in the gauge group and, similarly,

$$\begin{array}{rcl} \left[T_{-\bar{i}}, T_{-\bar{j}} \right] & = & 2\sqrt{2}\, g \left(\dfrac{1}{\sqrt{1+\tilde{\varphi}^2}}\, \epsilon_{\bar{i}\bar{j}}{}^{\bar{k}}\, T_{-\bar{k}} + c\, \dfrac{1-\tilde{\varphi}^2}{1+\tilde{\varphi}^2}\, \epsilon_{\bar{i}\bar{j}}{}^{k}\, T_{-k} \right) \, , \\ [\, T_{-\bar{i}}, T_{-j} \,] & = & 2\sqrt{2}\, g\, \dfrac{1}{\sqrt{1+\tilde{\varphi}^2}}\, \epsilon_{\bar{i}j}{}^{k}\, T_{-k} \, , \\ [\, T_{-i}, T_{-j} \,] & = & 0 \, , \end{array} \tag{8.18}$$

for the $\mathrm{ISO}(3)_2$ factor. There is the limiting case $\tilde{\varphi} \to \pm\infty$ for which the embedding tensor (8.13) stays regular and the $\mathrm{ISO}(3)_{1,2}$ factors become nilpotent. More concretely, they reduce to the nilpotent algebra denoted $n(3.5)$ in Table 4 of [130]. The drastic change in the four-dimensional algebra structure at $\tilde{\varphi} \to \pm\infty$ suggests a drastic change in the interpretation of the corresponding supergravity solutions as well as of their possible uplifts to ten or eleven dimensions.

It is worth mentioning that the class of half-maximal $\mathrm{G} = \mathrm{ISO}(3)_1 \times \mathrm{ISO}(3)_2$ gaugings specified by the embedding tensor (8.13) depends on three arbitrary parameters $(g, c, \tilde{\varphi})$. However the most general class of gaugings of $\mathrm{G} = \mathrm{ISO}(3)_1 \times \mathrm{ISO}(3)_2$ in half-maximal supergravity involves eight parameters (up to gauge fixing) and is discussed in detail in Appendix B. The study of the structure of AdS_4 solutions in this more general class of models goes beyond the scope of this work and is postponed for the future.

8.2 $\mathbb{Z}_2^2$-invariant sector

As in our study of vacua in maximal supergravity, the scalar potential (3.113) is a complicated function of the $2+36$ scalars spanning (3.109). We will thus again study a simpler setup which is the direct generalisation of the $\mathbb{Z}_2^3$ sector of chapter 4 in the context of half maximal supergravity. In this sector, only $2+12$ scalars are kept and the rest are set to zero. We will also consider a $\mathrm{U}(1)_{\mathrm{R}}$-invariant sector with $2+12$ scalars in the next section. Moreover, although we will find extrema of the scalar potential in the setups with $2+12$ scalars, we will provide the full mass spectrum for all the bosonic and fermionic fields in half-maximal supergravity using [36, 131] as we did for the maximal supergravity setup. This supergravity spectrum maps to the spectrum of operators in the would-be dual CFT_3's.

8.2.1 The $\mathcal{N}=1$ seven-chiral model

The $\mathbb{Z}_2^2$-invariant sector of half-maximal supergravity was investigated in [132]. It can be recast as a minimal $\mathcal{N}=1$ supergravity coupled to seven chiral multiplets. We will denote z_I, with $I=1,\dots,7$, the seven complex scalars in the chiral multiplets. One of them, we choose it to be z_7, is the one parameterising the $M_{\alpha\beta}\in\mathrm{SL}(2)$ element in (3.114). The remaining six complex fields specify the G and B matrices in (3.115) from which the $M_{MN}\in\mathrm{SO}(6,6)$ element is constructed. More concretely,

$$G=\begin{pmatrix} G_1 & 0 & 0\\ 0 & G_2 & 0\\ 0 & 0 & G_3\end{pmatrix} \qquad \text{and} \qquad B=\begin{pmatrix} B_1 & 0 & 0\\ 0 & B_2 & 0\\ 0 & 0 & B_3\end{pmatrix} \tag{8.19}$$

are block-diagonal matrices with components

$$G_i=\frac{\mathrm{Im}z_{i+3}}{\mathrm{Im}z_i}\begin{pmatrix} 1 & \mathrm{Re}z_i\\ \mathrm{Re}z_i & |z_i|^2\end{pmatrix} \quad , \quad B_i=\begin{pmatrix} 0 & \mathrm{Re}z_{i+3}\\ -\mathrm{Re}z_{i+3} & 0\end{pmatrix} \quad , \quad i=1,2,3\ , \tag{8.20}$$

that depend on the complex scalars $z_1,\dots,z_6$. The scalar kinetic terms for this sector of the theory take the form

$$\mathcal{L}_{\text{kin}}=-\frac{1}{4}\sum_{I=1}^{7}\left(d\varphi_I^2+e^{2\varphi_I}\,d\chi_7^2\right) \qquad \text{with} \qquad z_I=-\chi_I+i\,e^{-\varphi_I}\ . \tag{8.21}$$

The scalar manifold invariant under the $\mathbb{Z}_2^2$ discrete symmetry is therefore identified with the special Kähler (SK) factorised geometry

$$\mathcal{M}_{\text{scal}}^{\mathbb{Z}_2^2}=\left[\frac{\mathrm{SL}(2)}{\mathrm{SO}(2)}\right]^7\subset\frac{\mathrm{SL}(2)}{\mathrm{SO}(2)}\times\frac{\mathrm{SO}(6,6)}{\mathrm{SO}(6)\times\mathrm{SO}(6)}\ . \tag{8.22}$$

In the $\mathbb{Z}_2^2$-invariant sector of half-maximal supergravity, the scalar potential takes a lengthy but more tractable expression in terms of the seven complex scalars z_I. We will find families of $\mathcal{N}=2$ supersymmetric AdS_4 extrema analytically in this setup.

8.2.2 Warming up: the φ-family of $\mathcal{N}=2$ AdS_4 solutions of [56]

The half-maximal gauged supergravity specified by the undeformed ($\tilde{\varphi}=\pm1$) embedding tensor (8.11) possesses a one-parameter φ-family of $\mathcal{N}=2$ supersymmetric

AdS$_4$ solutions. This φ-family of solutions lies at the loci

$$z_1 = -\bar{z}_3 = i\,c\sqrt{\frac{1+\varphi^2}{2}} \ \ , \ \ z_2 = i\,c \ \ , \ \ z_4 = -\bar{z}_6 = \frac{-\varphi + i}{\sqrt{1+\varphi^2}} \ \ , \ \ z_5 = z_7 = \frac{\mp 1 + i}{\sqrt{2}} \ \ , \tag{8.23}$$

where the $\mp$ sign in (8.23) is correlated with the $\pm$ sign in (8.11). Having $\text{Im}z_{1,2,3} > 0$ then requires $c > 0$. The vacuum energy turns out to be independent of φ and given by

$$V_0 = -3\,g^2\,c^{-1} \ . \tag{8.24}$$

Since we have fixed $\tilde{\varphi} = \pm 1$, this solutions can be uplifted to $\mathcal{N} = 8$ supergravity and are exactly the φ-family constructed in [56]. In the context of maximal supergravity, this family possesses two extra flat-deformations we discussed at the end of section 6.2. We will come back to the axion-like flat deformation χ in section 8.3 when exploring a $\text{U}(1)_\text{R}$-invariant sector of the theory. In the maximal supergravity setup, this family of solutions presents symmetry enhancements at $\varphi = 0$ (to $\mathcal{N} = 2$ & $SU(2) \times U(1)$), and at $\varphi = 1$ (to $\mathcal{N} = 4$ & $SO(4)$) whose uplifts were constructed in section 5.4.

However, in the $\mathbb{Z}_2$ truncated, $\mathcal{N} = 4$ theory, these points manifest themselves as enhancement to

- **Point** $\varphi = 0$**:** At this value the AdS$_4$ solution preserves $\mathcal{N} = 2$ supersymmetry and a $\text{U}(1)_\text{R} \times \text{U}(1)_\text{F}$ symmetry within half-maximal supergravity.

- **Point** $\varphi = \pm 1$**:** At these values the AdS$_4$ solution preserves $\mathcal{N} = 3$ supersymmetry and an $\text{SO}(3)_\text{R}$ symmetry within the half-maximal supergravity.

This means that the vectors and gravitini responsible for the full (super)-symmetry enhancement in maximal supergravity have been truncated away in this setup.

Marginal deformation and $\mathfrak{osp}(2|4)$ superconformal multiplets

Being a flat direction in the scalar potential, φ was identified with a marginal deformation specifying a direction in an $\mathcal{N} = 2$ conformal manifold of S-fold CFT$_3$'s [56]. Interestingly, there are unprotected operators in the $\mathcal{N} = 2$ S-fold CFT$_3$'s whose conformal dimensions depend on φ.

According to the AdS$_4$/CFT$_3$ correspondence, the mass spectrum of the full set of half-maximal supergravity fields at the $\mathcal{N} = 2$ AdS$_4$ solutions in (8.23) can be arranged into multiplets of the $\mathfrak{osp}(2|4)$ superconformal symmetry of the dual $\mathcal{N} = 2$ CFT$_3$'s. Following the notation[3] of [25] for a superconformal multiplet $[j]_\Delta^R$, where j and R are the Lorentz and R-symmetry Dynkin labels of the highest weight state (HWS) in the multiplet and Δ is its conformal dimension, the spectrum contains five unprotected long multiplets

$$L\bar{L}[0]^0_{\Delta_1} \quad , \quad L\bar{L}[\tfrac{1}{2}]^0_{\Delta_\pm} \quad , \quad L\bar{L}[0]^0_{\tilde{\Delta}_\pm} \ , \tag{8.25}$$

[3] For the $\mathcal{N} = 2$ supermultiplets in three dimensions, our conventions for the Lorentz and R-symmetry Dynkin labels differ from the one in [25]: $j = \frac{1}{2} j_{[25]}$ and $R = r_{[25]}$.

with conformal dimensions given by

$$\begin{array}{rclcrcl} \Delta_1 & = & \frac{1}{2} + \frac{1}{2}\sqrt{\frac{17+33\varphi^2}{1+\varphi^2}} & , & \Delta_{\pm} & = & \frac{1}{2} + \frac{2+(\varphi\pm1)\varphi}{\sqrt{2(1+\varphi^2)}} \quad , \\ \tilde{\Delta}_- & = & \frac{1}{2} + \frac{1}{2}\sqrt{1+8\varphi^2} & , & \tilde{\Delta}_+ & = & \frac{1}{2} + \frac{1}{2}\sqrt{\frac{17+\varphi^2}{1+\varphi^2}} \quad . \end{array} \tag{8.26}$$

In addition there are one short and two semi-short protected multiplets with integer conformal dimension $\Delta = 2$, namely,

$$A_1\bar{A}_1[1]_2^0 \quad , \quad L\bar{B}_1[0]_2^2 \quad , \quad B_1\bar{L}[0]_2^{-2} \, , \tag{8.27}$$

where $A_1\bar{A}_1[1]_2^0$ is the stress-energy tensor multiplet of the $\mathcal{N}=2$ CFT_3's.

The multiplets in (8.25)-(8.27) describe a $\mathbb{Z}_2$-invariant subset of the spectrometry performed in [56] within the context of the $\text{G}_{\text{max}} = [\text{SO}(1,1)\times\text{SO}(6)]\ltimes\mathbb{R}^{12}$ maximal supergravity. The two semi-short multiplets $L\bar{B}_1[0]_2^2$ and $B_1\bar{L}[0]_2^{-2}$ in (8.27) contain the two real marginal operators investigated in [56]. They are $\mathbb{Z}_2$-even enabling us to capture them also within the context of half-maximal supergravity. The scalar modulus φ in (8.28) is dual to one such marginal operators. The other marginal operator is dual to a different modulus χ that will be studied in detail in Section 8.3. Last but not least, the spectrum in (8.25)-(8.27) does *not* contain a $\mathfrak{u}(1)_{\text{F}}$ flavour current multiplet $A_2\bar{A}_2[0]_1^0$ present in [56]. This multiplet is $\mathbb{Z}_2$-odd and therefore projected away when truncating from maximal to half-maximal supergravity. This fact has some consequences we touch upon in the conclusions.

8.2.3 A $(\varphi\,;\tilde{\varphi})$-family of $\mathcal{N}=2$ AdS_4 solutions

Let us now consider the effect of turning on the deformation parameter $\tilde{\varphi}$, *i.e.* $\tilde{\varphi}\neq\pm1$ in the embedding tensor (8.13). As already anticipated, turning on this parameter produces new $\mathcal{N}=2$ supersymmetric AdS_4 solutions which can still be found analytically.

At generic values of the deformation parameter $\tilde{\varphi}$, the locus of the $\mathcal{N}=2$ AdS_4 solutions in (8.23) changes to

$$z_1 = -\bar{z}_3 = i\,c\sqrt{\frac{1+\varphi^2}{1+\tilde{\varphi}^2}} \quad , \quad z_2 = i\,c \quad , \quad z_4 = -\bar{z}_6 = \frac{-\varphi+i}{\sqrt{1+\varphi^2}} \quad , \quad z_5 = z_7 = \frac{\mp\tilde{\varphi}+i}{\sqrt{1+\tilde{\varphi}^2}} \, , \tag{8.28}$$

with the $\mp$ sign in (8.28) being correlated with the $\pm$ sign in (8.13). Notice that having $\text{Im}z_{1,2,3}>0$ still requires $c>0$, and also the reflection symmetry $\tilde{\varphi}\to-\tilde{\varphi}$. The vacuum energy turns out to be independent of the embedding tensor deformation $\tilde{\varphi}$ and, therefore, still given by

$$V_0 = -3\,g^2\,c^{-1} \, . \tag{8.29}$$

Taking the limit $\tilde{\varphi}\to\pm\infty$ becomes pathological as $\text{Im}z_{1,3,5,7}=0$ hinting at some decompactification regime. This decompactification regime resonates well with the fact that taking $\tilde{\varphi}\to\pm\infty$ changes the gauge group to a new one being nilpotent (see discussion below (8.17)-(8.18)).

