Universidad de Oviedo

Programa de Doctorado en Materiales

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Teoría efectiva de Wilson loops y dualidad gauge-gravedad

Wilson loops effective theory and gauge-gravity duality

TESIS DOCTORAL

Diego Gútiez Bravo

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Directores de tesis
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RESUMEN DEL CONTENIDO DE TESIS DOCTORAL

1.- Título de la Tesis	
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RESUMEN (en español)

Esta tesis estudia cómo obtener una acción efectiva para el dual holográfico de líneas de Wilson en teorías de campos no conformes y fuertemente acopladas y algunas aplicaciones del mecanismo obtenido. Encontramos una acción efectiva para una cuerda dual a la línea de Wilson usando el mecanismo de renormalización de Wilson donde integramos los grados de libertad más próximos a la frontera de geometrías que asintóticamente se aproximan a AdS. La integración proporciona una contribución para la acción en el cutoff dependiente de coeficientes que pueden ser determinados por una ecuación del flujo del grupo de renormalización.

Empleamos esta técnica para estudiar observables fenomenológicos como el potencial quark-antiquark y proporcionamos dos tipos de ejemplos: Teorías en 3+1 dimensiones con un RG flow que acaba en un punto fijo en el infrarrojo y en las teorías confinantes de Witten QCD y el modelo de Klebanov-Strassler. También aplicamos este formalismo para calcular las fuerzas que experimenta un quark moviéndose en un plasma de quark-gluones, que modelamos con una cuerda moviendose en una geometría en una brana negra.

Finalmente estudiamos como afecta este formalismo a la invarianza bajo reparametrizaciones de los Wilson Loops, también conocida como simetría 'zig-zag'. Probamos que las reparametrizaciones de los Wilson loops pueden ser identificadas con transformaciones conformes en la hoja de mundo de la cuerda. La integración se lleva a cabo hasta un punto de corte en la dirección holográfica que puede estar asociado a la geometría de fondo o a la hoja de mundo. Cuando empleamos el primero rompemos la simetría bajo difeomorfismos y transformaciones de Weyl de la hoja de mundo, pero conservamos transformaciones conformes, sin embargo el segundo método rompe la invarianza conforme e induce una acción de defecto en la escala del punto de corte.

RESUMEN (en Inglés)

This thesis studies how to obtain an effective theory for the holographic dual of Wilson lines in strongly coupled non-conformal field theories and some applications of this mechanism. An effective action is found for a string dual to the Wilson line using the Wilsonian renormalization scheme where we integrate out the degrees of freedom that are close to the boundary in asymptotically AdS spaces. This integration results in a contribution to the action at the cutoff that depends on coefficients that are determined by an RG flow equation.

We use this technique to study some phenomenological observables such as the quark-antiquark potential and provide two kinds of examples: 3+1 dimensional theories with an RG Flow that ends in an IR fixed point and confining theories, specifically Witten QCD and Klebanov-Strassler models. We also apply this formalism to compute the forces a quark

experiences when moving through a quark-gluon plasma, modeled by a string moving in a black brane geometry.

We finally study how does this affect the reparametrization invariance of Wilson loops, also known as 'zig-zag' symmetry. We show that Wilson loop reparametrizations can be mapped to conformal transformations of the string worldsheet. The integration is done up to a cutoff in the holographic direction that can be anchored to either the background geometry or the worldsheet. When we perform the former, we break worldsheet diffeomorphisms and Weyl invariance, but we preserve conformal transformations, however the latter breaks conformal invariance and induces a defect action at the cutoff scale.

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Abstract

Esta tesis estudia como obtener una acción efectiva para el dual holográfico de líneas de Wilson en teorías de campos no conformes y fuertemente acopladas y algunas aplicaciones del mecanismo obtenido. Encontramos una acción efectiva para una cuerda dual a la línea de Wilson usando el mecanismo de renormalización de Wilson donde integramos los grados de libertad más próximos a la frontera de geometrías que asintóticamente se aproximan a AdS. La integración proporciona una contribución para la acción en el cutoff dependiente de coeficientes que pueden ser determinados por una ecuación del flujo del grupo de renormalización.

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Chapter 1

Motivation

1.1 Quantum Cromodynamics

The Standard model is, up to date, the most precise model we have in physics to describe the behaviour of the Universe. In this description, the fundamental interactions of nature are governed by the electroweak and strong forces, governed by symmetry groups $SU(2)_L \times U(1)$ and $SU(3)$ respectively. The theory describing the latter is known as Quantum Chromodynamics (QCD) and it is described by the Lagrangian:

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + \sum_{f=1}^{N_f} \bar{q}_f (i\gamma^\mu D_\mu - m_f) q_f \tag{1.1}$$

Where q_f are quark fields, A^a_μ the gluon fields, $G^a_{\mu\nu}$ is the QCD field strength tensor and D_μ the covariant derivative. The coupling strength appears at the field strength and the covariant derivative. The quarks are also charged under the $SU(2) \times U(1)$, and they have 6 distinct flavours. The quarks masses are acquired through the Higgs mechanism in the electroweak sector, with the 3 flavours (up, down and strange) being relatively light and the other 3 (charm, bottom and top) heavier. If we approximate the first three as massless and the remaining ones as infinitely massive as in the review [1], we get a running coupling

$$g^2(q^2) = \frac{16\pi^2}{b_0 \log(q^2/\Lambda^2_{QCD})} \tag{1.2}$$

Where Λ_{QCD} depends on the renormalization scheme, and takes the value $\approx 200\,\mathrm{MeV}$ in the Modified Minimal Substraction. This simplified computation provides us with an insight of the behaviour of QCD at different scales, as it features a dependence with the dimensionful parameter Λ_{QCD}. At low energies, we see that the theory is strongly coupled, giving rise to the characteristic confinement of QCD (we do not find isolated quarks in nature) and it becomes weakly coupled at high energies (asymptotic freedom).

This behaviour generates a very rich phase diagram that we have not been able to determine either theoretically or via experiments. Experimentally, a quark-gluon plasma is produced at RHIC and CERN (see for example [2]), and some information can be inferred from observations of neutron stars. Theoretically, we can use perturbation theory at high energies (or temperatures), and lattice computations are useful when the configuration has zero chemical potential regardless of the temperature. However, regions of both finite temperature and chemical potential remain unexplored.

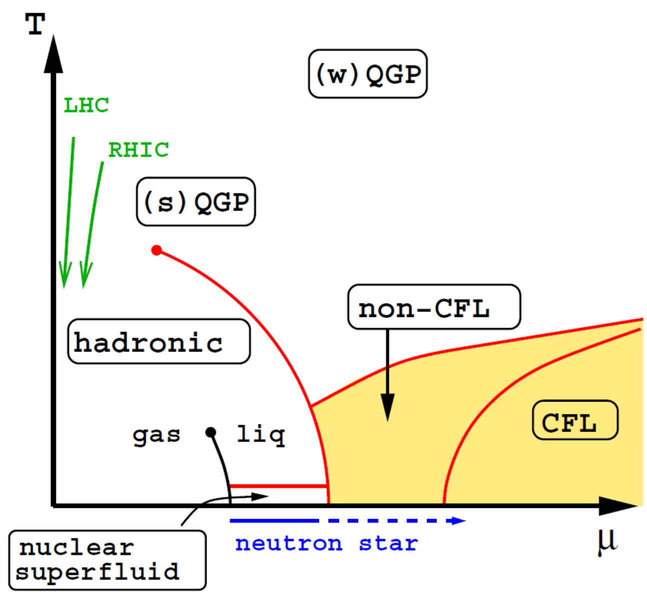

Figure 1.1: QCD phase diagram. Extracted from: [1]

A useful probe into the properties of this phase diagram are Wilson lines, which are operators that take this form in the fundamental representation:

$$\mathcal{W}(\mathcal{C}) = \frac{1}{N} \mathrm{Tr}\, \mathcal{P} \left(e^{i \oint_{\mathcal{C}} d\tau (\dot{x}^{\mu} A_{\mu})} \right) \tag{1.3}$$

This operator is analogous to an infinitely massive particle in the fundamental representation moving along a trajectory determined by $\mathcal{C}$, where A_μ are the gauge fields.