8.2.4 $\mathfrak{osp}(2|4)$ superconformal multiplets

The full half-maximal supergravity spectrum at this $(\varphi\,;\tilde{\varphi})$-family of AdS_4 solutions can be arranged into multiplets of the $\mathfrak{osp}(2|4)$ superconformal symmetry of the dual

$\mathcal{N}=2$ CFT$_3$'s. The spectrum contains five unprotected long multiplets

$$L\bar{L}[0]^0_{\Delta_1} \quad , \quad L\bar{L}[\tfrac{1}{2}]^0_{\Delta_\pm} \quad , \quad L\bar{L}[0]^0_{\tilde{\Delta}_\pm} \, , \tag{8.30}$$

with conformal dimensions given by

$$\begin{aligned}
\Delta_1 &= \tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{9+25\tilde{\varphi}^2+\varphi^2(17+49\tilde{\varphi}^2)}{(1+\varphi^2)(1+\tilde{\varphi}^2)}} \, , \\
\Delta_\pm &= 1 + \tfrac{1}{2}\sqrt{\tfrac{4\varphi^4+\varphi^2(13\tilde{\varphi}^2+9)+4\tilde{\varphi}^4+9\tilde{\varphi}^2+5}{(\varphi^2+1)(\tilde{\varphi}^2+1)} - \tfrac{4(\varphi^2+\tilde{\varphi}^2+1)\pm 8\varphi^3\tilde{\varphi}\pm 4\varphi\tilde{\varphi}\left(2\tilde{\varphi}^2+2-\sqrt{(\varphi^2+1)(\tilde{\varphi}^2+1)}\right)}{\sqrt{(\varphi^2+1)(\tilde{\varphi}^2+1)}}} \, , \\
\tilde{\Delta}_\pm &= \tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{5(1+\tilde{\varphi}^2)+8(\varphi^4+\tilde{\varphi}^4)+\varphi^2(\tilde{\varphi}^2+9)\pm 4\sqrt{4\varphi^8+8\varphi^6-4\varphi^4(\tilde{\varphi}^4+3\tilde{\varphi}^2-1)-4\varphi^2\tilde{\varphi}^2(3+\tilde{\varphi}^2)+(1+\tilde{\varphi}^2+2\tilde{\varphi}^4)^2}}{(1+\varphi^2)(1+\tilde{\varphi}^2)}} \, .
\end{aligned} \tag{8.31}$$

In addition, there are one short and two semi-short protected multiplets with integer conformal dimension $\Delta=2$. These are the same as in (8.27), namely,

$$A_1\bar{A}_1[1]^0_2 \quad , \quad L\bar{B}_1[0]^2_2 \quad , \quad B_1\bar{L}[0]^{-2}_2 \, , \tag{8.32}$$

where $A_1\bar{A}_1[1]^0_2$ is the stress-energy tensor multiplet of the dual $\mathcal{N}=2$ CFT$_3$'s.

8.2.5 Special loci

The $(\varphi\,;\tilde{\varphi})$-family of $\mathcal{N}=2$ AdS$_4$ solutions in (8.28) generically preserves a U(1)$_\text{R}$ symmetry. The latter can be seen from the normalised gravitino masses which are given by

$$mL \;=\; 1 \;\; (\times 2) \quad , \quad \frac{1+\varphi^2+\tilde{\varphi}^2 \pm \varphi\,\tilde{\varphi}}{\sqrt{(1+\varphi^2)\,(1+\tilde{\varphi}^2)}} \, . \tag{8.33}$$

As a result, the marginal deformation φ turns out to be compatible with the embedding tensor deformation $\tilde{\varphi}$. Moreover, they both enter the AdS$_4$ solutions (8.28) and the normalised gravitino masses (8.33) in a very symmetric fashion.

A detailed inspection of the normalised gravitino masses in (8.33) singles out four special cases to be further investigated:

$$\begin{array}{llcll}
i) & \varphi=\tilde{\varphi}=0 & \Rightarrow & mL=1 \;\; (\times 4) & , \\
ii) & \varphi=\pm\tilde{\varphi}\neq 0 & \Rightarrow & mL=1 \;\; (\times 3) \;\; , \;\; 3-\dfrac{2}{1+\tilde{\varphi}^2} & , \\
iii) & \varphi=0 & \Rightarrow & mL=1 \;\; (\times 2) \;\; , \;\; \sqrt{1+\tilde{\varphi}^2} \;\; (\times 2) & , \\
iv) & \tilde{\varphi}=0 & \Rightarrow & mL=1 \;\; (\times 2) \;\; , \;\; \sqrt{1+\varphi^2} \;\; (\times 2) & .
\end{array} \tag{8.34}$$

Note that the case $i)$ sits at the intersection of the one-dimensional slicings $ii)$, $iii)$ and $iv)$. A diagram of the $(\varphi\,;\tilde{\varphi})$-family of AdS$_4$ solutions in (8.28) is shown in Figure 8.1. In the figure, and in the rest of the work, we have denoted by $\mathcal{N}\,\&\,\text{G}_0$ the number $\mathcal{N}$ of four-dimensional supersymmetries and the residual symmetry group G$_0$ of a given AdS$_4$ solution.

$\mathcal{N}=3$ line of supersymmetry enhancement

The two involutions $\varphi=\pm\tilde{\varphi}$ respectively yield $\Delta_\mp=\frac{3}{2}$ so that the corresponding long multiplet in (8.30) hits the unitary bound. We will set $\varphi=\tilde{\varphi}$ (red dashed line in

Figure 8.1) without loss of generality by virtue of the reflection symmetry $\tilde{\varphi} \to -\tilde{\varphi}$ of (8.28) and (8.31).

The conformal dimensions in (8.31) reduce in this case to[4]

$$\Delta_1 = 4 - \tfrac{2}{1+\tilde{\varphi}^2} \quad , \quad \Delta_- = \tfrac{3}{2} \quad , \quad \Delta_+ = \tfrac{7}{2} - \tfrac{2}{1+\tilde{\varphi}^2} \quad , \quad \tilde{\Delta}_- = 2 \quad , \quad \tilde{\Delta}_+ = 3 - \tfrac{2}{1+\tilde{\varphi}^2} \, . \tag{8.35}$$

As a result, the long multiplet $L\bar{L}[\frac{1}{2}]^0_{\Delta_-}$ hits the unitarity bound and splits into one short and two semi-short multiplets. The multiplets in (8.30) then reduce to

$$\begin{aligned}
L\bar{L}[0]^0_{\Delta_1} \quad &\to \quad L\bar{L}[0]^0_{4-\frac{2}{1+\tilde{\varphi}^2}} \\
L\bar{L}[\tfrac{1}{2}]^0_{\Delta_-} \oplus L\bar{L}[\tfrac{1}{2}]^0_{\Delta_+} \quad &\to \quad \Big[A_1\bar{A}_1[\tfrac{1}{2}]^0_{\frac{3}{2}} \oplus A_2\bar{L}[0]^{-1}_2 \oplus L\bar{A}_2[0]^1_2 \Big] \oplus L\bar{L}[\tfrac{1}{2}]^0_{\frac{7}{2}-\frac{2}{1+\tilde{\varphi}^2}} \\
L\bar{L}[0]^0_{\tilde{\Delta}_-} \oplus L\bar{L}[0]^0_{\tilde{\Delta}_+} \quad &\to \quad L\bar{L}[0]^0_2 \oplus L\bar{L}[0]^0_{3-\frac{2}{1+\tilde{\varphi}^2}}
\end{aligned} \tag{8.36}$$

The enhancement to $\mathcal{N} = 3$ supersymmetry originates from the short multiplet $A_1\bar{A}_1[\frac{1}{2}]^0_{\frac{3}{2}}$ in (8.36) which contains a massless gravitino.

Alternatively, the mass spectrum in (8.32) and (8.36) can be arranged into unitary superconformal multiplets of the $\mathfrak{osp}(3|4)$ algebra. Following closely the notation[5] of [25], we find a set of multiplets

$$L[0]^0_{3-\frac{2}{1+\tilde{\varphi}^2}} \quad , \quad B_1[0]^2_2 \quad , \quad A_1[\tfrac{1}{2}]^0_{\frac{3}{2}} \, , \tag{8.37}$$

with $A_1[\frac{1}{2}]^0_{\frac{3}{2}}$ corresponding to the stress-energy multiplet of the dual $\mathcal{N} = 3$ CFT$_3$. The unprotected long multiplet in (8.37) is simply a rearrangement of the $\tilde{\varphi}$-dependent long multiplets in (8.36).

$\mathcal{N} = 4$ points of supersymmetry enhancement

There are two isolated points in the space of conformal dimensions (8.35) at which Δ_1 and $\tilde{\Delta}_\pm$ corresponding to a $[j] = [0]$ HWS are integer valued whereas $\Delta_\pm$ corresponding to a $[j] = [\frac{1}{2}]$ HWS are half-integer valued.[6] Let us look at these two special points in more detail.

- **Point $\varphi = \tilde{\varphi} = 1$:** At this point (red/blue circle in Figure 8.1), the conformal dimensions in (8.35) simplify to

$$\Delta_1 = 3 \quad , \quad \Delta_- = \tfrac{3}{2} \quad , \quad \Delta_+ = \tfrac{5}{2} \quad , \quad \tilde{\Delta}_\pm = 2 \, . \tag{8.38}$$

[4]One has that $\tilde{\Delta}_- = 2$ and $\tilde{\Delta}_+ = 3 - \frac{2}{1+\tilde{\varphi}^2}$ for $|\tilde{\varphi}| \geq 1$ whereas $\tilde{\Delta}_+ = 2$ and $\tilde{\Delta}_- = 3 - \frac{2}{1+\tilde{\varphi}^2}$ for $|\tilde{\varphi}| \leq 1$.

[5]Our convention for the Lorentz and R-symmetry Dynkin labels differs from the one in [25]: $j = \frac{1}{2} j_{[25]}$ and $R = \frac{1}{2} R_{[25]}$.

[6]There is also the limit $\tilde{\varphi} \to \pm\infty$ for which the gauging changes drastically pointing at a decompactification regime (see discussion below (8.17)-(8.18)). Studying this limit goes beyond the scope of this work.

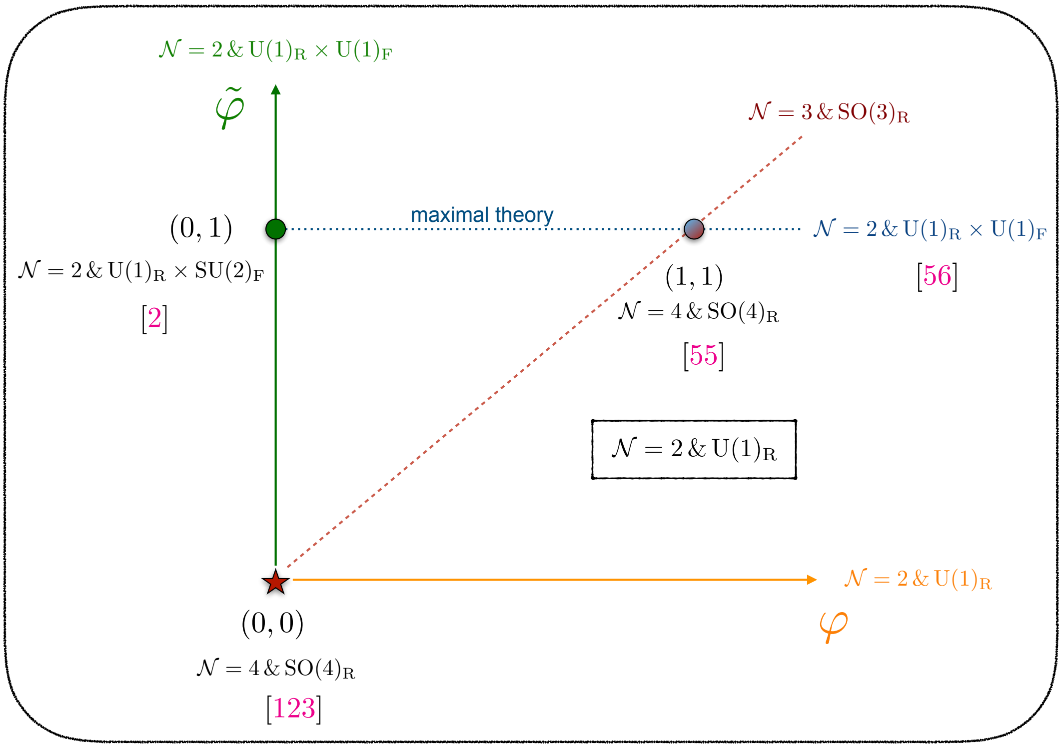

FIGURE 8.1: Web of $\mathcal{N} = 2$ CFT$_3$'s dual to the $(\varphi\,;\tilde{\varphi})$-family of AdS$_4$ solutions in (8.28). CFT$_3$'s at $\tilde{\varphi} = 1$ (dotted blue line) are dual to AdS$_4$ solutions of the $[\mathrm{SO}(1,1) \times \mathrm{SO}(6)] \ltimes \mathbb{R}^{12}$ maximal supergravity and have been studied and characterised in [56]. The green and blue/red circles have a type IIB dual: the S-fold backgrounds in [2] and [55], respectively. The $\mathcal{N} = 4$ CFT$_3$ sitting at $\varphi = \tilde{\varphi} = 0$ (red star) is dual to an exotic AdS$_4$ solution originally presented in [123] and classified as non-geometric therein. A generic CFT$_3$ in the diagram features $\mathcal{N} = 2 \,\&\, \mathrm{U}(1)_{\mathrm{R}}$ symmetry.

Therefore, there are no further long multiplets in (8.36) hitting the unitarity bound. Instead, they simply reduce to

$$
\begin{aligned}
L\bar{L}[0]^0_{\Delta_1} &\;\to\; L\bar{L}[0]^0_3 \\
L\bar{L}[\tfrac{1}{2}]^0_{\Delta_-} \oplus L\bar{L}[\tfrac{1}{2}]^0_{\Delta_+} &\;\to\; \left[A_1\bar{A}_1[\tfrac{1}{2}]^0_{\frac{3}{2}} \oplus A_2\bar{L}[0]^{-1}_2 \oplus L\bar{A}_2[0]^1_2\right] \oplus L\bar{L}[\tfrac{1}{2}]^0_{\frac{5}{2}} \\
L\bar{L}[0]^0_{\tilde{\Delta}_-} \oplus L\bar{L}[0]^0_{\tilde{\Delta}_+} &\;\to\; L\bar{L}[0]^0_2 \;\; (\times 2)
\end{aligned}
\tag{8.39}
$$

The multiplets in (8.32) and (8.39) precisely describe the $\mathbb{Z}_2$-even sector of the $\mathcal{N} = 4 \,\&\, \mathrm{SO}(4)_{\mathrm{R}}$ S-fold of [55]. In other words, the field content of half-maximal supergravity only captures an $\mathcal{N} = 3$ subsector of the $\mathcal{N} = 4$ S-fold CFT$_3$ of [55].

In $\mathfrak{osp}(3|4)$ language, the unprotected long multiplet in (8.37) does not split and the spectrum (8.37) simply reduces to

$$
L[0]^0_2 \quad , \quad B_1[0]^2_2 \quad , \quad A_1[\tfrac{1}{2}]^0_{\frac{3}{2}} \; . \tag{8.40}
$$

• **Point** $\varphi = \tilde{\varphi} = 0$**:** At this special point (red star in Figure 8.1) the conformal dimensions in (8.35) reduce to

$$\Delta_1 = 2 \quad , \quad \Delta_\pm = \frac{3}{2} \quad , \quad \tilde{\Delta}_- = 2 \quad , \quad \tilde{\Delta}_+ = 1 \, . \tag{8.41}$$

This implies that the three long multiplets $L\bar{L}[\frac{1}{2}]^0_{\Delta_\pm}$ and $L\bar{L}[0]^0_{\tilde{\Delta}_+}$ hit the unitarity bound and split, each of them producing one short and two semi-short multiplets. More concretely, the multiplets in (8.36) decompose as

$$\begin{aligned} L\bar{L}[0]^0_{\Delta_1} &\rightarrow L\bar{L}[0]^0_2 \\ L\bar{L}[\tfrac{1}{2}]^0_{\Delta_\pm} &\rightarrow \Big[A_1\bar{A}_1[\tfrac{1}{2}]^0_{\frac{3}{2}} \oplus A_2\bar{L}[0]^{-1}_2 \oplus L\bar{A}_2[0]^1_2 \Big] \quad (\times 2) \\ L\bar{L}[0]^0_{\tilde{\Delta}_-} \oplus L\bar{L}[0]^0_{\tilde{\Delta}_+} &\rightarrow L\bar{L}[0]^0_2 \oplus \Big[A_2\bar{A}_2[0]^0_1 \oplus B_1\bar{L}[0]^{-2}_2 \oplus L\bar{B}_1[0]^2_2 \Big] \end{aligned} \tag{8.42}$$

There is this time an enhancement to $\mathcal{N} = 4$ supersymmetry originating from the two short multiplets $A_1\bar{A}_1[\frac{1}{2}]^0_{\frac{3}{2}}$ in (8.42) each one containing a massless gravitino. In addition, there is the shortening associated with $\tilde{\Delta}_+ = 1$ which provides an additional massless vector multiplet.