These operators are able to show us different properties of the interactions in QCD that we can match with experimental data. One such configuration is that of an infinite rectangle in the time direction, that can be thought of as 2 static particles separated by a distance L.

From this operator we can extract the potential between two quarks

$$V_{q\bar{q}} \propto \log(\langle \mathcal{W} \rangle) \tag{1.4}$$

which in turn is a good predictor of IR confinement as we expect confining theories to have a potential that grows linearly with the distance.

Wilson lines can also be useful to estimate how the energy of a heavy particle is dissipated when moving through a medium. A state of quark-gluon plasma has been generated in LHCb and RHIC by colliding two heavy nuclei (Au-Au and Pb-Pb). At the beginning of this process some on shell partons (quarks and gluons) can be produced, which then travel through the plasma generated afterwards. As per usual, quarks and gluons cannot be seen isolated in nature, and after they are produced they usually decay into a shower of particles that move at high speeds known as jets. Due to conservation of momentum, jets are produced in pairs that carry the same amount of energy. However, if these partons are produced somewhere inside the plasma, one of them will have to travel through it for a longer distance and some of its energy will be lost in the process. This difference in energy can be measured through the nuclear modification factor

$$R_{AA} = \frac{\sigma_{NN}}{\langle N_{bin} \rangle} \frac{d^2 N_{AA}/dp_T d\eta}{d^2 \sigma_{pp}/dp_T d\eta} \tag{1.5}$$

which can then be used to characterize the properties of the plasma. In this expression $\langle N_{bin} \rangle$ is the average number of binary nucleon-nucleon collisions and σ are the cross section for the nucleon-nucleon collision and proton-proton collision. N_{AA} is the yield in nucleus-nucleus collisions . p_T and η are the transverse momentum and the pseudorapidity.

1.2 Holography

Holography first appeared in the late 90's in [3, 4] as a derivation from string theory, proposing a duality between $\mathcal{N} = 4$ Super Yang Mills theory in 4 dimensions and type II B supergravity in $AdS_5 \times S_5$. This concept was then extended to dualities between more general Quantum Field Theories (QFTs) being associated to different geometries with a boundary, where the gravity theory has one more dimension than the QFT we are dualising. This suggests a possibility: Could holography be used to make predictions in QCD if we find an appropriate dual geometry? The proposal is indeed interesting, as generating thermal states in the dual gravity theory can be done by introducing a black hole in the bulk of the geometry. The holographic dictionary then states that the temperature and entropy of the black hole are related to those of the field theory thermal bath, while the charge of the black hole is dual to the chemical potential, therefore allowing us to probe into different sections of the phase diagram.

There are, of course, complications to this approach, of which maybe the most clear one is that we have not found a gravity dual to QCD, hence all of the predictions that come out of holography can be, for the most part, qualitative in nature. In this direction, some theories such as Witten QCD and the Klebanov Strassler model [5, 6] have been able to produce theories that present confinement, and recent work by [7] has found a fully backreacted gravity dual to a thermodynamic bath with baryons in 2+1 dimensions. Good predictions have also been made about a universal contribution at large 't Hooft coupling to the shear viscosity [8, 9, 10].

One of the possible interpretations of the $AdS - CFT$ duality is that the extra dimension in the gravity side is analogous to an energy scale, where the deeper you go in the bulk, the lower the energy scale of the process is. This is interesting, as some of the properties we are interested in are highly dependent on the deep bulk properties of the theory [11]. On the field theory side, this way of looking only at the low energy interactions can remind us of the concept of an RG flow, which will be the main focus of this thesis. We expect these results to be useful in hybrid models, allowing us to ignore the UV details of the geometry but still making it possible to generate predictions.

The thesis will then follow this structure: Chapter 2 will be a review of the tools existing in the literature that will allow us to tackle this problem. In chapter 3 we will present the results of this thesis: a mechanism to generate an effective action and different applications to it. Chapters 4, 5 and 6 include the articles that compose this thesis and chapter 7 provides the conclusions.

Chapter 2

Renormalization and Wilson lines in holography

2.1 Wilson RG Flow

Wilson renormalization starts from a theory which we will consider valid up to a certain cutoff Λ_0. For the sake of simplicity, we will focus on a theory with one scalar field with a renormalized partition function:

$$Z = \int \prod_{||p||<\Lambda_0} d\phi e^{i\int d^dx\left((\partial\phi)^2+\sum g_i(\Lambda_0)\mathcal{O}_i\right)} \tag{2.1}$$

Using a Fourier transformation, we can split these fields into high and low energy modes with respect to a certain scale Λ:

$$\phi(x) = \int_{\Lambda_0>||p||>\Lambda} \hat{\phi}(p)e^{ipx} + \int_{||p||<\Lambda} \hat{\phi}(p)e^{ipx} = \phi_H(x) + \phi_l(x) \tag{2.2}$$

Inserting this into (2.1), we can write the partition function as:

$$Z = \int \prod d\phi_H d\phi_l e^{i\int d^dx(\mathcal{L}_l+\mathcal{L}_H+\mathcal{L}_{int})} \tag{2.3}$$

Where the high energy modes of the field are then considered as heavy fields and then integrated out of the action. At tree level, this amounts to solving the equations of motion for the heavy fields and substituting them in the original action, while loop computations can be achieved with several approaches (see for example [12]) . The result will then be a theory valid up to the energy scale Λ containing new couplings that depend on the cutoff.

$$\int \prod d\phi_l e^{i(S_0+S_W)} = \int \prod d\phi_l e^{i\int d^dx\left((\partial\phi_l)^2+g_i(\Lambda)\mathcal{O}_i\right)} \tag{2.4}$$

Where S_0 takes the form of $\int \mathcal{L}_l$ and the operators $\mathcal{O}$ arise from the integration of the heavy modes with modified couplings g_i. The β functions that determine the running of the couplings are:

$$\beta_i = \Lambda\frac{\partial g_i}{\partial\Lambda} \tag{2.5}$$

2.2 $AdS - CFT$ and Holographic renormalization

The $AdS - CFT$ correspondence [13] states that the generating functional of the field theory is related to the supergravity action by:

$$W[\Phi_0] = \mathcal{S}_{SUGRA}\big|_{\lim\limits_{z\to 0} z^{A(\Delta,d)}\Phi(z,x)=\Phi_0(x)} \tag{2.6}$$

Where $\Phi_0(x)$ is the source field of an operator in the field theory, Δ is the conformal dimension of the operator and A is a function that relates Δ, the dimensions of the theory d and the asymptotic behaviour of the field near the boundary in AdS. This relation is obtained by matching the representations of the superconformal algebra to the mode expansion in the internal space in the supergravity side. For example, chiral primary 1/2 BPS operators $\mathcal{O}_\Delta$ are sourced by scalar fields in AdS that behave asymptotically as $\phi(x,z) = z^{d-\Delta}\phi_0(x)$, where we are using the Poincaré patch in AdS and z is the holographic coordinate. The boundary of AdS is at $z \to 0$.

The process of computing correlation functions in the conformal theory can be done by finding a classical solution of a field with appropriate boundary conditions in AdS and then computing the variation of the on-shell partition function with respect to the boundary field. This computation is usually divergent, but can be regularized by adding some counterterms to the action.

We then find that in order to compute correlation functions, our partition function looks like:

$$Z = e^{i(\mathcal{S}_{SUGRA}+\mathcal{S}_{ct})} \tag{2.7}$$

Wilson renormalization can then be implemented in this holographic setup in quite an intuitive way. We start by proposing that the boundary theory with a cutoff Λ_0 can be identified with a bulk geometry that extends up to a cutoff in the holographic direction $z = z_{\Lambda_0}$. The equivalent of performing a Wilson renormalization up to a new cutoff Λ' is to integrate out the geometry up to a new $z_{(\mu)} < z_{\Lambda_0}$. This results in a boundary term S_B. Notice that due to regularization requirements, the original action in the bulk had already a boundary term, and the process of integrating out gives out a similar bulk theory but with a new boundary and new boundary conditions. This new boundary term can be interpreted as the Wilsonian action we described above.

The new boundary conditions are obtained by demanding the total action to be independent from the cutoff, which leads to a group of equations over the couplings that are interpreted as the RG flow equations.

2.3 Wilson loops in holography

When we work in the $AdS - CFT$ duality, we can define 1/2 BPS Wilson lines in SYM that preserve some of the symmetries of the theory. In order to get an infinitely massive particle in the fundamental representation, we start from a configuration of $N + 1$ D3 branes and we take one of them an infinite distance away. The N remaining branes generate the expected $AdS_5 \times S_5$ geometry, and the original $SU(N + 1)$ symmetry is broken to $SU(N) \times U(1)$. The separated brane is then situated in the boundary of AdS and strings can hang from it to the other D3 branes. From a field theory perspective, what is seen is an infinitely massive particle that generates a Wilson loop [14]

$$\mathcal{W}_{BPS}(\mathcal{C}) = \frac{1}{N}\mathrm{Tr}\,\mathcal{P}\,\left(e^{i\oint_{\mathcal{C}} d\tau\left(\dot{x}^\mu A_\mu + |\dot{x}|\theta_I \Phi^I\right)}\right), \quad \theta^2 = 1 \tag{2.8}$$

In the gravity side what is seen is a string that follows the Wilson line trajectory at the boundary and propagates into the interior of the bulk. In the large N limit, the expectation value reduces to the on-shell action of a Nambu-Goto string:

$$\langle \mathcal{W}(\mathcal{C}) \rangle = e^{-S^{NG}_{on-shell}} \tag{2.9}$$

This interpretation can be extended to other geometries, where we just let a Nambu-Goto string propagate on the bulk, and we compute its action to obtain the expectation value in the dual theory.

When we set two parallel Wilson lines in order to compute the potential between quarks our string hangs from one of the lines at the boundary, reaching a tipping point in the IR where it turns towards the other line as indicated in the figure:

As we are focusing on geometries that present some sort of RG-flow, the holographic dual can have a general metric:

$$ds^2_{10} = \Delta(\theta, r)dr^2 + \Sigma(\theta, r)\eta_{\mu\nu}dx^\mu dx^\nu + d\tilde{\mathcal{M}}_5^2 \tag{2.10}$$

Where Δ and Σ are warping factors that depend on the radial and internal space $\tilde{M}_5$ coordinates. We will focus on metrics that can be put in a domain wall form where the boundary is at $r \to \infty$:

$$ds^2_{10} = \frac{dr^2}{f(r)} + e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + d\tilde{\mathcal{M}}_5^2 \tag{2.11}$$

If we identify the metric coordinates r, t with the string coordinates σ, τ respectively, the Nambu-Goto action is given by:

$$S_{NG} = -\frac{\beta}{2\pi\alpha'}\int d\sigma \frac{e^A}{\sqrt{f}}\sqrt{1 + fe^{2A}(x')^2} \tag{2.12}$$

β is the time extension of the rectangular Wilson loop, and it is taken to $\beta \to \infty$. It is immediate to see that there is a conserved quantity in the action that allows us to compute the equation of motion for the embedding:

$$\pi_x = \frac{\delta S_{NG}}{\delta x'} = \frac{\beta}{2\pi\alpha'}p, \quad x' = -p\frac{e^{-3A}}{\sqrt{f}\sqrt{1 - e^{-4A}p^2}} \tag{2.13}$$

Which can be related to the lowest point in the string profile, as the square root should vanish in order to make $x' \to \infty$.

Using (1.4) and (2.9) we can identify the on-shell Nambu-Goto action of the string with the potential between quarks. Taking into account that the separation L between quarks varies when we modify the boundary value of x, we can now find the force between quarks as:

$$\mathcal{F}_x = \frac{\delta V_{q\bar{q}}}{\delta L} = \frac{\delta S^{NG}}{\delta x_{(\infty)}} = \frac{1}{2\pi\alpha'}p \tag{2.14}$$

Where $x_{(\infty)}$ is the position of the string at the boundary and the distance between quarks can be obtained by integrating x' in (2.13).

Chapter 3

Results

In the thesis we analyze the properties of Wilson loops duals in effective field theories through Wilsonian renormalization. As seen in the introduction, the classical string profile needs knowledge of the full geometry. For large separations, however, we expect the profile of the string to be mostly below some finite value of a certain cutoff $r_{(\mu)}$, where as in section 2.2 we use the subindex (μ) to identify the coordinate where we introduce the cutoff. In the regions close to the boundary (which are analogous to high energies in the field theory) quarks should not feel each other and the profile of the string should be that of a single isolated quark, not deviating much from a straight line configuration. This profile should persist into the interior until far below the cutoff, where it starts turning parallel to the boundary to meet the other end of the string. This allows us to divide the string in the following way:

$$S_{string} = S^{>}_{string} + S^{<}_{NG} = S^{>}_{NG} + S_{c.t.} + S^{<}_{NG} \tag{3.1}$$

$S^{>}_{string}$ is the action of the string above the cutoff, and it includes the Nambu-Goto action $S^{>}_{NG}$ and a counterterm that renormalizes the action $S_{c.t.}$. As the string does not deviate a lot from the straight profile we can expand the action to quadratic order in x' and write that contribution to the action (including counterterms) as:

$$S^{>}_{string} \simeq -\frac{\beta}{2\pi\alpha'}\left[M_{(\mu)} + \frac{1}{2a_{(\mu)}}x^2_{(\mu)}\right] \tag{3.2}$$