In $\mathfrak{osp}(3|4)$ language, the unprotected long multiplet in (8.37) hits the unitarity bound and splits into two short multiplets as $L[0]^0_1 \rightarrow A_2[0]^0_1 \oplus B_1[0]^2_2$. The spectrum in (8.37) then reduces to

$$A_2[0]^0_1 \quad , \quad B_1[0]^2_2 \;\; (\times 2) \quad , \quad A_1[\tfrac{1}{2}]^0_{\frac{3}{2}} \, , \tag{8.43}$$

as a consequence of the supersymmetry enhancement to $\mathcal{N} = 4$ in the exotic CFT_3.

$\mathrm{U}(1)_\mathrm{F}$ flavour symmetry enhancement

The identification $\varphi = 0$ (vertical axis in Figure 8.1) gives rise to a $\mathrm{U}(1)_\mathrm{F}$ flavour symmetry enhancement in the corresponding CFT_3's. At this value it occurs that $\tilde{\Delta}_- = 1$, the long multiplet $L\bar{L}[0]^0_{\tilde{\Delta}_-}$ in (8.30) hits the unitarity bound and splits again into one short and two semi-short multiplets. The multiplets in (8.30) reduce to

$$\begin{aligned} L\bar{L}[0]^0_{\Delta_1} &\rightarrow L\bar{L}[0]^0_{\frac{1}{2}+\frac{1}{2}\sqrt{25-\frac{16}{1+\tilde{\varphi}^2}}} \\ L\bar{L}[\tfrac{1}{2}]^0_{\Delta_\pm} &\rightarrow L\bar{L}[\tfrac{1}{2}]^0_{1+\frac{1}{2}\sqrt{5+4\tilde{\varphi}^2-4\sqrt{\tilde{\varphi}^2+1}}} \quad (\times 2) \\ L\bar{L}[0]^0_{\tilde{\Delta}_-} \oplus L\bar{L}[0]^0_{\tilde{\Delta}_+} &\rightarrow \Big[A_2\bar{A}_2[0]^0_1 \oplus L\bar{B}_1[0]^2_2 \oplus B_1\bar{L}[0]^{-2}_2 \Big] \oplus L\bar{L}[0]^0_{\frac{1}{2}+\frac{1}{2}\sqrt{9+\frac{16\,\tilde{\varphi}^4}{1+\tilde{\varphi}^2}}} \end{aligned} \tag{8.44}$$

where $A_2\bar{A}_2[0]^0_1$ is a massless vector multiplet reflecting the $\mathrm{U}(1)_\mathrm{F}$ flavour symmetry enhancement in the dual $\mathcal{N} = 2$ CFT_3's.

Note that the two degenerated long multiplets $L\bar{L}[\frac{1}{2}]^0_{\Delta_\pm}$ in (8.44) hit the unitarity bound at the special point $\tilde{\varphi} = 0$. At this point, each of them splits as

$$L\bar{L}[\tfrac{1}{2}]^0_{\frac{3}{2}} \quad \rightarrow \quad A_1\bar{A}_1[1]^0_{\frac{3}{2}} \quad \oplus \quad A_2\bar{L}[0]^{-1}_2 \quad \oplus \quad L\bar{A}_2[0]^1_2 \tag{8.45}$$

recovering the exotic $\mathcal{N} = 4$ CFT$_3$.

8.2.6 General axion deformations of the exotic $\mathcal{N} = 4$ CFT$_3$

Setting $\varphi = \tilde{\varphi} = 0$ in (8.28) reproduces the exotic $\mathcal{N} = 4$ AdS$_4$ solution with SO$(4)_\text{R}$ symmetry originally reported in [123] (red star in Figure 8.1). Let us recall that it is located at the (rescaled by c) origin of the scalar geometry, namely,

$$z_{1,2,3} = i\,c \qquad , \qquad z_{4,5,6,7} = i \ , \tag{8.46}$$

and preserves the compact part of the gauging, namely,

$$\text{SO}(4)_\text{R} \sim \text{SO}(3)_1 \times \text{SO}(3)_2 \subset \text{ISO}(3)_1 \times \text{ISO}(3)_2 \ . \tag{8.47}$$

Three new scalar moduli can be turned on at this AdS$_4$ solution which are parameterised by three axions $\chi_{1,2,3} \in \mathbb{R}$ of the axion-like type investigated for the S-fold solutions in [95] and in chapter 6. Activating $\chi_{1,2,3}$ changes the location of the AdS$_4$ solution (8.46) to

$$z_{1,2,3} = c\,(-\chi_{1,2,3} + i) \qquad , \qquad z_{4,5,6,7} = i \ , \tag{8.48}$$

keeping the vacuum energy at the value $V_0 = -3\,g^2 c^{-1}$. Therefore, the moduli fields χ_i $(i = 1, 2, 3)$ are naturally identified with marginal deformations of the $\mathcal{N} = 4$ exotic CFT$_3$.

The explicit computation of the normalised gravitino masses at the AdS$_4$ solution (8.48) yields

$$mL \;=\; \sqrt{1+\omega_1^2} \quad , \quad \sqrt{1+\omega_2^2} \quad , \quad \sqrt{1+\omega_3^2} \quad , \quad \sqrt{1+\omega_4^2} \ , \tag{8.49}$$

with

$$\chi_i = \omega_j + \omega_k \quad (i \neq j \neq k) \qquad \text{and} \qquad \omega_1 + \omega_2 + \omega_3 + \omega_4 = 0 \ . \tag{8.50}$$

It proves very convenient to introduce a set $\omega_A = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ of deformation parameters subject to the constraint $\sum_{A=1}^4 \omega_A = 0$. The twelve normalised vector masses can then be very symmetrically written as

$$m^2L^2 = 1 + \omega_A^2 + \omega_B^2 \pm \sqrt{1 + (\omega_A + \omega_B)^2 + 4\,\omega_A^2\,\omega_B^2} \qquad \text{with} \qquad A < B \ , \tag{8.51}$$

whereas the normalised scalar masses are given by

$$\begin{aligned} m^2L^2 \;&=\; -1 + \omega_A^2 + \omega_B^2 \pm \sqrt{1 + (\omega_A + \omega_B)^2 + 4\,\omega_A^2\,\omega_B^2} \qquad \text{with} \qquad A < B \ , \\ &\quad 4 \ , \ -2 \ (\times 2) \ , \ 0 \ (\times 15) \ , \ \lambda_{1,\dots,8} \ (\times 1) \ . \end{aligned} \tag{8.52}$$

We cannot provide a closed form for the eight normalised scalar masses $\lambda_{1,\ldots,8}$. They correspond to the roots of the degree-eight characteristic polynomial

$$P(\omega) = \sum_{\alpha=0}^{8} p_\alpha\, \lambda^\alpha \ , \tag{8.53}$$

with

$$\begin{aligned}
p_0 &= -5e_2^4 - 36e_2^3e_4 - 6e_2^3 + 32e_2^2e_3^2 - 64e_2^2e_4^2 - 24e_2^2e_4 \\
&\quad +60e_2e_3^2e_4 + 30e_2e_3^2 + 64e_2e_4^2 - 35e_3^4 + 96e_3^2e_4^2 - 80e_3^2e_4 + 256e_4^3 \ , \\
p_1 &= -20e_2^4 - 64e_2^3e_4 - 40e_2^3 + 60e_2^2e_3^2 + 32e_2^2e_4^2 - 76e_2^2e_4 - 14e_2^2 + 48e_2e_3^2e_4 \\
&\quad +104e_2e_3^2 + 224e_2e_4^2 - 16e_2e_4 - 16e_3^4 - 128e_3^2e_4^2 - 60e_3^2e_4 + 30e_3^2 - 128e_4^3 + 128e_4^2 \ , \\
p_2 &= -16e_2^4 + 48e_2^3e_4 - 74e_2^3 - 40e_2^2e_3^2 + 64e_2^2e_4^2 + 16e_2^2e_4 - 64e_2^2 - 128e_2e_3^2e_4 + 18e_2e_3^2 \\
&\quad -80e_2e_4^2 + 12e_2e_4 - 10e_2 + 64e_3^4 + 120e_3^2e_4 + 42e_3^2 - 256e_4^3 + 128e_4^2 + 8e_4 \ , \\
p_3 &= 16e_2^4 + 64e_2^3e_4 - 16e_2^3 - 64e_2^2e_3^2 + 96e_2^2e_4 - 82e_2^2 - 120e_2e_3^2 - 192e_2e_4^2 + 56e_2e_4 \\
&\quad -36e_2 + 64e_3^2e_4 - 22e_3^2 - 112e_4^2 + 20e_4 - 2 \ , \\
p_4 &= 16e_2^4 + 56e_2^3 + 32e_2^2e_4 - 4e_2^2 - 64e_2e_3^2 + 12e_2e_4 - 40e_2 - 50e_3^2 - 112e_4^2 + 8e_4 - 7 \ , \\
p_5 &= 32e_2^3 + 52e_2^2 - 16e_2e_4 - 4e_2 - 16e_3^2 - 12e_4 - 8 \ , \\
p_6 &= 24e_2^2 + 18e_2 - 8e_4 - 2 \ , \\
p_7 &= 2 + 8e_2 \ , \\
p_8 &= 1 \ ,
\end{aligned} \tag{8.54}$$

written in terms of the elementary symmetric polynomials

$$e_k(\omega_1, \omega_2, \omega_3, \omega_4) = \sum_{1\le A_1<\cdots<A_k\le 4} \omega_{A_1}\cdots\omega_{A_k} \ . \tag{8.55}$$

Note that e_1 vanishes identically by virtue of the second equation in (8.50).

It follows from (8.51) that the number of massless vectors is given by the number n_p of pairs $\omega_A = \omega_B$ with $A < B$. This yields the following classification of CFT$_3$ duals in terms of the four parameters ω_A:

- All four ω's are zero: $\mathcal{N} = 4$ with $\mathrm{SO}(4)_\mathrm{R}$ symmetry
- Three ω's are zero: same case as before by virtue of $\sum \omega_A = 0$
- Exactly two ω's are zero: $\mathcal{N} = 2$ with $\mathrm{SO}(2)_\mathrm{R}$ symmetry
- Exactly one ω is zero: $\mathcal{N} = 1$ with
  - $\mathrm{SO}(2)_\mathrm{F}$ flavour symmetry if two of the remaining ω's are equal
  - no flavour symmetry otherwise
- All ω's are non-zero: $\mathcal{N} = 0$ with
  - $\mathrm{SO}(2)_\mathrm{F} \times \mathrm{SO}(2)_\mathrm{F}$ if there are two pairs of ω's of equal value, one pair with the opposite sign of the other
  - $\mathrm{SO}(3)_\mathrm{F}$ if three ω's are identified
  - $\mathrm{SO}(2)_\mathrm{F}$ if only two ω's are identified

– no flavour symmetry otherwise

In summary, the number of supersymmetries (both of the AdS_4 solution and the dual CFT_3) matches the number of parameters $\omega_A = 0$, and the AdS_4 solution features an orthogonal symmetry group of dimension n_p. Amongst the n_p pairs, each pair $\omega_A = \omega_B = 0$ adds one generator to the orthogonal R-symmetry group of the dual CFT_3 whereas each pair $\omega_A = \omega_B \neq 0$ adds one generator to an orthogonal flavour symmetry group in the CFT_3. Finally, observe that there is a case compatible with a single axion, let us denote it χ, that preserves $\mathcal{N} = 2$ and $SO(2)_R$ symmetry within half-maximal supergravity. The consequences of turning on this axion χ will be investigated in detail in Section 8.3.

Relation to axion-like deformations of S-folds?

The way the χ's enter the AdS_4 solution in (8.48) is identical to the known examples of axion-like deformations of S-folds discussed in chapter 6. However one must be cautious about giving the same interpretation to all the χ's in (8.48). The reason is twofold:

i) We find three different axion deformations $\chi_{1,2,3}$ for the exotic $\mathcal{N} = 4$ AdS_4 solution with $SO(4)_R$ symmetry and not two, as one would naively expect from the number of Cartan generators of $SO(4)_R$.

ii) Generic values of $\chi_{1,2,3}$ totally break the $SO(4)_R$ symmetry of the undeformed solution: the Cartan subgroup of $SO(4)_R$ is generically not preserved. As a result, the pattern of symmetry breaking is very different from the one induced by the axion-like deformations in the S-fold solutions.

As a result, while the $\mathcal{N} = 0$ case with $SO(2)_F \times SO(2)_F$ symmetry preserves the Cartan subgroup of $SO(4)_R$ and, therefore, stands a chance of having a geometrical interpretation alike the axion-like deformations of S-folds, the other cases appear (at first glance) to be different as the Cartan subgroup of $SO(4)_R$ is not preserved. Of special interest will be the case of the single axion χ mentioned above, which we move to discuss in the next section. This axion preserves $\mathcal{N} = 2$ supersymmetry, an $SO(2)_R \subset SO(4)_R$ symmetry within half-maximal supergravity, and combines with the modulus φ to generalise the conformal manifold of $\mathcal{N} = 2$ CFT_3's in (8.3) to arbitrary values of the parameter $\tilde{\varphi}$.

8.3 $U(1)_R$-invariant sector

In this section we construct a particular $U(1)_R$ invariant sector of half-maximal supergravity that suffices to capture the modulus χ dual to the second marginal deformation spanning with φ the $\mathcal{N} = 2$ conformal manifold in (8.3).