Where

$$M_{(\mu)} = \lim_{r_{(\Lambda)}\to\infty}\int_{r_{(\mu)}}^{r_{(\Lambda)}} d\sigma \frac{e^A}{\sqrt{f}} - e^{A(r_{(\mu)})} \tag{3.3}$$

$$a_{(\mu)} = \int_{r_{(\mu)}}^{\infty} \frac{e^{-3A}}{\sqrt{f}} \tag{3.4}$$

The on-shell action should be stationary for small perturbations in the profile that do not modify the boundary conditions. Imposing this at the cutoff fixes the conjugate momentum for the solution below the cutoff value:

$$p = \frac{x_{(\mu)}}{a_{(\mu)}} \tag{3.5}$$

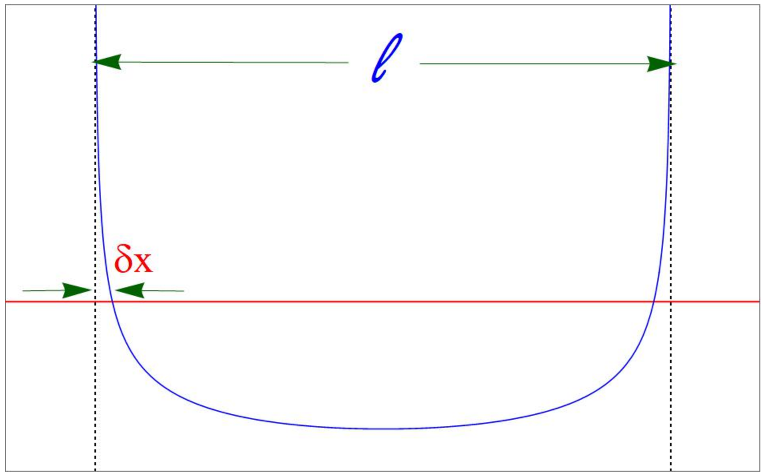

Figure 3.1: Profile of a string dual to a $q\bar{q}$ pair separated a distance ℓ (blue line). The vertical direction corresponds to the holographic radial coordinate, with the asymptotic boundary (UV) at the top. A cutoff is introduced at an IR scale (horizontal red line) and degrees of freedom above the cutoff are integrated out. The separation in the field theory directions between the endpoint of the string at the boundary and at the cutoff is denoted by δx, it corresponds to $x_{(\mu)}$ in the text.

And if we regard $r_{(\mu)}$ as an RG scale, the RG-flow equations of the parameters $M_{(\mu)}$ and $a_{(\mu)}$ are given by:

$$\partial_{r_{(\mu)}} a_{(\mu)} = -\frac{e^{-3A_{(\mu)}}}{\sqrt{f_{(\mu)}}}, \quad \partial_{r_{(\mu)}} M_\mu = -\frac{e^{A_{(\mu)}}}{\sqrt{f_{(\mu)}}}. \tag{3.6}$$

The coefficients cannot be determined from the IR theory, we need to match them with a UV theory or making a fit measuring the force at a separation L since the conditions at the cutoff imply that:

$$p = e^{2A(\sigma_*)}, \quad x_{(\mu)} = a_{(\mu)}, \quad \frac{L}{2} = x_{(\mu)} + p\int_{\sigma_*}^{r_{(\mu)}} \frac{e^{-3A}}{\sqrt{f}\sqrt{1-e^{-4A}p^2}} \tag{3.7}$$

3.1 Effective field theory potentials

3.1.1 IR fixed point

We will first study a theory with an IR fixed point that flows towards a UV CFT through an irrelevant operator of conformal dimension Δ. In this case the force between quarks is given by:

$$\mathcal{F}_x \simeq \frac{R^2}{2\pi\alpha'}\frac{c_0^2}{L^2}\left[1 + \frac{2a_0}{c_0}\left(\frac{c_0 R}{L}\right)^3 + \frac{2a_{\Delta-d}}{c_0}\left(\frac{c_0 R}{L}\right)^{2(\Delta-d)}\right], \quad \Delta - d \neq \frac{3}{2}, \tag{3.8}$$

or

$$\mathcal{F}_x = \frac{R^2}{2\pi\alpha'}\frac{c_0^2}{L^2}\left[1 + \frac{2\tilde{a}_0}{c_0}\left(\frac{c_0 R}{L}\right)^3 + \frac{2a_{3/2}}{c_0}\left(\frac{c_0 R}{L}\right)^3 \log\left(\frac{c_0^2 R^2}{p_0 L^2}\right)\right], \quad \Delta - d = \frac{3}{2}, \tag{3.9}$$

The coefficients are given by:

$$
\begin{aligned}
& c_0=\frac{2\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}, \quad a_0=2a_{(\mu)}-\frac{2}{3}e^{-3r_{(\mu)}/R}\left(1-\frac{9}{4}\frac{\alpha^2}{2(\Delta-d)-3}e^{-2(\Delta-d)(r_{(M)}-r_{(\mu)})/R}\right), \\
& a_{\Delta-d}=\frac{\alpha^2}{2}e^{-2(\Delta-d)r_{(M)}/R}\frac{(\Delta-d)\sqrt{\pi}\Gamma\left(\frac{3}{4}-\frac{\Delta-d}{2}\right)}{2\Gamma\left(\frac{5}{4}-\frac{\Delta-d}{2}\right)}, \\
& \tilde{a}_0=2a_{(\mu)}-\frac{2}{3}e^{-3r_{(\mu)}/R}\left(1-\frac{9\alpha^2}{4}e^{-3(r_{(M)}-r_{(\mu)})/R}\frac{r_{(\mu)}}{R}\right), \\
& a_{3/2}=-\frac{3\alpha^2}{4}e^{-3r_{(M)}/R}, \quad p_0=2e^{-2/3}.
\end{aligned} \tag{3.10}
$$

Where $a_{(\mu)}$ is defined as in (3.4), R is the AdS radius of the space corresponding to the IR fixed point and $r_{(M)}$ denotes the point where the geometry deviates significantly from AdS. The first term in both expressions is determined by the conformal length-dependence of the IR CFT, and the last one is given by the contribution from the irrelevant operator that deforms the CFT. The second term is a bit less intuitive, it appears as a consequence of adding the boundary action in the effective field theory. From a defect theory perspective, this introduces a double-trace deformation on the string. The operator producing this double trace deformation can be identified with the electric field strength in the x direction E_x. When we are close to the IR fixed point, conformal invariance fixes the contribution of this deformation to the potential as.

$$
\Delta V_{q\bar{q}} \propto c_{E^2}\left\langle E_x^2\right\rangle \sim \frac{c_{E^2}}{L^4}. \tag{3.11}
$$

3.1.2 Confining theories

We also applied this method to top-down confining gravity duals, namely Witten QCD and the Klevanov-Strassler model. As stated in [11], field theories are confining if the dual geometry ends at some point in the IR (we look for a collapsing cycle deep into the bulk). For both of these geometries we find a cycle that collapses, and the string becomes almost paralell to the field theory directions near the end of the geometry as long as L is large enough.

The WQCD model can be written as in (2.11) with coefficients

$$
e^{2A(r)}=\left(\frac{r}{4R}\right)^6, \quad f(r)=1-\frac{{r_{(M)}}^{12}}{r^{12}}, \tag{3.12}
$$

and we find that the force between quarks is given by:

$$
\mathcal{F}_x=\sigma_s\left(1+q_M e^{-ML}\right), \tag{3.13}
$$

where the string tension and the coefficient of the exponential term are

$$
\sigma_s=\frac{p_M}{2\pi\alpha'}=\frac{2}{27\pi}\lambda_{YM}M^2, \quad q_M=6\sqrt{3}e^{-\frac{\pi}{2\sqrt{3}}}e^{-a_0/c_0}. \tag{3.14}
$$

The coefficient $c_0=\frac{2}{3}\left(\frac{4R}{r_{(\mu)}}\right)$ only depends on the characteristics of the geometry (R is the curvature radius and $r_{(M)}$ is the lowest point of the geometry), and $a_0=2a_{(\mu)}-\left(\frac{4R}{r_{(\mu)}}\right)^8$.