8.3.1 The $\mathcal{N} = 2$ three-vector and two-hyper model

To our knowledge, there is no explicit construction of the $U(1)_R$ invariant sector of half-maximal supergravity of relevance for this work. So we will present it in some detail. In order to construct it, let us first introduce the set of $SO(6, 6)$ generators

$$[t_{MN}]_P{}^Q = 2\,\eta_{P[M}\,\delta^Q_{N]}\ , \tag{8.56}$$

where $M = 1\ldots,6,\bar{1},\ldots\bar{6}$ is a fundamental $\mathrm{SO}(6,6)$ index in the light-cone basis. We choose the specific $\mathrm{U}(1)_{\mathrm{R}}$ generator to be

$$t_{\mathrm{U}(1)_{\mathrm{R}}} = (t_{5\bar{1}} - t_{1\bar{5}}) - (t_{6\bar{2}} - t_{2\bar{6}}) \ , \tag{8.57}$$

which is embedded in the duality group of half-maximal supergravity as

$$\mathrm{SL}(2)\times\mathrm{SO}(6,6) \supset \mathrm{SL}(2)\times\mathrm{SO}(2,2)\times\mathrm{SO}(4,4) \supset \mathrm{SL}(2)\times\mathrm{SO}(2,2)\times\mathrm{SU}(2,2)\times\mathrm{U}(1)_{\mathrm{R}} \, . \tag{8.58}$$

From the commutant of $\mathrm{U}(1)_{\mathrm{R}}$ in the embedding chain (8.58), the scalar manifold invariant under $\mathrm{U}(1)_{\mathrm{R}}$ is identified with

$$\mathcal{M}_{\mathrm{scal}}^{\mathrm{U}(1)_{\mathrm{R}}} = \underbrace{\frac{\mathrm{SL}(2)}{\mathrm{SO}(2)} \times \frac{\mathrm{SO}(2,2)}{\mathrm{SO}(2)\times\mathrm{SO}(2)}}_{\left[\frac{\mathrm{SL}(2)}{\mathrm{SO}(2)}\right]^3} \times \frac{\mathrm{SU}(2,2)}{\mathrm{S}\left(\mathrm{U}(2)\times\mathrm{U}(2)\right)} \, . \tag{8.59}$$

This $\mathrm{U}(1)_{\mathrm{R}}$-invariant sector of half-maximal supergravity can be described as an $\mathcal{N}=2$ gauged supergravity coupled to three vector multiplets and two hypermultiplets. Within this $\mathcal{N}=2$ sector, the gauge group is

$$\mathrm{G}_{\mathcal{N}=2} = \mathrm{U}(1)_{\mathrm{R}} \times \mathrm{U}(1)_{\gamma} \times \mathbb{R}_{a} \times \mathbb{R}_{\epsilon} \subset \mathrm{ISO}(3)\times\mathrm{ISO}(3) \ , \tag{8.60}$$

with all the fields being inert under the $\mathrm{U}(1)_{\mathrm{R}}$ factor. The three complex scalars in the vector multiplets span the $[\mathrm{SL}(2)/\mathrm{SO}(2)]^3$ factor of the scalar geometry (8.59). They are identified with z_7 and (z_2, z_5) in the $\mathbb{Z}_2^2$-invariant sector of Section 8.2. The scalar matrix $M_{\alpha\beta}$ spanned by z_7 was given in (3.114). The part of the M_{MN} matrix in (3.115) spanned by (z_2, z_5) was constructed in terms of the 2×2 blocks G_2 and B_2 in (8.20). Alternatively, it can directly be constructed from the coset representative

$$\mathcal{V}_{\mathrm{SO}(2,2)} = e^{-\chi_2\, t_{4\bar{3}} + \chi_5\, t_{43}} \ e^{-\frac{1}{2}\left[\varphi_2\,(t_{4\bar{4}} - t_{3\bar{3}}) + \varphi_5 (t_{4\bar{4}} + t_{3\bar{3}})\right]} \ , \tag{8.61}$$

such that $z_{2,5} = -\chi_{2,5} + i e^{-\varphi_{2,5}}$. There is also the part of the scalar matrix M_{MN} that depends on the scalars in the quaternionic Kähler (QK) space $[\mathrm{SU}(2,2)/\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(2))] \sim [\mathrm{SO}(2,4)/(\mathrm{SO}(2)\times\mathrm{SO}(4))]$. Following the coset construction of [133], we will first introduce the generators

$$\begin{array}{rclcrcl}
H_1 &=& t_{1\bar{1}} + t_{5\bar{5}} + t_{6\bar{6}} + t_{2\bar{2}} & , & H_2 &=& t_{1\bar{1}} + t_{5\bar{5}} - t_{6\bar{6}} - t_{2\bar{2}} \\
{E_2}^3 &=& -t_{15} & , & V^{23} &=& t_{26} \\
{U_1}^3 &=& -t_{12} + t_{56} + t_{16} - t_{25} & , & {U_1}^2 &=& t_{2\bar{1}} + t_{2\bar{5}} + t_{6\bar{1}} - t_{6\bar{5}} \\
{U_2}^3 &=& t_{12} - t_{56} + t_{16} - t_{25} & , & {U_2}^2 &=& t_{2\bar{1}} - t_{2\bar{5}} - t_{6\bar{1}} - t_{6\bar{5}}
\end{array} \tag{8.62}$$

and construct the coset representative as

$$\mathcal{V}_{\mathrm{SO}(2,4)} = e^{\frac{1}{2\sqrt{2}}\,U} \ e^{a\,V^{23}} \ e^{h\,{E_2}^3} \ e^{-\frac{1}{4}\left[(\phi_2+\phi_1)H_1 + (\phi_2-\phi_1)H_2\right]} \ , \tag{8.63}$$

with

$$U = -(\tilde{\zeta}_0 - \tilde{\zeta}_1)\,{U_1}^2 - (\zeta^0 + \zeta^1)\,{U_2}^2 - (\zeta^0 - \zeta^1)\,{U_1}^3 + (\tilde{\zeta}_0 + \tilde{\zeta}_1)\,{U_2}^3 \, . \tag{8.64}$$

The scalar matrix $M_{MN} \in \mathrm{SO}(6,6)$ is then obtained as $M = \mathcal{V}\,\mathcal{V}^t$ using the factorised coset representative

$$\mathcal{V} = \mathcal{V}_{\mathrm{SO}(2,2)}\,\mathcal{V}_{\mathrm{SO}(2,4)}\ . \tag{8.65}$$

In order to complete the characterisation of the $2+12$ scalars in the U(1)$_\mathrm{R}$-invariant sector, we will now look at the metric on the scalar manifold (8.59) which can be extracted from the kinetic terms in (3.116). An explicit computation gives

$$\begin{array}{rcl}
\mathcal{L}_{\mathrm{kin}} & = & -\frac{1}{4}\,(d\varphi_7^2 + e^{2\varphi_7}\,d\chi_7^2) - \frac{1}{4}\,(d\varphi_2^2 + e^{2\varphi_2}\,d\chi_2^2) - \frac{1}{4}\,(d\varphi_5^2 + e^{2\varphi_5}\,d\chi_5^2)\\
& - & \frac{1}{4}\left(D\phi_2^2 + e^{2\phi_2}\,Dh^2\right)\\
& - & \frac{1}{4}\left[d\phi_1^2 + e^{2\phi_1}\left(Da + \frac{1}{2}(\zeta^0 D\tilde{\zeta}_0 + \zeta^1 D\tilde{\zeta}_1 - \tilde{\zeta}_0 D\zeta^0 - \tilde{\zeta}_1 D\zeta^1)\right)^2\right]\\
& - & \frac{1}{8}\left[e^{\phi_1-\phi_2}\,(D\zeta^0 + D\zeta^1)^2 + e^{\phi_1-\phi_2}\,(D\tilde{\zeta}_0 - D\tilde{\zeta}_1)^2\right.\\
& & \quad + e^{\phi_1+\phi_2}\left(D\zeta^0 - D\zeta^1 + h\,(D\tilde{\zeta}_0 - D\tilde{\zeta}_1)\right)^2\\
& & \quad \left. + e^{\phi_1+\phi_2}\left(D\tilde{\zeta}_0 + D\tilde{\zeta}_1 - h\,(D\zeta^0 + D\zeta^1)\right)^2\right]\ ,
\end{array} \tag{8.66}$$

where we have introduced the kinetic term notation $dX\,dY \equiv \partial_\mu X\,\partial^\mu Y$ and $DX\,DY \equiv D_\mu X\,D^\mu Y$ for two generic scalars X and Y. The covariant derivatives in (8.66) include a gauge connection for the gauge group in (8.60). They can be straightforwardly computed from (3.117) and take the form

$$D\phi_2 = d\phi_2 - 4g\,A^{(\epsilon)}\,h \quad , \quad Dh = dh - 2g\,A^{(\epsilon)}\,\left(e^{-2\phi_2} - h^2\right) \quad , \quad Da = da - 2g\,A^{(a)}\ , \tag{8.67}$$

together with

$$\begin{array}{ll}
D\zeta^0 = d\zeta^0 - g\,A^{(\epsilon)}\,(\tilde{\zeta}_0 + \tilde{\zeta}_1) - 2g\,A^{(\gamma)}\,\tilde{\zeta}_0 \quad , & D\zeta^1 = d\zeta^1 - g\,A^{(\epsilon)}\,(\tilde{\zeta}_0 + \tilde{\zeta}_1) + 2g\,A^{(\gamma)}\,\tilde{\zeta}_1\ ,\\
D\tilde{\zeta}_0 = d\tilde{\zeta}_0 - g\,A^{(\epsilon)}\,(\zeta^1 - \zeta^0) + 2g\,A^{(\gamma)}\,\zeta^0 \quad , & D\tilde{\zeta}_1 = d\tilde{\zeta}_1 - g\,A^{(\epsilon)}\,(\zeta^0 - \zeta^1) - 2g\,A^{(\gamma)}\,\zeta^1\ ,
\end{array} \tag{8.68}$$

in terms of three linear combinations of vectors ${A_\mu}^{\alpha M}$ given by

$$\begin{array}{rcl}
A_\mu^{(\epsilon)} & \equiv & \sqrt{2}\,c\,\frac{1-\tilde{\varphi}^2}{1+\tilde{\varphi}^2}\,{A_\mu}^{+3} + \frac{\sqrt{2}}{\sqrt{1+\tilde{\varphi}^2}}\left({A_\mu}^{+\bar{3}} + c\,\tilde{\varphi}\,{A_\mu}^{-3}\right)\ ,\\
A_\mu^{(a)} & \equiv & \sqrt{2}\,c\,\frac{1-\tilde{\varphi}^2}{1+\tilde{\varphi}^2}\,{A_\mu}^{-\bar{4}} + \frac{\sqrt{2}}{\sqrt{1+\tilde{\varphi}^2}}\left({A_\mu}^{-4} + c\,\tilde{\varphi}\,{A_\mu}^{+\bar{4}}\right)\ ,\\
A_\mu^{(\gamma)} & \equiv & \frac{\sqrt{2}}{\sqrt{1+\tilde{\varphi}^2}}\left({A_\mu}^{+3} + {A_\mu}^{-\bar{4}}\right)\ .
\end{array} \tag{8.69}$$

The vectors $A_\mu^{(\epsilon)}$, $A_\mu^{(a)}$ and $A_\mu^{(\gamma)}$ in (8.69) respectively gauge the factors $\mathbb{R}_\epsilon$ and $\mathbb{R}_a$ and $\mathrm{U}(1)_\gamma$ in (8.60). There is an additional vector field $A_\mu^{(\mathrm{R})}$ associated with the U(1)$_\mathrm{R}$ generator in (8.57) under which all the scalars in this sector of the theory are invariant.

8.3.2 Warming up: the (φ, χ)-family of $\mathcal{N}=2$ AdS$_4$ solutions of [56]

Let us start by recovering the two-parameter (φ, χ)-family of $\mathcal{N}=2$ AdS$_4$ solutions of the $[\mathrm{SO}(1,1)\times\mathrm{SO}(6)]\ltimes\mathbb{R}^{12}$ maximal supergravity put forward in [56]. As explained in Section 8.1.2, we must first of all set $\tilde{\varphi}^2 = 1$ in order to make contact with the maximal theory. Then, the two-parameter (φ, χ)-family of $\mathcal{N} = 2$ AdS_4 solutions of [56] is recovered within the $\text{U}(1)_{\text{R}}$-invariant sector of half-maximal supergravity as follows. The SK scalars are fixed to

$$z_5 = z_7 = \frac{\mp 1 + i}{\sqrt{2}} \qquad \text{and} \qquad z_2 = i\,c\,, \tag{8.70}$$

where, as before, the $\mp$ sign in (8.70) is correlated with the $\pm$ sign in (8.13). The QK scalars are fixed to

$$\begin{aligned} h + i\,e^{-\phi_2} = i\,\frac{\sqrt{2}}{c\,(1+\varphi^2)} \quad &, \quad e^{-\phi_1} = \frac{c}{\sqrt{2}}\,, \\ a = 0 \quad , \quad \zeta^0 = \zeta^1 = \frac{c\,\chi}{\sqrt{2}} \quad &, \quad \tilde{\zeta}_0 = \tilde{\zeta}_1 = \frac{\varphi}{\sqrt{1+\varphi^2}}\,. \end{aligned} \tag{8.71}$$

This $\mathcal{N} = 2$ family of AdS_4 solutions comes along with a vacuum energy

$$V_0 = -3g^2 c^{-1}\,, \tag{8.72}$$

for any value of the moduli fields (φ, χ) thus identifying them with marginal deformations in the dual CFT_3's. It is worth emphasising that (8.70)-(8.71) provides an explicit realisation of the (φ, χ)-family of $\mathcal{N} = 2$ AdS_4 solutions of [56] in a specific supergravity model.[7]

Marginal deformation and $\mathfrak{osp}(2|4)$ superconformal multiplets

The explicit computation of the normalised mass spectrum recovers the half-maximal ($\mathbb{Z}_2$-invariant) subset of multiplets within the maximal content of [56]. This consists of five unprotected long multiplets

$$L\bar{L}[0]^0_{\Delta_1} \quad , \quad L\bar{L}[\tfrac{1}{2}]^0_{\Delta_\pm} \quad , \quad L\bar{L}[0]^0_{\tilde{\Delta}_\pm}\,, \tag{8.73}$$

with conformal dimensions[8]

$$\begin{aligned} \Delta_1 &= \tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{17+33\varphi^2}{1+\varphi^2}}\,, \\ \Delta_\pm &= \tfrac{1}{2} + \frac{\sqrt{(2+\varphi^2)^2+\chi^2} \pm \varphi}{\sqrt{2(1+\varphi^2)}}\,, \\ \tilde{\Delta}_\pm &= \tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{9+4\varphi^4+5\varphi^2+4\chi^2 \pm 4(\varphi^4+\varphi^2+\chi^2-2)}{\varphi^2+1}}\,. \end{aligned} \tag{8.74}$$

There are also the short and semi-short protected multiplets given in (8.27), namely,

$$A_1\bar{A}_1[1]^0_2 \quad , \quad L\bar{B}_1[0]^2_2 \quad , \quad B_1\bar{L}[0]^{-2}_2\,, \tag{8.75}$$

where $A_1\bar{A}_1[1]^0_2$ is identified with the stress-energy tensor multiplet of the $\mathcal{N} = 2$ CFT_3's.

[7] This two-parameter (φ, χ)-family of solutions was constructed in [56] by applying a χ-dependent $\text{E}_{7(7)}$ duality transformation to the one-parameter φ-family in (8.23).

[8] The conformal dimensions $(\Delta_1, \Delta_+, \Delta_-, \tilde{\Delta}_+, \tilde{\Delta}_-)$ map into $(\beta_3, \beta_5, \beta_4, \beta_1, \beta_2)$ in eq.(4.5) of [56] upon the identification $\chi = \sqrt{1+\tilde{\varphi}^2}\,\chi_{[56]}$ with $\tilde{\varphi} = 1$.