The KS model has the metric of a deformed conifold, and has a collapsing cycle in its internal manifold. Near the end of the geometry, the metric can be written in the ansatz (2.11) with:

$$e^{2A(r)} = h^{-1/2}(\varepsilon^{-2/3}r), \quad f(r) = 6h^{-1/2}(\varepsilon^{-2/3}r)(K(\varepsilon^{-2/3}r))^2. \tag{3.15}$$

Where the functions h and k are given near the limit where the geometry collapses by:

$$K(\tau) \simeq \left(\frac{2}{3}\right)^{1/3}\left(1-\frac{\tau^2}{10}\right), \quad h(\tau) = \left(\frac{2}{3}\right)^{1/3}\alpha\left(\hat{h}_M - \frac{\tau^2}{6}\right), \tag{3.16}$$

The force computation can be written as in (3.13) with coefficients:

$$\sigma_s = \frac{p_M}{2\pi\alpha'} = \frac{3^{1/6}}{2\pi}\hat{h}_M^{3/2}\lambda_{YM}M^2, \quad q_M = \frac{\hat{h}_M}{3}e^{-a_0/c_0}. \tag{3.17}$$

In this case, the glueball mass scale is defined as:

$$M = \frac{\varepsilon^{-2/3}}{c_0 p_M} = \frac{1}{(12)^{1/6}\hat{h}_M}\frac{\varepsilon^{2/3}}{g_s N_c \alpha'}. \tag{3.18}$$

The first contribution to the force is the expected one for confining theories. The exponential correction can be interpreted if we look at the string as a flux tube in the IR with sources at the points where the string curves towards the boundary. This allows us to identify that contribution to the force with an internal massive mode corresponding to excitations of the string along the holographic direction.

3.2 Quark moving through plasma

The holographic dual of a strongly coupled plasma in 3+1 dimensions is realized by a 5 dimensional geometry with an event horizon that extends along 4 dimensions, hence we will work with a background metric of the form

$$ds^2 = G_{MN}dx^M dx^N = G_{zz}(z)dz^2 + G_{tt}(z)dt^2 + G_{xx}(z)\delta_{ij}dx^i dx^j \tag{3.19}$$

We will pick our coordinates so that there is a horizon characterized by $G_{tt}(z_h) = 0$ with the boundary located at $z \to \infty$

Our quark will be modelled by a NG string and we will only consider movement in one direction on the field theory.

We can consider both a static trajectory in the boundary with small perturbations or a high speed moving quark. In both cases the lagrangian can be expanded to second order as:

$$\mathcal{L}_{NG} = L_0(v) + L_{1z}(v)X' - L_{1t}(v)\dot{X} - \frac{1}{2}g_v(z)\left(\dot{X}\right)^2 + \frac{1}{2}f_v(z)\left(X'\right)^2 \tag{3.20}$$

Where v is the velocity of the quark moving through the plasma (we will supress the subindex from now on), the first order terms are total derivatives so they do not contribute to the equations of motion for the fluctuations. The two L_{1i} coefficients are also linear in v, which means they vanish

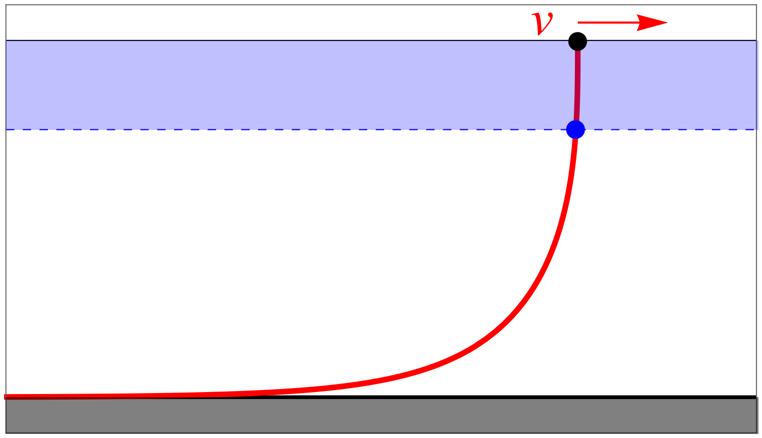

Figure 3.2: The holographic dual of a heavy quark moving at speed v is a string (red curve) ending at the asymptotic boundary at the position of the quark (black dot). The strings extends from the asymptotic boundary at the top to the black brane horizon at the bottom of the figure. A cutoff (dashed blue line) is introduced and the shaded region between the boundary and the cutoff is "integrated out". One is left with the string in the region between the cutoff and the horizon and determined boundary conditions for the endpoint of the string at the cutoff (blue dot).

for a slow moving quark that is perturbed around a rest frame. The equations of motion can be written in general as:

$$(fX')' - g\ddot{X} = 0. \tag{3.21}$$

With

$$g(z) = (|G_{tt}|G_{xx}G_{zz})^{1/2}\frac{\left(|G_{tt}|G_{xx} - p_0^2\right)^{1/2}}{\left(|G_{tt}| - G_{xx}v^2\right)^{3/2}}, \quad f(z) = (|G_{tt}|G_{xx}G_{zz})^{-1/2}\frac{\left(G_{tt}G_{xx} - p_0^2\right)^{3/2}}{\left(|G_{tt}| - G_{xx}v^2\right)^{1/2}}. \tag{3.22}$$

These solutions can be found by expanding the profile of the string to the order in time derivatives, the first orders are given by

$$X^{(0)}(t,z) = x(t) + p^{(0)}(t)a(z), \quad X^{(1)}(t,z) = p^{(1)}(t)a(z), \quad a(z) = \int_0^z \frac{du}{f(u)}. \tag{3.23}$$

$$X^{(n)}(t,z) = p^{(n)}(t)a(z) + \int_0^z \frac{du}{f(u)}\int_{z_{(\mu)}}^u dvg(v)\ddot{X}^{(n-2)}(t,v). \tag{3.24}$$

$p^{(n)}(t)$ are integration constants fixed by the boundary conditions:

$$X(t, z=0) = x(t), \quad \partial_z X\Big|_{z=z_{(\mu)}} = \frac{1}{f(z_{(\mu)})}\left(p^{(0)}(t) + p^{(1)}(t) + p^{(2)}(t) + \cdots\right) \equiv \frac{p}{f(z_{(\mu)})}. \tag{3.25}$$

The force can be determined from these solutions. When we work with a fast moving quark, the linear terms in the Lagrangian add a contribution:

$$\mathcal{F}_x^v = -T_s p_0. \tag{3.26}$$

Where we $T_s = 1/(2\pi\alpha')$ is the tension string. The force due to perturbations can be computed in both cases with the coefficients f and g from the quadratic terms in the expansion.

$$F(t) = p + \ddot{x}A(0) + \dddot{p}B(0) + O(\partial_t^4 x, \partial_t^4 p). \tag{3.27}$$

Where we have defined the functions

$$A(z) = \frac{R^2}{z_{(\mu)}} + \int_{z_{(\mu)}}^{z} \left(g(v) - \frac{R^2}{v^2} \right), B(z) = \int_{z_{(\mu)}}^{z} g(v)a(v). \tag{3.28}$$

Once the general theory parameters are defined, we can apply the effective field formalism described previously. The action above the cutoff can be expressed as:

$$S_{UV} \simeq T_s \int dt \left[M_{(\mu)} - \frac{1}{2}K_{(\mu)}\dot{x}^2 - \frac{1}{2a_{(\mu)}}(x_{(\mu)} - x)^2 + \frac{1}{2}m_{(\mu)}(\dot{x}_{(\mu)} - \dot{x})^2 - \kappa_{(\mu)}\ddot{x}(x_{(\mu)} - x) + O(\partial_t^4 x, \partial_t^4 x_{(\mu)}) \right]. \tag{3.29}$$