8.3.3 A $(\varphi, \chi\,;\tilde{\varphi})$-family of $\mathcal{N}=2$ AdS$_4$ solutions

The two-parameter family of $\mathcal{N}=2$ AdS$_4$ solutions in (8.70)-(8.71) can be generalised to arbitrary values of the deformation parameter $\tilde{\varphi}$. The SK scalars are given by

$$z_5 = z_7 = \frac{\mp\tilde{\varphi}+i}{\sqrt{1+\tilde{\varphi}^2}} \qquad \text{and} \qquad z_2 = i\,c\,, \tag{8.76}$$

whereas the QK scalars take the form

$$\begin{array}{c} h+i\,e^{-\phi_2} = i\,\dfrac{\sqrt{1+\tilde{\varphi}^2}}{c\,(1+\varphi^2)} \quad , \quad e^{-\phi_1} = \dfrac{c}{\sqrt{1+\tilde{\varphi}^2}}\,, \\ a=0 \quad , \quad \zeta^0=\zeta^1 = \dfrac{c\,\chi}{\sqrt{1+\tilde{\varphi}^2}} \quad , \quad \tilde{\zeta}_0=\tilde{\zeta}_1 = \dfrac{\varphi}{\sqrt{1+\varphi^2}}\,. \end{array} \tag{8.77}$$

The vacuum energy at this $\mathcal{N}=2$ family of AdS$_4$ solutions is still given by

$$V_0 = -3g^2c^{-1}\,, \tag{8.78}$$

for any value of the deformation parameter $\tilde{\varphi}$ as well as of the moduli fields (φ,χ) dual to marginal deformations. It is worth highlighting that, at any value of $\tilde{\varphi}$, turning on (φ,χ) activates the hypermultiplet scalars $(\zeta^0,\zeta^1;\tilde{\zeta}_0,\tilde{\zeta}_1)$ spanning the Heisenberg fiber of the QK geometry. Activating these scalars automatically breaks the compact $\mathrm{U}(1)_\gamma$ factor of the gauge group (8.60), as it can be seen from (8.68).

8.3.4 $\mathfrak{osp}(2|4)$ superconformal multiplets

The half-maximal supergravity spectrum *at generic* $\tilde{\varphi}$ of the $\mathcal{N}=2$ (φ,χ)-family of AdS$_4$ solutions can be arranged into multiplets of the $\mathfrak{osp}(2|4)$ superconformal symmetry of the would-be dual $\mathcal{N}=2$ CFT$_3$'s. The spectrum contains five unprotected long multiplets

$$L\bar{L}[0]^0_{\Delta_1} \quad , \quad L\bar{L}[\tfrac{1}{2}]^0_{\Delta_\pm} \quad , \quad L\bar{L}[0]^0_{\tilde{\Delta}_\pm}\,, \tag{8.79}$$

with conformal dimensions given by

$$\begin{aligned} \Delta_1 &= \tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{9+25\tilde{\varphi}^2+\varphi^2(17+49\tilde{\varphi}^2)}{(1+\varphi^2)(1+\tilde{\varphi}^2)}}\,, \\ \Delta_\pm &= \tfrac{1}{2}+\frac{\sqrt{(1+\varphi^2+\tilde{\varphi}^2)^2+\chi^2}\pm\varphi\tilde{\varphi}}{\sqrt{(1+\varphi^2)(1+\tilde{\varphi}^2)}}\,, \\ \tilde{\Delta}_\pm &= \tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{5+8\varphi^4+\varphi^2(\tilde{\varphi}^2+9)+8\tilde{\varphi}^4+5\tilde{\varphi}^2+8\chi^2\pm4\sqrt{\Theta}}{(\varphi^2+1)(\tilde{\varphi}^2+1)}}\,, \end{aligned} \tag{8.80}$$

where

$$\begin{aligned} \Theta \;=\;& 4\varphi^8+8\varphi^6-4\varphi^4\left(\tilde{\varphi}^4+3\tilde{\varphi}^2-1-2\chi^2\right)-4\varphi^2\left(\tilde{\varphi}^4+3\tilde{\varphi}^2-2\chi^2\right) \\ & \left(\tilde{\varphi}^2\left(1+2\tilde{\varphi}^2\right)+1-2\chi^2\right)^2+4\left(\tilde{\varphi}^2-1\right)^2\chi^2. \end{aligned} \tag{8.81}$$

In addition, there are one short and two semi-short protected multiplets with integer conformal dimension $\Delta=2$. These are the same as in (8.27), namely,

$$A_1\bar{A}_1[1]^0_2 \quad , \quad L\bar{B}_1[0]^2_2 \quad , \quad B_1\bar{L}[0]^{-2}_2\,, \tag{8.82}$$

where $A_1\bar{A}_1[1]_2^0$ is the stress-energy tensor multiplet of the dual $\mathcal{N}=2$ CFT$_3$'s. The two moduli fields φ and χ in (8.77) belong to the semi-short multiplets $L\bar{B}_1[0]_2^2$ and $B_1\bar{L}[0]_2^{-2}$.

8.3.5 Special loci

The computation of the four gravitino masses in the half-maximal theory yields

$$m\,L = 1 \quad (\times 2) \quad , \quad \frac{\sqrt{(1+\varphi^2+\tilde{\varphi}^2)^2+\chi^2} \pm \varphi\,\tilde{\varphi}}{\sqrt{(1+\varphi^2)\,(1+\tilde{\varphi}^2)}} \,, \tag{8.83}$$

which consistently reduces to (8.33) when setting $\chi=0$. Notice that, even when turning on $\chi \neq 0$, the marginal deformation φ and the embedding tensor parameter $\tilde{\varphi}$ continue entering the gravitino masses (8.83) in a symmetric fashion. A quick inspection of (8.83) shows that supersymmetry enhancement to $\mathcal{N}>2$ is no longer possible whenever $\chi \neq 0$. Still we will look at two special cases. The first case is $\tilde{\varphi}=0$ which accounts for the $\mathcal{N}=2$ marginal deformations of the $\mathcal{N}=4$ exotic CFT$_3$. The second one is $\varphi=0$ which describes the effect of the modulus χ in a genuine half-maximal supergravity at generic $\tilde{\varphi}$.

(φ,χ)-deformations of the $\mathcal{N}=4$ exotic CFT$_3$

Setting $\tilde{\varphi}=0$ in the general expressions of the previous section one is left with the $\mathcal{N}=2$ (φ,χ) marginal deformations of the exotic $\mathcal{N}=4$ CFT$_3$. More concretely, we find in this case five unprotected long multiplets

$$L\bar{L}[0]^0_{\Delta_1} \quad , \quad L\bar{L}[\tfrac{1}{2}]^0_{\Delta} \quad (\times 2) \quad , \quad L\bar{L}[0]^0_{\tilde{\Delta}_\pm} \,, \tag{8.84}$$

with conformal dimensions

$$\begin{aligned} \Delta_1 &= \tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{9+17\varphi^2}{1+\varphi^2}} \,, \\ \Delta &= \tfrac{1}{2}+\tfrac{\sqrt{(1+\varphi^2)^2+\chi^2}}{\sqrt{1+\varphi^2}} \,, \\ \tilde{\Delta}_\pm &= \tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{5+8\chi^2+8\varphi^4+9\varphi^2\pm 4\sqrt{1+4(\varphi^4+\varphi^2+\chi^2)^2}}{1+\varphi^2}} \,. \end{aligned} \tag{8.85}$$

Notice that, unlike for the $\mathcal{N}=4\,\&\,\mathrm{SO}(4)_{\mathrm{R}}$ S-fold in (8.73), the multiplets $L\bar{L}[\tfrac{1}{2}]^0_{\Delta_\pm}$ get degenerated in this case. In addition, there are also the short and semi-short protected multiplets given in (8.82). The long multiplets $L\bar{L}[\tfrac{1}{2}]^0_{\Delta}$ and $L\bar{L}[0]^0_{\tilde{\Delta}_-}$ hit the unitarity bound at the special value $\varphi=\chi=0$ and split as

$$\begin{aligned} L\bar{L}[\tfrac{1}{2}]^0_{\frac{3}{2}} &\rightarrow A_1\bar{A}_1[\tfrac{1}{2}]^0_{\frac{3}{2}} \oplus A_2\bar{L}[0]_2^{-1} \oplus L\bar{A}_2[0]_2^1 \\ L\bar{L}[0]^0_1 &\rightarrow A_2\bar{A}_2[0]^0_1 \oplus B_1\bar{L}[0]_2^{-2} \oplus L\bar{B}_1[0]_2^2 \end{aligned} \tag{8.86}$$

recovering the undeformed exotic $\mathcal{N}=4$ CFT$_3$.

$(\chi\,;\tilde{\varphi})$-family of $\mathcal{N}=2$ AdS$_4$ solutions

In Section 8.2.3 we characterised the $(\varphi\,;\tilde{\varphi})$-family of $\mathcal{N}=2$ CFT$_3$'s at $\chi=0$. Let us now take the complementary case $\varphi=0$ and characterise the $(\chi\,;\tilde{\varphi})$-family of $\mathcal{N}=2$ CFT$_3$'s. Since the axion-like deformations are by now well understood geometrically for the S-fold backgrounds at $\tilde{\varphi} = 1$ [3], it would be interesting to investigate whether a higher-dimensional geometric interpretation at arbitrary values of $\tilde{\varphi}$ is still be possible.

Setting $\varphi = 0$, the half-maximal spectrum of dual operators contains the five unprotected long multiplets

$$L\bar{L}[0]^0_{\Delta_1} \quad , \quad L\bar{L}[\tfrac{1}{2}]^0_{\Delta} \ (\times 2) \quad , \quad L\bar{L}[0]^0_{\tilde{\Delta}_\pm} \ , \tag{8.87}$$

this time with conformal dimensions given by

$$\begin{aligned} \Delta_1 &= \tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{9+25\tilde{\varphi}^2}{1+\tilde{\varphi}^2}} \ , \\ \Delta &= \tfrac{1}{2} + \tfrac{\sqrt{(1+\tilde{\varphi}^2)^2+\chi^2}}{\sqrt{1+\tilde{\varphi}^2}} \ , \\ \tilde{\Delta}_\pm &= \tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{8\tilde{\varphi}^4+5\tilde{\varphi}^2+8\chi^2+5\pm 4\sqrt{\Theta_{\varphi\to 0}}}{1+\tilde{\varphi}^2}} \ , \end{aligned} \tag{8.88}$$

with

$$\Theta_{\varphi\to 0} = \left(\tilde{\varphi}^2\left(1+2\tilde{\varphi}^2\right) + 1 - 2\chi^2\right)^2 + 4\left(\tilde{\varphi}^2 - 1\right)^2\chi^2 . \tag{8.89}$$

There are also the short and semi-short protected multiplets given in (8.82). As a check of consistency, the χ-family of $\mathcal{N} = 2$ S-folds in [2] is recovered at $\tilde{\varphi} = 1$. We also recover the results of Section 8.2.6 upon setting $\tilde{\varphi} = 0$ together with $\chi_1 = -\chi_3 = \chi$ and $\chi_2 = 0$ (up to a $\mathrm{U}(1)_\gamma$ transformation (see eq.(8.91)) of angle $\gamma = -\frac{\pi}{4}$).

8.3.6 On the conformal manifold of $\mathcal{N} = 2$ CFT$_3$'s

Given the supergravity model in Section 8.3.1, which includes vector fields and gaugings of scalar isometries, the holographic Zamolodchikov metric on a conformal manifold of CFT$_3$'s cannot be obtained simply by direct substitution of the AdS$_4$ solution (8.76)-(8.77) into the scalar kinetic terms (8.66). The reason being that a solution like (8.76)-(8.77) can be brought to a different, but physically equivalent, form upon a gauge transformation.

The infinitesimal $\mathrm{U}(1)_\gamma \times \mathbb{R}_a \times \mathbb{R}_\epsilon$ gauge transformations entering the covariant derivatives in (8.68) – recall that all the fields within this sector are invariant under $\mathrm{U}(1)_\mathrm{R}$ – can be integrated to finite transformations. To describe such finite transformations we will introduce three complex fields

$$z = h + i\,e^{-\phi_2} \quad , \quad \psi_0 = \zeta_0 + i\,\tilde{\zeta}_0 \quad , \quad \psi_1 = \tilde{\zeta}_1 + i\,\zeta_1 \ , \tag{8.90}$$

in terms of which the compact $\mathrm{U}(1)_\gamma$ transformation acts as

$$\psi_0 \to e^{i\gamma}\,\psi_0 \qquad \text{and} \qquad \psi_1 \to e^{i\gamma}\,\psi_1 \ , \tag{8.91}$$

the non-compact $\mathbb{R}_a$ acts as a shift

$$a \to a + c_a \ , \tag{8.92}$$

and $\mathbb{R}_\epsilon$ acts as a fractional linear transformation

$$z \to \frac{z}{\epsilon\, z + 1} \qquad \text{and} \qquad \begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix} \to \begin{pmatrix} 1 - i\,\frac{\epsilon}{2} & \frac{\epsilon}{2} \\ \frac{\epsilon}{2} & 1 + i\,\frac{\epsilon}{2}\end{pmatrix}\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix} . \tag{8.93}$$

As a result, the gauge-fixed solution in (8.76)-(8.77) can be gauge-released by acting on it with (8.91)-(8.93). This action introduces three additional (yet unphysical) parameters (γ, c_a, ϵ) in the solution (8.76)-(8.77), and the naive pull-back of the metric (8.66) on different gauge-fixed solutions will depend on the choice of gauge. For example, choosing $c_a = f(\varphi, \chi)$ in (8.92) and performing the pull-back of the metric (8.66), one encounters different Zamolodchikov metrics for different choices of the function $f(\varphi, \chi)$. Of course, the catch is that we are considering solutions which are gauge-equivalent, and therefore physically equivalent, as different. In order to perform the gauge-fixing properly, one should not pick up a subspace within the gauge-released space of solutions and perform the naive pull-back of the metric onto it. Instead, one must study the quotient of the gauge-released space of solutions by the gauge group. This quotient space is the one being dual to a conformal manifold with a uniquely defined Zamolodchikov metric.

For the sake of concreteness, let us particularise to our specific gauge-released or ambient scalar geometry as described by the kinetic terms in (8.66). We first need to identify three independent one-forms on the scalar geometry which are to be declared as "pure gauge" or unphysical, and then quotient the scalar geometry by them. How to identify such three one-forms in field space is a physical question. And the answer to that question comes from the vector sector which, despite being set to zero at the supergravity solution, still provides equations of motion that must hold. In short, we must quotient the geometry by the one-form currents $J^{(\gamma)}$, $J^{(a)}$ and $J^{(\epsilon)}$ acting as sources for the vector fields that have been set to zero at the supergravity solution. This implies that we should first put those one-forms to zero in the kinetic terms (8.66) before reading off the Zamolodchikov metric by performing the pull-back of the ambient metric on any gauge-fixed subspace of solutions.

Let us illustrate the procedure described above by looking at the gauge-fixing of the $\mathbb{R}_a$ shift symmetry in (8.92) spanned by the vector field $A^{(a)}$. The scalar a plays the role of a Stückelberg field for the massive vector $A^{(a)}$ – recall that $Da = da + 2\,g\,A^{(a)}$ in (8.67) – and the associated current computed from (8.66) reads

$$J^{(a)} \equiv g\,e^{2\phi_1} * \left[Da + \tfrac{1}{2}(\zeta^0 D\tilde{\zeta}_0 + \zeta^1 D\tilde{\zeta}_1 - \tilde{\zeta}_0 D\zeta^0 - \tilde{\zeta}_1 D\zeta^1)\right] . \tag{8.94}$$

Quotienting the scalar geometry by this (field space) one-form implies that (8.94) must be set to zero identically, *i.e.* $J^{(a)} = 0$, when evaluating any quantity at a gauge-fixed solution like (8.76)-(8.77). In particular, its contribution to the third line in the kinetic terms (8.66) must be removed before reading off the Zamolodchikov metric from it. Proceeding similarly with the contributions coming from the remaining one-form currents $J^{(\epsilon)}$ and $J^{(\gamma)}$ acting as sources for $A^{(\epsilon)}$ and $A^{(\gamma)}$, the resulting Zamolodchikov metric becomes independent of the deformation parameter $\tilde{\varphi}$ and reads

$$ds^2_{\text{CM}} = \frac{1+2\varphi^2}{2(1+\varphi^2)^2}\left(d\varphi^2 + (1+\varphi^2)\,d\chi^2\right) . \tag{8.95}$$

This metric in the conformal manifold of $\mathcal{N}=2$ CFT$_3$'s at generic $\tilde{\varphi}$ agrees with that of [56] upon the identification $\chi = \sqrt{1+\tilde{\varphi}^2}\,\chi_{[56]}$ with $\tilde{\varphi} = 1$.

The moduli (φ, χ) belong to the hypermultiplet sector in the AdS$_4$ solution (8.76)-(8.77). Recalling that, for AdS$_4$ solutions preserving $\mathcal{N}=2$, the hypermultiplet moduli space must be a Kähler submanifold of the quaternionic Kähler geometry [134], the conformal manifold in (8.95) must be Kähler. The Kählericity of the metric (8.95) can be checked as follows. Let us first introduce a set of so-called isothermal coordinates for which the metric is conformal to the Euclidean metric. These are given by $x = \chi$ and $y = \text{arcsinh}\,\varphi$ so that the Zamolodchikov metric in (8.95) is brought to the form

$$ds^2_{\text{CM}} = \tfrac{1}{2}\,\Omega(y)^2\left(dx^2 + dy^2\right) \qquad \text{with} \qquad \Omega(y)^2 \equiv 1 + \tanh^2 y\ . \tag{8.96}$$

Introducing the complex coordinate $z = x + iy$ one arrives at

$$ds^2_{\text{CM}} = g_{z\bar{z}}\, dz\, d\bar{z} \qquad \text{with} \qquad g_{z\bar{z}} = \tfrac{1}{2}\left(1 + \tanh^2\left[\tfrac{-i(z-\bar{z})}{2}\right]\right)\ , \tag{8.97}$$

where $g_{z\bar{z}} = \frac{\partial^2 K}{\partial z \partial \bar{z}}$ can be derived from the real Kähler potential

$$K(z,\bar{z}) = |z|^2 - \log\left[\cosh^2\left(\tfrac{-i(z-\bar{z})}{2}\right)\right]\ . \tag{8.98}$$

More on gauge-fixing and Zamolodchikov metric

The scalar manifold $\mathcal{M}_{\text{scal}}$ in (3.109) of half-maximal supergravity is endowed with a canonical Riemannian metric g, prior to any gauge fixing, and a left action of a gauge group G, *e.g.*, $\mathrm{G} = \mathrm{ISO}(3)_1 \times \mathrm{ISO}(3)_2$ in our case. In general, the action of G on $\mathcal{M}_{\text{scal}}$ is not free – there are fixed loci under the action of the compact part of G – and thus the quotient space $\mathrm{G}\backslash\mathcal{M}_{\text{scal}}$ is not a manifold. In supergravity, the action of G on $\mathcal{M}_{\text{scal}}$ is well-behaved and we can chop the fixed loci out of $\mathcal{M}_{\text{scal}}$ to define a new manifold $\tilde{M}_{\text{scal}}$ on which G has a free action.

In order to establish a connection with the $\mathcal{N} = 2$ supergravity model of Section 8.3.1, we will focus on the $\mathrm{U}(1)_{\mathrm{R}}$-invariant sector of half-maximal supergravity for which $\mathrm{G} \rightarrow \mathrm{G}_{\mathcal{N}=2} = \mathrm{U}(1)_{\mathrm{R}} \times \mathrm{U}(1)_{\gamma} \times \mathbb{R}_a \times \mathbb{R}_{\epsilon}$ and $\mathcal{M}_{\text{scal}}$ in (3.109) reduces to the one in (8.59). From (8.68), the $\mathrm{U}(1)_{\mathrm{R}} \times \mathrm{U}(1)_{\gamma}$ compact part of $\mathrm{G}_{\mathcal{N}=2}$ leaves invariant the scalar locus defined by the condition $\tilde{\zeta}_0 = \tilde{\zeta}_1 = \zeta^0 = \zeta^1 = 0$.[9] Starting from the gauge-released solution extending the gauged-fixed one in (8.76)-(8.77) with three parameters (γ, c_a, ϵ) (see discussion below (8.93)) and removing the fixed locus under $\mathrm{U}(1)_{\gamma}$, we can finally define a manifold of supergravity solutions $\mathcal{S} \subset \tilde{\mathcal{M}}_{\text{scal}}$ on which $\mathrm{G}_{\mathcal{N}=2}$ acts freely. This gives a structure of principal bundle

$$\pi : \mathcal{S} \rightarrow \mathrm{G}_{\mathcal{N}=2}\backslash\mathcal{S}\ . \tag{8.99}$$

The quotient space $\mathrm{G}_{\mathcal{N}=2}\backslash\mathcal{S}$ is the object dual to the $\mathcal{N} = 2$ conformal manifold of CFT_3's, and it is on this quotient space that we must define metric g_{CM} dual to the Zamolodchikov metric in (8.95).

We can always decompose the tangent space of $\mathcal{S}$ as $T\mathcal{S} = V\mathcal{S} \oplus H\mathcal{S}$ where $V\mathcal{S} = \ker \pi_*$ and $H\mathcal{S} = [V\mathcal{S}]^{\perp}$ is the orthogonal complement of $V\mathcal{S}$ with respect to the metric g. The tangent space $T\mathcal{S}$ should then be understood as the space of *all* small deformations: $H\mathcal{S}$ corresponding to *physical* deformations and $V\mathcal{S}$ being the space of *unphysical* deformations. The latter are exactly the infinitesimal gauge transformations, and a projector $\mathrm{Pr}_{H\mathcal{S}}$ onto $H\mathcal{S}$ can be defined that projects them away. Finally, for any $x \in \mathrm{G}_{\mathcal{N}=2}\backslash\mathcal{S}$ there is a $p \in \mathcal{S}$ such that $\pi(p) = x$. Then, for any pair of vectors (v, v') in $T(\mathrm{G}_{\mathcal{N}=2}\backslash\mathcal{S})$ we can choose vectors (w, w') in $T\mathcal{S}$ such that $\pi_* w = v$ and $\pi_* w' = v'$. In this manner the Zamolodchikov metric can be defined as the map

$$g_{\text{CM}} : T(\mathrm{G}_{\mathcal{N}=2}\backslash\mathcal{S})^{\otimes 2} \rightarrow \mathbb{R} : v \otimes v' \rightarrow g_{\text{CM}}(v, v') = g\left(\mathrm{Pr}_{H\mathcal{S}} w,\, \mathrm{Pr}_{H\mathcal{S}} w'\right)\ . \tag{8.100}$$

[9] In the gauge-fixed supergravity solution of (8.76)-(8.77), this locus corresponds to setting to zero the marginal deformations, *i.e.*, $\varphi = \chi = 0$.

This is a well defined metric on $G_{\mathcal{N}=2}\backslash\mathcal{S}$ and it is equivalent to the prescription we have given in our example. Importantly, it does not depend on the choice of gauge fixing nor the invariant subsector used to find the supergravity solution within the full theory.

8.4 Remarks

In the present paper we have initiated a holographic study of new CFT_3's with $\mathcal{N}=2,3,4$ supersymmetry using an effective four-dimensional gauged supergravity approach. The rich structure of multi-parametric families of supersymmetric AdS_4 solutions we have just started to identify in the half-maximal gauged supergravities with $\mathrm{ISO}(3)_1\times\mathrm{ISO}(3)_2$ gaugings raises some immediate questions.

Perhaps the most obvious question is whether or not the $\tilde{\varphi}$-family of $\mathrm{ISO}(3)_1\times\mathrm{ISO}(3)_2$ gaugings of half-maximal supergravity we have presented in Sections 8.1.2 and 8.1.3 (and, more ambitiously, its generalisation in Appendix B) describes consistent truncations of ten- or eleven-dimensional supergravity down to four dimensions. In this respect, since turning on the embedding parameter $\tilde{\varphi}$ (*i.e.* $\tilde{\varphi}\neq\pm1$ in our parameterisation) only affects the embedding of the non-compact translational generators in the gauge algebra (see *e.g.* (8.17)-(8.18)), the family of $\mathcal{N}=2$ AdS_4 solutions we have found may stand a chance of being upliftable to new (possibly only locally geometric) type II or M-theory backgrounds. However, it could still happen – as for the ω deformation of the SO(8)-gauged supergravity [42] – that only very specific values of $\tilde{\varphi}$ enjoy a higher-dimensional interpretation, the natural ones being $\tilde{\varphi}=\pm1$ and 0. The case $\tilde{\varphi}=\pm1$ is by now known to uplift to type IIB S-fold backgrounds. Examples are the $\mathcal{N}=4$ S-fold of [55] and its marginal deformations we reviewed in this thesis. The case $\tilde{\varphi}=0$ remains to be understood. But it would certainly be disappointing if a supergravity solution like the exotic AdS_4 vacuum of [123] with such a (conjectured but) highly symmetric $\mathcal{N}=4$ CFT_3 dual – together with its marginal deformations presented in Sections 8.2.6 and 8.3.5 – ended up being in the Swampland.

Let us further comment on the exactly marginal deformation of the exotic $\mathcal{N}=4$ CFT_3 dual to the modulus χ in (8.77) preserving $\mathcal{N}=2$ supersymmetry. As discussed in Section 8.2.6, turning on χ breaks the original $\mathrm{SO}(4)_{\mathrm{R}}$ symmetry down to a $\mathrm{U}(1)_{\mathrm{R}}$ factor within the Cartan subgroup $\mathrm{U}(1)_{\mathrm{R}}\times\mathrm{U}(1)_{\mathrm{F}}\subset\mathrm{SO}(4)_{\mathrm{R}}$. As a result, the whole Cartan subgroup is not preserved and a geometric interpretation of the modulus χ along the lines of the axion-like deformations of S-folds seems a priori unplausible within the context of half-maximal supergravity. However, the situation is more subtle: when setting $\tilde{\varphi}=\pm1$, the embedding of this solution into maximal supergravity provides an additional flavour current multiplet $A_2\bar{A}_2[0]_1^0$ that accounts for the $\mathrm{U}(1)_{\mathrm{F}}$ symmetry [56]. In this case, the $\mathrm{U}(1)_{\mathrm{F}}\subset\mathrm{SO}(4)_{\mathrm{R}}$ is actually not broken but projected out by the $\mathbb{Z}_2$ symmetry truncating maximal to half-maximal supergravity. Therefore, if a ten- or eleven-dimensional uplift exists for the AdS_4 solution (8.76)-(8.77) at $\tilde{\varphi}=0$ (or more generically at $\tilde{\varphi}\neq\pm1$), it is still possible that some symmetries have been truncated away in the half-maximal effective description. Needless to say, an uplift (if any) to ten or eleven dimensions is required in order to settle this question.

The effective four-dimensional gauged supergravity approach adopted in this work provides us with some guidance in order to guess what the potential higher-dimensional realisation of the AdS_4 solutions could be. Remarkably, amongst the extra quadratic constraints in (3.112), only those in the $(\mathbf{1}, \mathbf{462'})$ irrep of $\mathrm{SL}(2) \times \mathrm{SO}(6,6)$ are violated in the AdS_4 solutions with $\tilde{\varphi} \neq \pm 1$.[10] Assuming (as for the S-folds) a type IIB origin, and since in a type IIB duality frame the $\mathrm{SL}(2)$ factor of the duality group is identified with S-duality, one possibility is the presence of $\mathrm{SL}(2)$-singlet branes (or bound states) in the corresponding ten-dimensional backgrounds. It could be interesting to look at possible uplifts incorporating such branes in a smeared limit. For example, try to add smeared D3-branes to the S-fold setups previously investigated in the literature. This could shed some light on how to incorporate sources in S-fold backgrounds. Another possibility is that the breaking of supersymmetries in the four-dimensional supergravity Lagrangian (from maximally to half-maximally supersymmetric) is not related to the inclusion of sources but, instead, it stems from geometry. This would be more in the spirit of the half-maximal consistent truncations of [137] and, perhaps, some generalised frame could be constructed for these half-maximal supergravities along the lines of [135] in order to systematically uplift any four-dimensional solution.

Another interesting line to explore is the possible relation between the AdS_4 solutions of the $\mathrm{ISO}(3)_1 \times \mathrm{ISO}(3)_2$ gaugings of half-maximal supergravity and various classes of type IIB backgrounds of the form $\mathrm{AdS}_4 \times \mathrm{M}_6$ that have been constructed directly in ten dimensions using different techniques: pure spinor formalism, G-structures, non-abelian T-duality, ... (see [138, 139, 140, 141] for an incomplete list). Identifying the field theory duals of these ten-dimensional solutions is a laborious and generically non-systematic task: one first makes an educated guess for the field theory duals and then runs as many holographic tests as possible. Thinking along these lines, it would be interesting to establish whether or not the general eight-parameter family of $\mathrm{ISO}(3)_1 \times \mathrm{ISO}(3)_2$ gaugings of half-maximal supergravity we have presented in Appendix B describes classes of consistent truncations of type IIB supergravity on $\mathrm{M}_6 = \mathrm{S}^2 \times \mathrm{S}^2 \times \Sigma$ with Σ being a Riemann surface. If such a connection exists and is well established via generalised geometry or extended field theory techniques, then exploiting the four-dimensional effective description would provide a way to characterise the CFT_3's dual to such type IIB solutions (presumably related to IR fixed points of Gaiotto–Hanany–Witten-like brane constructions) without having to work out their ten-dimensional uplift explicitly. For example, as we have done in this work, the conformal dimensions of the low lying operators in the dual CFT_3's could be extracted directly using four-dimensional data, namely, from the mass spectrum of the supergravity fields in the half-maximal $\mathrm{ISO}(3)_1 \times \mathrm{ISO}(3)_2$ gauged supergravity. Also the new techniques for Kaluza-Klein (KK) spectrometry put forward in [85] could be applied to the type IIB backgrounds of the form $\mathrm{AdS}_4 \times \mathrm{S}^2 \times \mathrm{S}^2 \times \Sigma$ (see [95, 142, 5, 143] for a study of the spectrum of KK modes around the type IIB S-folds at $\tilde{\varphi} = \pm 1$) upon suitable adjustment of the techniques to the context of half-maximal supergravity.

Finally, the analysis performed in Appendix B shows that the results in the main text can be straightforwardly generalised to include different embedding parameters $\tilde{\varphi}_{1,2}$ and $c_{1,2}$, as well as independent gauge couplings $g_{1,2}$, for each of the $\mathrm{ISO}(3)_{1,2}$ factors in the gauge group. In particular, having two independent gauge couplings $g_{1,2}$ permits to collapse or flatten-out one S^2 while keeping the other S^2 at finite size. This suggests an a priori much larger structure of supersymmetric AdS_4 solutions with new potentially interesting CFT_3 duals. Also going beyond the $\mathbb{Z}_2^2$ and $\mathrm{U}(1)_{\mathrm{R}}$ invariant sectors investigated in this work could accommodate new families of AdS_4

[10] The extra quadratic constraints in (3.112) living in the $(\mathbf{3}, \mathbf{1})$ irrep are not violated in our half-maximal supergravity models. These constraints appeared as the four-dimensional incarnation of the $\mathrm{SL}(2)$ extension [135] of the so-called *section constraint* in double field theory [136].

solutions with additional flat directions dual to new marginal deformations in the dual CFT_3's. These are all open questions and aspects we plan to continue exploring in the future.