Where we have defined the coefficients as

$$M_{(\mu)} = \frac{R^2}{z_{(\mu)}},\ K_{(\mu)} = A(0),\ a_{(\mu)} = a(z_{(\mu)}),\ \ m_{(\mu)} = \frac{1}{a_{(\mu)}^2} \int_0^{z_{(\mu)}} dvg(v)a(v)^2,\ \ \kappa_{(\mu)} = \frac{1}{a_{(\mu)}} \int_0^{z_{(\mu)}} dvg(v)a(v). \tag{3.30}$$

with RG flow equations:

$$\begin{aligned}
\partial_{z_{(\mu)}} M_{(\mu)} &= -\frac{R^2}{z_{(\mu)}^2}, \\
\partial_{z_{(\mu)}} K_{(\mu)} &= -g(z_{(\mu)}), \\
\partial_{z_{(\mu)}} a_{(\mu)} &= \frac{1}{f(z_{(\mu)})}, \\
\partial_{z_{(\mu)}} m_{(\mu)} &= -\frac{2}{f(z_{(\mu)})}\frac{m_{(\mu)}}{a_{(\mu)}} + g(z_{(\mu)}), \\
\partial_{z_{(\mu)}} \kappa_{(\mu)} &= -\frac{1}{f(z_{(\mu)})}\frac{\kappa_{(\mu)}}{a_{(\mu)}} + g(z_{(\mu)}).
\end{aligned} \tag{3.31}$$

The force can then be expressed in terms of these coefficients:

$$F(t) \simeq \frac{1}{a_{(\mu)}}(x_{(\mu)} - x) + (m_{(\mu)} - \kappa_{(\mu)})\ddot{x}_{(\mu)} + (K_{(\mu)} - m_{(\mu)} + 2\kappa_{(\mu)})\ddot{x} + O(\partial_t^4 x, \partial_t^4 x_{(\mu)}). \tag{3.32}$$

This formalism can be applied to a theory with an IR fixed point. In this case the physics are dominated by the IR conformal theory and we can approximate the holographic dual by an AdS_5 black brane.

$$G_{tt}(z) = -\frac{R^2}{z^2}h(z),\quad G_{zz}(z) = \frac{R^2}{z^2 h(z)},\quad G_{xx} = \frac{R^2}{z^2},\quad h(z) = 1 - \frac{z^4}{z_h^4}. \tag{3.33}$$

Then for the slow moving quark the force is given by:

$$F(t) = \frac{R^2}{z_h^3} \sum_{i=1}^{3} F_i \,(z_h \partial_t)^i x + O(\partial_t^4 x). \tag{3.34}$$

With coefficients

$$
\begin{aligned}
F_1 &= \frac{s_1}{\widehat{a}_{(\mu)}} = -1, \\
F_2 &= \widehat{K}_{(\mu)} + \widehat{\kappa}_{(\mu)} + \frac{s_2}{\widehat{a}_{(\mu)}} = \widehat{a}_{(\mu)} + \widehat{K}_{(\mu)} + H_2(u_{(\mu)}), \\
F_3 &= (\widehat{m}_{(\mu)} - \widehat{\kappa}_{(\mu)})s_1 + \frac{s_3}{\widehat{a}_{(\mu)}} = \widehat{a}_{(\mu)}(2\widehat{\kappa}_{(\mu)} - \widehat{a}_{(\mu)}) - (c_1(u_{(\mu)}) + 2\widehat{a}_{(\mu)})H_2(u_{(\mu)}) + H_3(u_{(\mu)}).
\end{aligned}
\tag{3.35}
$$

Where $c_1(u_{(\mu)}) = -\frac{1}{4}log(1 - u_{(\mu)}^4)$ and the other coefficients can be written as a function of the cutoff:

$$
\begin{aligned}
\widehat{a}_{(\mu)} &= \frac{1}{4}\log\frac{1+u_{(\mu)}}{1-u_{(\mu)}} - \frac{1}{2}\tan^{-1}u_{(\mu)} = \frac{1}{2}\left(\tanh^{-1}u_{(\mu)} - \tan^{-1}u_{(\mu)}\right) + a_{UV}, \\
\widehat{K}_{(\mu)} &= \frac{1}{u_{(\mu)}} - \widehat{a}_{(\mu)} + K_{UV}, \\
\widehat{\kappa}_{(\mu)} &= -\frac{1}{u_{(\mu)}} + \frac{\widehat{a}_{(\mu)}}{2} + \frac{1}{2\widehat{a}_{(\mu)}}\tanh^{-1}(u_{(\mu)}^2) + \frac{\kappa_{UV}}{\widehat{a}_{(\mu)}}, \\
H_2(u_{(\mu)}) &= -\frac{1}{u_{(\mu)}} + 1, \\
H_3(u_{(\mu)}) &= \frac{1}{4}(\pi - \log 4) - \frac{c_1(u_{(\mu)})}{u_c} + \widehat{a}_{(\mu)} - \frac{1}{2}\tan^{-1}u_{(\mu)} + \frac{1}{4}\left(2\log(1+u_{(\mu)}) - 3\log\left(1+u_{(\mu)}^2\right)\right).
\end{aligned}
\tag{3.36}
$$

The integration constants a_{UV}, K_{UV}, κ_{UV} depend on the specific geometry of the UV region, they are zero for the AdS_5 black brane, and they could be fixed with lattice computations or experimental data. If we identify the geometry parameters with the corresponding field theory constants using the holographic dictionary

$$
T_s R^2 = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}, \quad z_h = \frac{1}{\pi T}. \tag{3.37}
$$

The force acting on the heavy quark is, to third order in derivatives of the trajectory

$$
\mathcal{F}_x \simeq \frac{\sqrt{\lambda}}{2\pi}\left(-(\pi T)^2\partial_t x + \pi T\, F_2\, \partial_t^2 x + F_3\, \partial_t^3 x\right) + O(\partial_t^4 x). \tag{3.38}
$$

Where the coefficient of the term proportional to $\partial_t x$ agrees with the drag force [15, 16]. The coefficient proportional to the acceleration can be interpreted as a thermal correction to the mass of the quark and the coefficient of the jerk $\partial_t^3 x$ can be interpreted as a combination of the Abraham-Lorentz force produced by Larmor radiation emission ([17, 18] and a viscosity contribution from the surrounding plasma. The fast moving quark can be similarly obtained

$$
\mathcal{F}_x \simeq \frac{\sqrt{\lambda}}{2\pi}\left(-(\pi T)^2\gamma v - (\pi T)^2\gamma^3\partial_t x + \pi T\, F_2\, \gamma^{7/2}\partial_t^2 x + F_3\, \gamma^4\partial_t^3 x\right) + O(\partial_t^4 x). \tag{3.39}
$$

The γ factors appearing in higher derivative terms imply that this expansion requires time derivatives to be much smaller than the temperature for very fast quarks $\partial_t \ll \gamma^{-1/2}\pi T$.

3.3 Symmetries of the Wilson loop

3.3.1 Wilson loop reparametrization in holography

Wilson lines are determined by the holonomy of the gauge field along a closed curve $\mathcal{C}$. This curve can be parametrized in any way, resulting in the zig-zag symmetry [19]. For half BPS Wilson loops the coupling to the scalars breaks this symmetry, but it is recovered at strong coupling [20]. In this section we will discuss how does our Wilson renormalization mechanism in the holographic dual affect this symmetry, we will start from the simple case of a straight Wilson line along a spatial direction.

We will describe the dual fundamental string with the renormalized Polyakov action in the Poincaré patch in dimensionless coordinates

$$\begin{aligned} ds^2 &= G_{MN}dx^M dx^N = \frac{L^2}{z^2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right). \\ S_P &= \frac{T_s L^2}{2}\int d^2\sigma\sqrt{h}h^{ab}g_{ab} + \phi_0\chi_E - T_s L^2\int_{\sigma=\epsilon} d\tau\sqrt{h_{\tau\tau}}. \end{aligned} \tag{3.40}$$

Where g_{ab} and h_{ab} the induced and worldsheet metrics respectively. We will work with worldsheet coordinates $(\tau,\sigma)$and embedding functions $X^M(\tau,\sigma)$. For the straight string we will use $X^1 = X, \quad X^z = Z$. χ_E is the Euler characteristic and has a coefficient proportional to the constant dilaton $\phi_0 = \log g_s$, and the last contribution is a counterterm to regularize the action. We will introduce an arbitrary parametrization of the line at the boundary

$$\lim_{\sigma\to 0} X = x_0(\tau), \tag{3.41}$$

without modifying the shape of the string in the embedding space. This results in the non trivial embedding functions $X = X(\tau,\sigma)$, $Z = Z(\tau,\sigma)$. Diffeomorphisms and Weyl transformations allow us to fix the gauge of the worldsheet metric to the conformal gauge $h_{ab} = \frac{1}{\sigma^2}\delta_{ab}$. Then the embedding functions must be compatible with the induced metric and solve the equations of motion.

$$g_{ab} - \frac{1}{2}h_{ac}h^{bd}g_{cd} = 0. \tag{3.42}$$

$$\frac{1}{\sqrt{h}}\partial_a\left(\sqrt{h}h^{ab}\frac{\partial_b X^M}{Z^2}\right) + \frac{2}{Z}h^{ab}g_{ab}\delta_z^M = 0. \tag{3.43}$$

The first condition is met by any conformally flat metric and both of them are solved by:

$$Z' = \dot{X}, \quad X' = -\dot{Z}, \quad X'' + \ddot{X} = 0, \quad Z'' + \ddot{Z} = 0. \tag{3.44}$$

When the derivatives of x_0 are small compared to $1/\sigma$ the solutions to these equations can be expanded as:

$$\begin{aligned} X &= \cos\left(\sigma\frac{d}{d\tau}\right)x_0(\tau) = x_0 - \frac{1}{2}\sigma^2\ddot{x}_0 + \frac{1}{24}\sigma^4 x_0^{(4)} + \cdots, \\ Z &= \sin\left(\sigma\frac{d}{d\tau}\right)x_0(\tau) = \sigma\dot{x}_0 - \frac{1}{6}\sigma^3\dddot{x}_0 + \frac{1}{120}\sigma^5 x_0^{(5)} + \cdots. \end{aligned} \tag{3.45}$$

The conformal factor in the induced metric can be given in terms of the Schwarzian derivative:

$$\Omega = \frac{1}{\sigma^2} - \frac{2}{3}\{x_0, \tau\} + \sigma^2\left(\frac{1}{15}\partial_\tau^2\{x_0,\tau\} + \frac{4}{15}(\{x_0,\tau\})^2\right) + \cdots .$$
$$\{x_0, \tau\} = \frac{\dddot{x}_0}{\dot{x}_0} - \frac{3}{2}\left(\frac{\ddot{x}_0}{\dot{x}_0}\right)^2 . \tag{3.46}$$

The Schwarzian is invariant under $GL(2,\mathbb{R})$ reparametrizations of the form

$$x_0(\tau) \longrightarrow \frac{ax_0 + b}{cx_0 + d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc \neq 0. \tag{3.47}$$

Which are the symmetries induced by AdS_2 isometries.

Full reparametrization invariance on the boundary can be reproduced however in the following way: We start performing a worldsheet diffeomorphism

$$\tau = \tau(\bar{\tau}, \bar{\sigma}), \quad \sigma = \sigma(\bar{\tau}, \bar{\sigma}) \tag{3.48}$$

where $\bar{\sigma} = Z, \quad \bar{\tau} = X$. This transforms the induced and worldsheet metrics as:

$$\bar{g}_{ab} = \frac{1}{\bar{\sigma}^2}\delta_{ab}, \quad \bar{h}_{ab} = \bar{\Omega}\delta_{ab}, \tag{3.49}$$

Then we can use a Weyl transformation to recover the original worldsheet metric, which shows that we can use a conformal transformation to produce any arbitrary reparametrization from the trivial embedding.

3.3.2 Induced anomalies in the cutoff action

When we work with a non-trivial embedding of the string, we have two possible choices for the cutoff, it can either be in the worldsheet coordinate $\sigma = 1/(L\Lambda)$ or introduce a cutoff in the radial coordinate $z = 1/(L\Lambda)$. If we take the former option, we obtain a cutoff action

$$S_\Lambda = T_s L^2 \int d\tau \left(-L\Lambda - \frac{2}{3}\frac{1}{L\Lambda}\{x_0,\tau\} + \frac{1}{3}\frac{1}{(L\Lambda)^3}\left(\frac{1}{15}\partial_\tau^2\{x_0,\tau\} + \frac{2}{5}(\{x_0,\tau\})^2\right) + \cdots\right) + \frac{\phi_0}{2\pi}\int d\tau L\Lambda. \tag{3.50}$$

which is not invariant under reparametrizations. However from the bulk perspective, the string extended beyond this cutoff is reparametrization invariant up to boundary terms. This is compensated by the action at the cutoff, allowing us to identify the terms depending on the Schwarzian as a reparametrization anomaly at the cutoff. On the other hand, a cutoff in the radial coordinate is readily identified with an energy scale in the field theory dual. Fixing this radial cutoff implies integrating the string action up to a value of the coordinate given by

$$Z(\tau, \sigma_\Lambda(\tau)) = 1/(L\Lambda) \tag{3.51}$$

The integrated action up to this value is just a reparametrization of the worldline coordinate $d\tau \to d\tau_\Lambda = d\tau \dot{x}_\Lambda$ where

$$x_\Lambda = x_0 + \frac{1}{2}\frac{1}{(L\Lambda)^2}\frac{\ddot{x}_0}{\dot{x}_0} + \frac{1}{72}\frac{1}{(L\Lambda)^4}\frac{4\ddot{x}_0\,\dddot{x}_0 - \dot{x}_0 x_0^{(4)}}{(\dot{x}_0)^5} + \cdots . \tag{3.52}$$

However, the contribution from the Ricci scalar in the Euler characteristic gives a non-trivial contribution

$$S_\Lambda = T_s L^2 \int d\tau_\Lambda \left(-L\Lambda\right) + \frac{\phi_0}{2\pi}\int d\tau_\Lambda \left(L\Lambda + \frac{2}{3}\frac{1}{L\Lambda}\{t(\tau_\Lambda),\tau_\Lambda\} + \cdots\right). \tag{3.53}$$

Where t is the inverse of x_0. This result can also be reproduced performing a worldsheet diffeomorphism (3.48). This indicates that there is an anomaly at the cutoff compensating the non invariance of the string under worldsheet diffeomorphisms, but as we saw before this can be removed using a Weyl transformation, indicating these transformations have an associated anomaly in such a way that both anomalous terms cancel out.