Chapter 9

Conclusions

In this thesis, we have studied AdS_4 solutions of the dyonic $[SO(6) \times SO(1,1)] \ltimes \mathbb{R}^{12}$ gauging of maximal supergravity in four dimensions and uplifted these to solutions of type IIB supergravity on $AdS_4 \times S^1 \times S^5$ with an SL(2, $\mathbb{Z}$)-monodromy along the S^1 [1, 2]. This SL(2, $\mathbb{Z}$)-monodromy is responsible for the non-geometric nature of the type IIB backgrounds. These AdS_4 solutions are conjectured to be dual to new classes of strongly coupled CFT_3's dubbed S-fold CFT_3's.

We have shown that the four-dimensional supergravity scalar potential features flat directions allowing us to deform our S-fold solutions. Such deformations, dubbed "flat deformations", are conjectured to be dual to exactly marginal deformations of the dual CFT_3, at least in the large N limit. Uplifting the deformed solutions, we understood them geometrically in terms of an object called the mapping torus [3]. This mapping torus encodes a geometric-monodromy of the five-sphere on the S^1 factor of the geometry. The mapping torus construction allowed us to show that such deformations always exist when the internal geometry is of the form $S^1 \times M_{\text{int}}$ and possesses a Lie group symmetry (independently of the theory of gravity considered). Surprisingly, the flat deformations can break the residual (super-)symmetry of a given solution. This method gives us a controlled mechanism for supersymmetry breaking that could be used to extract exact results in non-supersymmetric field theories.

Focussing on the $\mathcal{N} = 4$ S-fold, we have studied a specific 2-parameter moduli space of solutions with no residual supersymmetry except at specific points of (super)symmetry enhancement [4]. These moduli are dual to exactly marginal deformations and provide holographic evidence for the existence of a conformal manifold of non-SUSY CFT's, at least in the large N limit. Using the recent method of KK spectrometry [102], we have been able to study the full towers of KK modes of these geometries and probe their perturbative stability at all KK-levels. These deformations do not lead to perturbative instabilities, and the deformed solutions have passed a number of checks of non-perturbative stability [5]. Our arguments for stability mostly rely on our geometric understanding of these deformations. This poses some challenges to the non-susy AdS conjecture [109]. This is also surprising from the CFT point of view since it shows evidence for an exact result in the absence of supersymmetry and known protection mechanisms.

We have also studied the domain-walls between the AdS_4 vacua using numerical and semi-analytical methods [3]. We have found a CFT_3 to CFT_3 flow as well as a flow that could be described, in the UV, as an anisotropic deformation of the compactified D3-brane solution. These domain-walls are conjectured, in light of the AdS/CFT correspondence, to be dual to RG-flows between the dual S-fold CFT_3's.

Finally, we have started a study of possibly more generic S-folds admitting an effective description within a half-maximal theory of gravity. In this setup we have found a web of $\mathcal{N} = 2$ vacua connected to a special point of supersymmetric enhancement with $\mathcal{N} = 4$. The string theory origin of this point, if any, is still lacking.

Future work

- The most obvious path forward is to produce new examples of the flat deformations we have uncovered, both in a different number of dimensions and in different theories of gravity (e.g. systematize the study we have started in section 8.2.6).

- The φ-family of solutions in the dyonic $[SO(6) \times SO(1,1)] \ltimes \mathbb{R}^{12}$ gauging of maximal supergravity in four dimensions (4.50) has not yet been uplifted and could be of interest to run tests, for example, of the CFT distance conjecture [57]. From unpublished results, we know that the ten-dimensional moduli space seems to be non-compact and feature a strange behaviour as $\varphi \to \infty$. For example, despite the uplift features a finite and non-zero internal volume, the theory seems to include free operators as well as operators of diverging conformal dimension. This requires more investigation since the global structure of the moduli space of CFT_3's could be different from the structure of the moduli of the AdS_4 vacua.

- It would certainly be interesting to known if the $[SO(6) \times SO(1,1)] \ltimes \mathbb{R}^{12}$ gauged maximal supergravity contains more $\mathcal{N} = 2$ solutions and, if so, uplift them to Type IIB supergravity. This would provide new examples of S-fold solutions.

- A proof of the non-perturbative stability of the non-supersymmetric solutions obtained from flat deformations is still missing. It would be interesting to understand if, and how, the axionic deformations appear in the D3-D5-NS5 brane systems and what is their interpretation from a holographic perspective. We have proposed a CFT dual for our flat deformations of the $\mathcal{N} = 4$ S-fold. Whether or not this proposition is correct, and the dual operator is exactly marginal, is to be assessed within a strongly coupled field theoretic context.

- Finding a string theoretical embedding of the exotic $\mathcal{N} = 4$ solution discussed in chapter 8 could lead to the construction of more generic S-folds. However, this would require a better understanding of consistent truncations of ExFT, for example along the lines of [137]. It is not yet clear if such a construction is possible for the gauged half-maximal supergravity under consideration in this thesis, or if the embedding tensor deformations are related to the presence of sources in the uplift.

In the long term, one goal of this research is to refine the understanding of Exceptional Field Theory and Exceptional Geometry. Not necessarily regarding their formal structure which has been fairly well studied, but regarding their interpretation in the context of string theory and their predictions in light of the AdS/CFT correspondence. In particular, I would like to understand how theories in four dimensions do or do not fit in string theory, how the global structure of their moduli space of AdS_4 vacua can or cannot be inferred from their 4d descriptions, and how to build their dual CFT's directly from a lower dimensional perspective. Understanding what information is encoded, and how, in a lower dimensional effective supergravity would simplify the study of solutions in string theory.

Chapter 10

Conclusiones

En esta tesis hemos estudiado soluciones AdS_4 de la supergravedad maximal gaugeada con grupo de gauge $[SO(6) \times SO(1,1)] \ltimes \mathbb{R}^{12}$ en cuatro dimensiones, y embebido estas soluciones en la supergravedad diez-dimensional de tipo IIB resultando en geometrías de la forma $AdS_4 \times S^1 \times S^5$ con una monodromía SL(2, $\mathbb{Z}$) a lo largo de la S^1 [1, 2]. Esta monodromía SL(2, $\mathbb{Z}$) es responsable del carácter no-geométrico de estas configuraciones de tipo IIB. Por otro lado, se conjectura que estas soluciones AdS_4 son duales a unas clases nuevas de CFT_3's fuertemente acopladas denominadas S-fold CFT_3's.

Hemos mostrado que el potencial escalar de la teoría de supergravedad cuatridimensional contiene direcciones planas a lo largo de las cuales podemos deformar nuestras soluciones de tipo S-fold. Estas deformationes, denominadas "deformaciones planas", se conjecturan duales a deformaciones exactamente marginales de las CFT_3's duales, al menos en el límite de N grande. Embeber las soluciones deformadas en la supergravedad de tipo IIB nos ha permitido entender las deformaciones planas de manera geométrica en términos de un objecto matemático llamado el *mapping torus* [3]. Este *mapping torus* codifica una monodromía geométrica de la cinco-esfera sobre el factor S^1 de la geometría. La interpretación en términos del *mapping torus* nos ha permitido mostrar que las deformaciones planas siempre existen si la geometría interna es de la forma $S^1 \times M_{\text{int}}$ y poseen la estructura de grupo de Lie (independientemente de la teoría de gravedad que se esté considerando). Sorprendentemente, las deformaciones planas proporcionan un mecanismo para romper la (super-)symmetría de una solución de manera controlada.

Centrándonos en el S-fold con supersimetría $\mathcal{N} = 4$, hemos estudiado un espacio de módulos bi-paramétrico que contiene soluciones no supersimétricas excepto en algunos puntos especiales donde la supersimetría se restaura [4]. Estos módulos son duales a deformaciones exactamente marginales y proporcionan evidencia (holográfica) acerca de la existencia de una variedad conforme de CFT's no supersimétricas, al menos en el límite de N grande. Utilizando técnicas novedosas de spectrometría KK [102], hemos sido capaces de estudiar las torres completas de modos de KK en estas geometrías y hemos probado la estabilidad de las soluciones frente a fluctuaciones perturbativas a todos los niveles en la torre de KK. Las deformaciones planas que hemos investigado no dan por tanto lugar a inestabilidades perturbativas, ni tampoco a varios tipos de inestabilidades no-perturbativas que también hemos investigado [5]. Por último, nuestros argumentos acerca de la estabilidad de las soluciones no supersimétricas se sustentan en la interpretación geométrica de las deformaciones planas. Esto plantea algunos retos a la conjectura acerca de la no existencia de soluciones AdS estables no-supersimétricas [109]. Los resultados presentados también son sorprendentes desde el punto de vista de las CFT's duales, ya que muestran evidencia de la existencia de resultados exactos en ausencia de supersimetría y, por tanto, de mecanismos de protección.

También hemos estudiado muros de dominio entre las soluciones AdS_4 usando métodos numéricos y semi-analíticos [3]. A la luz del diccionario holográfico, estos muros de dominio describen flujos del grupo de renormalización que conectan dos S-fold CFT_3's, así como flujos de renormalización que conectan una S-fold CFT en el infrarrojo con deformaciones anisotrópicas de la D3-brana (compactificada en la S^1) en el ultravioleta.

Finalmente, hemos iniciado un estudio acerca de la posible existencia de soluciones tipo S-folds más generales las cuales admiten una descripción efectiva en términos de una teoría de supergravedad semi-maximal. En este contexto, construimos una web de vacíos con supersimetría $\mathcal{N} = 2$ conectados de manera continua con un punto especial con supersimetría $\mathcal{N} = 4$. El origen en teoría de cuerdas de este punto espacial, si es que existe, es aún desconocido.

Líneas de investigación futuras

- La línea más inmediata de continuación consiste en identificar y construir nuevos ejemplos de deformaciones planas, tanto en dimensiones diferentes a cuatro como en otras teorías de supergravedad (*v.g.* sistematizar el estudio iniciado en la sección 8.2.6).

- La φ-familia de soluciones en la supergravedad cuatridimensional maximal diónica con grupo de gauge $[SO(6) \times SO(1,1)] \ltimes \mathbb{R}^{12}$ (4.50) no se ha embebido en teoría de cuerdas y podría ser de relevancia para llevar a cabo tests, por ejemplo, de la conjectura de la distancia en CFT [57]. A partir de resultados aún sin publicar, sabemos que el espacio de módulos diez-dimensional parece ser no-compacto y presenta un comportamiento extraño en el límite $\varphi \to \infty$. Por ejemplo, a pesar de que la geometría diez-dimensional presenta un volumen interno finito y no nulo, la teoría dual parece incluir operadores libres así como operadores con dimensiones conformes divergentes. Esto requiere un estudio más profundo de la estructura global del espacio de módulos de las CFT_3's ya que ésta puede diferir de la estructura de módulos de los duales AdS_4 gravitacionales.

- Sería interesante hacer una clasificación exhaustiva de soluciones $\mathcal{N} = 2$ en la supergravedad maximal con grupo de gauge $[SO(6) \times SO(1,1)] \ltimes \mathbb{R}^{12}$ y embeberlas en la supergravedad de tipo IIB. Esto podría proporcional ejemplos nuevos de soluciones tipo S-fold con duales interesantes en teoría de campos.

- La estabilidad no-perturbativa de las soluciones no-supersimétricas obtanidas mediante deformaciones planas está aún por demostrarse. Sería interesante entender cómo aparecen estas deformaciones de carácter axiónico en sistemas de branas del tipo D3-D5-NS5 y cómo se interpretarían de forma general en holografía. En esta dirección, hemos propuesto un dual en CFT para las deformaciones planas del S-fold con supersimetría $\mathcal{N} = 4$. Esta propuesta, así como la marginalidad exacta de las deformaciones planas, ha de ser aún confirmada/desmentida en el contexto de teoría de campos fuertemente acopladas.

- Encontrar cómo se embebe en teoría de cuerdas la solución exótica con supersimetría $\mathcal{N} = 4$ discutida en el capítulo 8 podría proporcionar una forma más general de construir soluciones de tipo S-fold. Sin embargo, esto requiere una mejor comprensión de las truncaciones consistentes en el contexto de las ExFT's, por ejemplo, en la línea de [137]. No está claro a día de hoy si construcciones de este tipo son posibles para supergravedades gaugeadas semi-maximales del tipo a las estudiadas en esta tesis, en las que el tensor de embedding podría estar relacionado con la presencia de fuentes en el background diez-dimensional.

A largo plazo, uno de los objetivos es el de refinar nuestra comprensión de la Teoría de Campos Excepcional así como de la Geometría Excepcional. No necesariamente en lo que respecta a su estructura formal, la cual ha sido estudiada en profundidad, sino en lo que respecta a su interpretación en teoría de cuerdas así como a sus predicciones a la luz de la correspondencia AdS/CFT. En particular, me gustaría estudiar cómo las teorías efectivas cuatridimensionales se conectan con la teoría de cuerdas, cómo la estructura global del espacio de módulos de los vacíos AdS_4 puede o no ser inferida a partir de propiedades cuatridimensionales, y cómo construir sus CFT's duales directamente desde una perspectiva de baja dimensionalidad. Entender qué información se haya codificada, y cómo, en una teoría de supergravedad de baja dimensión simplificaría en gran medida el estudio de soluciones en la teoría de cuerdas completa.

Appendix A

$E_{7(7)}$

The group $E_{7(7)}$ is the split-real form associated with the $\mathfrak{e}_7$ lie algebra. It is a 133 dimensional group whose maximal compact subgroup is $SU(8)/\mathbb{Z}$. The fundamental representation of $E_{7(7)}$ is the **56**, its adjoint representation is the **133**. Another representation of interest for supergravity is the $\mathbf{912} \subset \mathbf{133} \otimes \mathbf{56}$. The Cartan-Killing metric on $\mathfrak{e}_7$ is defined as $Tr(t^\alpha t^\beta)$, invariant under the action of $E_{7(7)}$, which induces a metric on $E_{7(7)}$. The pair $(E_{7(7)}, SU(8))$ is reducible[1] which means that we can split the algebra $\mathfrak{e}_7$ as

$$\mathfrak{e}_7 = \mathfrak{su}(8) \oplus \mathfrak{k}\,, \tag{A.1}$$

with $[\mathfrak{su}(8), \mathfrak{su}(8)] \subset \mathfrak{su}(8), [\mathfrak{su}(8), \mathfrak{k}] \subset \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{su}(8)$. Moreover, $\mathfrak{k}$ is orthogonal to $\mathfrak{su}(8)$ with respect to the Cartan-Killing metric.

In particular, this means that $\mathfrak{k}$ admits an $SU(8)$ representation. Under the branching $E_{7(7)} \to SU(8)$ the adjoint branches as

$$\mathbf{133} \to \mathbf{63} \oplus \mathbf{70}\,. \tag{A.2}$$

where **63** spans the $\mathfrak{su}(8)$ algebra and the $\mathfrak{k} = \mathbf{70}$ is a representation of $\mathfrak{su}(8)$. Moreover, this allows us to endow the quotient space $E_{7(7)}/SU(8)$ with a Riemannian structure induced by the Cartan-Killing metric.

WIn the SL(8) basis the adjoint representation of $\mathrm{E}_{7(7)}$ splits into $\mathbf{133} \to \mathbf{63} \oplus \mathbf{70}$ under $\mathrm{SL}(8) \subset \mathrm{E}_{7(7)}$. This implies a splitting of generators of the form $t_\alpha \to t_A{}^B \oplus t_{ABCD}$ with $t_A{}^A = 0$ and $t_{ABCD} = t_{[ABCD]}$. The fundamental representation of $\mathrm{E}_{7(7)}$ branches as $\mathbf{56} \to \mathbf{28} \oplus \mathbf{28'}$ so the fundamental $\mathrm{E}_{7(7)}$ index splits as $_{\mathbb{M}} \to_{[AB]} \oplus^{[AB]}$. Then, the **63** generators of SL(8) correspond with $\mathrm{E}_{7(7)}$ generators of the form

$$[t_A{}^B]_{\mathbb{M}}{}^{\mathbb{N}} = \frac{1}{\sqrt{12}} \begin{pmatrix} 2\,\delta_{[C}^{[E}\,[t_A{}^B]_{D]}{}^{F]} & 0 \\ 0 & -2\,\delta_{[E}^{[C}\,[t_A{}^B]_{F]}{}^{D]} \end{pmatrix} \quad \text{with} \quad [t_A{}^B]_C{}^D = 4\,\delta_A^C\,\delta_D^B - \frac{1}{2}\,\delta_A^B\,\delta_C^D\,, \tag{A.3}$$

whereas the remaining **70** generators extending SL(8) to $\mathrm{E}_{7(7)}$ take the form

$$[t_{ABCD}]_{\mathbb{M}}{}^{\mathbb{N}} = \sqrt{12} \begin{pmatrix} 0 & \epsilon_{ABCDEFGH} \\ 4!\,\delta_{ABCD}^{EFGH} & 0 \end{pmatrix}\,. \tag{A.4}$$

They are normalized such that $\mathrm{Tr}(t_\alpha t_\beta{}^t) = \delta_{\alpha\beta}$. The Killing-Cartan matrix is then given by

$$\mathcal{K}_{\alpha\beta} = \mathrm{Tr}(t_\alpha t_\beta) = \begin{cases} 1 & \text{if} \quad \beta = \alpha^t \\ 0 & \text{otherwise} \end{cases}\,, \tag{A.5}$$

[1]This is always the case for a pair (G, H) where H is the maximal compact subgroup of G.

where by α^t we refer to the generator $t_{\alpha^t} \equiv (t_\alpha)^t$. With the generators in (A.3) and (A.4) one has that

$$(t_A{}^B)^t = t_B{}^A \qquad \text{and} \qquad (t_{ABCD})^t = \frac{1}{4!}\,\epsilon^{ABCDEFGH}\,t_{EFGH}\ . \tag{A.6}$$

Note that if t_α is a positive root of the $\mathfrak{e}_{7(7)}$ algebra then t_{α^t} is the corresponding negative root.

Appendix B

ISO(3) × ISO(3) gaugings of half-maximal supergravity

In this appendix we analyse the set of possible embeddings of an $\mathrm{ISO}(3)_1 \times \mathrm{ISO}(3)_2$ gauging of half-maximal supergravity of the form

$$\mathrm{ISO}(3)_1 \times \mathrm{ISO}(3)_2 \;\subset\; \mathrm{SL}(2) \times \mathrm{SO}(3,3)_1 \times \mathrm{SO}(3,3)_2 \;\subset\; \mathrm{SL}(2) \times \mathrm{SO}(6,6) \;. \tag{B.1}$$

We have attached labels $_1$ and $_2$ to keep track of each repeated factor. Following the notation of [123], and building upon the results of [125], a gauging of this type is totally encoded in a set of eight embedding matrices [144]. Since $\mathrm{SO}(3,3)_1 \sim \mathrm{SL}(4)_1$, the most general embedding[1] of the first $\mathrm{ISO}(3)_1$ factor is encoded in four 4×4 matrices given by

$$Q_+^{(1)} = \begin{pmatrix} -a_0' & 0 \\ 0 & 0 \times \mathbb{I}_3 \end{pmatrix} \quad , \quad \tilde{Q}_+^{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{c}_1' \times \mathbb{I}_3 \end{pmatrix} , \tag{B.2}$$

$$Q_-^{(1)} = \begin{pmatrix} -b_0' & 0 \\ 0 & 0 \times \mathbb{I}_3 \end{pmatrix} \quad , \quad \tilde{Q}_-^{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{d}_1' \times \mathbb{I}_3 \end{pmatrix} , \tag{B.3}$$

with $\tilde{c}_1' \neq 0$. Equivalently, the first $\mathrm{ISO}(3)_1$ factor is specified by four embedding tensor components of the form

$$f_{+\bar{a}bc} = \tilde{c}_1' \, \epsilon_{\bar{a}bc} \quad , \quad f_{-abc} = -b_0' \, \epsilon_{abc} \quad , \quad f_{+abc} = -a_0' \, \epsilon_{abc} \quad , \quad f_{-\bar{a}bc} = \tilde{d}_1' \, \epsilon_{\bar{a}bc} \; . \tag{B.4}$$

Analogously, the second $\mathrm{ISO}(3)_2$ factor is encoded in another set of four 4×4 matrices of the form

$$Q_+^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{c}_2 \times \mathbb{I}_3 \end{pmatrix} \quad , \quad \tilde{Q}_+^{(2)} = \begin{pmatrix} a_3 & 0 \\ 0 & 0 \times \mathbb{I}_3 \end{pmatrix} , \tag{B.5}$$

$$Q_-^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{d}_2 \times \mathbb{I}_3 \end{pmatrix} \quad , \quad \tilde{Q}_-^{(2)} = \begin{pmatrix} b_3 & 0 \\ 0 & 0 \times \mathbb{I}_3 \end{pmatrix} , \tag{B.6}$$

with $\tilde{d}_2 \neq 0$. The components of the embedding tensor for the second $\mathrm{ISO}(3)_2$ factor are then given by

$$f_{-i\bar{j}\bar{k}} = \tilde{d}_2 \, \epsilon_{i\bar{j}\bar{k}} \quad , \quad f_{+\bar{i}\bar{j}\bar{k}} = a_3 \, \epsilon_{\bar{i}\bar{j}\bar{k}} \quad , \quad f_{-\bar{i}\bar{j}\bar{k}} = b_3 \, \epsilon_{\bar{i}\bar{j}\bar{k}} \quad , \quad f_{+i\bar{j}\bar{k}} = \tilde{c}_2 \, \epsilon_{i\bar{j}\bar{k}} \; . \tag{B.7}$$

[1] This is so up to equivalent solutions of the quadratic constraints in (3.111).

Together, (B.4) and (B.7) account for all the components of the embedding tensor $f_{\alpha MNP}$ that are activated in the class (B.1) of $\text{ISO}(3)_1 \times \text{ISO}(3)_2$ gaugings of half-maximal supergravity we investigate in this work.

B.0.1 Quadratic constraints and algebra structure

The embedding tensor components in (B.4) and (B.7) automatically satisfy the quadratic constraints of half-maximal supergravity. However, the computation of the additional constraints in (3.112) for this multi-parameteric family of $\text{ISO}(3)_1 \times \text{ISO}(3)_2$ gaugings yields

$$f_{\alpha MNP}\, f_{\beta}{}^{MNP} = 0 \qquad \text{and} \qquad \epsilon^{\alpha\beta}\, f_{\alpha[MNP}\, f_{\beta QRS]}\Big|_{\text{SD}} = 0 \;\Leftrightarrow\; \begin{cases} b_0'\, \tilde{c}_2 - a_0'\, \tilde{d}_2 = 0 \\ b_3\, \tilde{c}_1' - a_3\, \tilde{d}_1' = 0 \end{cases} . \tag{B.8}$$

The antisymmetry of the commutators $[\, T_{\alpha M} \,,\, T_{\beta N} \,] = f_{\alpha MN}{}^{P}\, T_{\beta P}$ for this general class of embeddings imposes a set of linear relations between the generators of the form

$$(\tilde{c}_1')^2\, T_{-a} = \tilde{d}_1'\, \tilde{c}_1'\, T_{+a} + (a_0'\, \tilde{d}_1' - b_0'\, \tilde{c}_1')\, T_{+\bar{a}} \qquad , \qquad \tilde{c}_1'\, T_{-\bar{a}} = \tilde{d}_1'\, T_{+\bar{a}} \;, \tag{B.9}$$

and

$$(\tilde{d}_2)^2\, T_{+\bar{i}} = \tilde{d}_2\, \tilde{c}_2\, T_{-\bar{i}} + (a_3\, \tilde{d}_2 - b_3\, \tilde{c}_2)\, T_{-i} \qquad , \qquad \tilde{d}_2\, T_{+i} = \tilde{c}_2\, T_{-i} \;. \tag{B.10}$$

Choosing the independent generators to be $(T_{+a}, T_{+\bar{a}})$ and $(T_{-i}, T_{-\bar{i}})$, one finds a set of non-trivial commutation relations of the form

$$\begin{aligned} [\, T_{+a}, T_{+b} \,] &= \tilde{c}_1'\, \epsilon_{ab}{}^{c}\, T_{+c} - a_0'\, \epsilon_{ab}{}^{\bar{c}}\, T_{+\bar{c}} \;, \\ \left[\, T_{+a}, T_{+\bar{b}} \,\right] &= \tilde{c}_1'\, \epsilon_{a\bar{b}}{}^{\bar{c}}\, T_{+\bar{c}} \;, \\ \left[\, T_{+\bar{a}}, T_{+\bar{b}} \,\right] &= 0 \;, \end{aligned} \tag{B.11}$$

for the first $\text{ISO}(3)_1$ factor in the gauge group and, similarly,

$$\begin{aligned} \left[\, T_{-\bar{i}}, T_{-\bar{j}} \,\right] &= \tilde{d}_2\, \epsilon_{\bar{i}\bar{j}}{}^{\bar{k}}\, T_{-\bar{k}} + b_3\, \epsilon_{\bar{i}\bar{j}}{}^{k}\, T_{-k} \;, \\ [\, T_{-\bar{i}}, T_{-j} \,] &= \tilde{d}_2\, \epsilon_{\bar{i}j}{}^{k}\, T_{-k} \;, \\ [\, T_{-i}, T_{-j} \,] &= 0 \;, \end{aligned} \tag{B.12}$$

for the second $\text{ISO}(3)_2$ factor. Note that, for each of the $\text{ISO}(3)_{1,2}$ factors, there are two parameters entering the commutation relations in (B.11) and (B.12) and two additional parameters specifying the linear combinations of generators in (B.9) and (B.10).

B.0.2 $\mathcal{N} = 1$ superpotentials

It is also interesting to investigate the dynamics of the seven moduli fields z_I, $I = 1, \ldots, 7$, in the $\mathbb{Z}_2^2$-invariant sector of half-maximal supergravity coupled to six vector multiplets. This sector is described by the $\mathcal{N} = 1$ supergravity multiplet coupled to seven chiral superfields with Kähler potential

$$K = -\sum_{I=1}^{7} \log[-i(z_I - \bar{z}_I)] \ , \tag{B.13}$$

and a superpotential given by

$$\begin{aligned} W \; = \; & \Big[- a_3 + a_0' \, z_4 z_5 z_6 - \tilde{c}_2 \, (z_1 z_4 + z_2 z_5 + z_3 z_6) + \tilde{c}_1' \, (z_1 z_5 z_6 + z_2 z_4 z_6 + z_3 z_4 z_5) \Big] \\ + \; & \Big[\; b_3 - b_0' \, z_4 z_5 z_6 + \tilde{d}_2 \, (z_1 z_4 + z_2 z_5 + z_3 z_6) - \tilde{d}_1' \, (z_1 z_5 z_6 + z_2 z_4 z_6 + z_3 z_4 z_5) \Big] \, z_7 \ . \end{aligned} \tag{B.14}$$

Note that the eight gauging parameters in (B.4) and (B.7) enter the superpotential of the model.

In this work we have made a simple choice of gauging parameters. More concretely, we have chosen the same embedding for the two $\text{ISO}(3)_{1,2}$ factors in the gauging. This choice drastically simplifies the analysis of supersymmetric vacua. These vacua satisfy the set of supersymmetric (or F-flatness) conditions

$$F_I \equiv D_I W = \partial_I W + (\partial_I K) W = 0 \ , \tag{B.15}$$

with $I = 1, \dots 7$.

B.0.3 Back to our $ISO(3) \times ISO(3)$ gauging

The specific model discussed in Section 8.1.2 corresponds to a simple fixing of the gauging parameters in (B.4) of the form $\tilde{d}_1' = \tilde{c}_2 = 0$ and

$$\begin{aligned} &\tilde{c}_1' = \tfrac{2\sqrt{2}\,g}{\sqrt{1+\tilde{\varphi}^2}} \ , \quad -b_0' = \pm 2\sqrt{2}\, g\, c\, \tfrac{\tilde{\varphi}}{\sqrt{1+\tilde{\varphi}^2}} \ , \quad -a_0' = -2\sqrt{2}\, g\, c\, \tfrac{\tilde{\varphi}^2-1}{\tilde{\varphi}^2+1} \ , \\ &\tilde{d}_2 = \tfrac{2\sqrt{2}\,g}{\sqrt{1+\tilde{\varphi}^2}} \ , \quad a_3 = \pm 2\sqrt{2}\, g\, c\, \tfrac{\tilde{\varphi}}{\sqrt{1+\tilde{\varphi}^2}} \ , \quad b_3 = -2\sqrt{2}\, g\, c\, \tfrac{\tilde{\varphi}^2-1}{\tilde{\varphi}^2+1} \ . \end{aligned} \tag{B.16}$$

Plugging (B.16) into (B.14) yields a superpotential

$$\begin{aligned} W \; = \; & \frac{2\sqrt{2}\,g}{\sqrt{1+\tilde{\varphi}^2}} \Big[z_1 z_5 z_6 + z_2 z_4 z_6 + z_3 z_4 z_5 + (z_1 z_4 + z_2 z_5 + z_3 z_6)\, z_7 \Big] \\ - \; & \frac{2\sqrt{2}\,g}{\sqrt{1+\tilde{\varphi}^2}}\, c \Big[\pm \tilde{\varphi}\, (1 - z_4 z_5 z_6 z_7) + \frac{1-\tilde{\varphi}^2}{\sqrt{1+\tilde{\varphi}^2}} (z_4 z_5 z_6 - z_7) \Big] \ . \end{aligned} \tag{B.17}$$

We have verified that the AdS_4 solutions in (8.28) solve the F-flatness equations in (B.15) constructed from (B.17).

Lastly, as a further check of consistency, setting $\tilde{\varphi}^2 = 1$ recovers the maximal theory. Namely, the superpotential in (B.17) reduces to

$$W = 2\, g \Big[z_1 z_5 z_6 + z_2 z_4 z_6 + z_3 z_4 z_5 + (z_1 z_4 + z_2 z_5 + z_3 z_6)\, z_7 \Big] \pm 2\, g\, c\, (1 - z_4 z_5 z_6 z_7) \ , \tag{B.18}$$

in agreement with the result of [2].

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