This analysis can also be performed for nonzero temperatures, where we have a black brane AdS_{d+1} solution.

$$ds^2 = \frac{L^2}{z^2}\left(\frac{dz^2}{f(z)} - f(z)(dx^0)^2 + \delta_{ij}dx^i dx^j\right), \quad f(z) = 1 - \left(\frac{z}{z_H}\right)^d. \tag{3.54}$$

Picking a new radial coordinate determined by

$$du = \frac{dz}{\sqrt{f(z)}} \tag{3.55}$$

and rescaling the coordinates $u \to u_H u, \quad z \to z_H z, \quad x^\mu \to u_H x^\mu$ yields the metric

$$ds^2 = \frac{\tilde{L}^2}{z(u)^2}\left(du^2 - f[z(u)](dx^0)^2 + \delta_{ij}dx^i dx^j\right), \quad f(z) = 1 - z^d, \quad z(u)^d = I_u^{-1}\left(\frac{1}{d},\frac{1}{2}\right), \tag{3.56}$$

Where $I_u^{-1}(a,b)$ is the inverse of the regularized incomplete Beta function and $\tilde{L} = Lu_H/z_H = B\left(\frac{1}{d},\frac{1}{2}\right)L/d$, $B_x(a,b)$ being the incomplete Beta function. Using the embedding

$$X^1 \equiv X = \tau, \quad X^u \equiv U = \sigma, \quad X^M = 0, M \neq 1, u. \quad h_{ab} = \frac{1}{\sigma^2}\delta_{ab}, \tag{3.57}$$

the problem is now analogous to the zero temperature case. One distinct feature, however, is that when fixing the radial cutoff in the geometry there is a physical cutoff at the horizon of the black brane. The effective action can then be integrated all the way to this cutoff yielding a Schwarzian term:

$$S_{Sch} = \frac{\phi_0}{12\pi^2}\frac{B\left(\frac{1}{d},\frac{1}{2}\right)}{T}\int d\bar{\tau}\,\{t(\bar{\tau}),\bar{\tau}\} \tag{3.58}$$

Where we have restored the units $x_0 \to x_0/u_H$ and $\bar{\tau} \to \bar{\tau}/u_H$

3.3.3 Applications to Circular Wilson loops

In the circular Wilson loop we will work using polar coordinates

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 + dr^2 + r^2 d\theta^2 + \sum_{\mu=3}^{d-1}(dx^\mu)^2\right). \tag{3.59}$$

and the generalized embedding

$$\Theta = q\tau + \theta(\tau,\sigma), \quad R = \frac{r_0}{\cosh S}, \quad Z = r_0 \tanh S, \quad S = q\sigma + s(\tau,\sigma). \tag{3.60}$$

Where the periodicity of τ is $2\pi p$ and p, q are nonzero integers, both θ and s are periodic functions in τ. Notice that setting $q \to 1$, $\theta \to 0$, $s \to 0$ we obtain the induced metric is that of global AdS_2 in conformally flat coordinates. We will select the string metric to be precisely this one:

$$h_{ab} = \frac{1}{\sinh^2 \sigma}\delta_{ab}. \tag{3.61}$$

the embedding functions have to satisfy the same set of equations as in the flat case (3.44), however in this case they have to satisfy periodic boundary conditions. The solutions can be expanded as:

$$\begin{aligned} \Theta &= \cos\left(\sigma \frac{d}{d\tau}\right)\Theta_0(\tau) = \Theta_0 - \frac{1}{2}\sigma^2\ddot{\Theta}_0 + \frac{1}{24}\sigma^4\Theta_0^{(4)} + \cdots, \\ S &= \sin\left(\sigma \frac{d}{d\tau}\right)\Theta_0(\tau) = \sigma\dot{\Theta}_0 - \frac{1}{6}\sigma^3\dddot{\Theta}_0 + \frac{1}{120}\sigma^5\Theta_0^{(5)} + \cdots. \end{aligned} \tag{3.62}$$

As in the straight case, this leads to a conformal metric, this time with a conformal factor:

$$\Omega = \frac{1}{\sigma^2} - \frac{2}{3}\left\{\tan\frac{\Theta_0}{2}, \tau\right\} + \sigma^2\left(\frac{1}{15}\partial_\tau^2\left\{\tan\frac{\Theta_0}{2}, \tau\right\} + \frac{4}{15}\left\{\tan\frac{\Theta_0}{2}, \tau\right\}^2\right) + \cdots. \tag{3.63}$$

Where the Schwarzian terms are now

$$\left\{\tan\frac{\Theta_0}{2}, \tau\right\} = \{\Theta_0, \tau\} + \frac{1}{2}\dot{\Theta}_0^2. \tag{3.64}$$

Which is invariant under boundary reparametrizations

$$e^{i\Theta_0(\tau)} \longrightarrow \frac{\alpha e^{i\Theta_0} + \bar{\beta}}{\beta e^{i\Theta_0} + \bar{\alpha}}, \quad \alpha, \beta \in \mathbb{C}, \; |\alpha|^2 - |\beta|^2 = 1. \tag{3.65}$$

As in 3.3.1, this can be interpreted as the boundary limit of the isometry transformations of the global AdS_2 metric $SU(1,1)$. The analysis from this point onwards is analogous to the straight Wilson line, replacing the Schwarzian in (3.50) and (3.53) with (3.64) for the cutoff in the worldsheet coordinate σ and the geometry cutoff respectively.

3.3.4 Polyakov Loop

We can also apply this method to a finite temperature Polyakov loop. The holographic dual is a string wrapped around the Euclidean time direction of a Wick rotated AdS_{d+1} black brane with metric:

$$ds^2 = \frac{L^2}{z^2}\left(\frac{dz^2}{f(z)} + f(z)dt_E^2 + \delta_{ij}dx^i dx^j\right), \quad f(z) = 1 - \left(\frac{z}{z_H}\right)^d. \tag{3.66}$$

The euclidean time direction has periodicity $\beta = 1/T$. The string has the topology of a disk as in the circular case. In this case the conformally flat metric is obtained through a change of variables

$$du = \frac{dz}{f(z)}, \quad u = \frac{1}{2\pi z_H T}r, \quad t_E = \frac{1}{2\pi z_H T}\theta \tag{3.67}$$

We can then choose a non trivial embedding

$$\Theta = q\tau + \theta(\tau, \sigma), \quad R = S = q\sigma + s(\tau, \sigma). \tag{3.68}$$

that coupled with an embedding

$$h_{ab} = \frac{1}{\sinh^2 \sigma} \delta_{ab}, \tag{3.69}$$

yields the same solutions for the embeddings and the symmetries of the worldsheet metric as the ones we found in the circular Wilson loop, obtaining the same result of a Schwarzian action for the worldsheet diffeomorphism.

Articles (Impact Factor)

This thesis is based on the following papers:

- Effective long distance $q\bar{q}$ potential in holographic RG flows [21]
- Holographic Wilsonian renormalization of a heavy quark moving through a strongly coupled plasmaHolographic Wilsonian renormalization of a heavy quark moving through a strongly coupled plasma [22]
- Holographic RG flow and reparametrization invariance of Wilson loops [23]

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Chapter 4

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Effective long distance $q\bar{q}$ potential in holographic RG flows

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Abstract: We study the $q\bar{q}$ potential in strongly coupled non-conformal field theories with a non-trivial renormalization group flow via holography. We focus on the properties of this potential at an inter-quark separation L large compared to the characteristic scale of the field theory. These are determined by the leading order IR physics plus a series of corrections, sensitive to the properties of the RG-flow. To determine those corrections, we propose a general method applying holographic Wilsonian renormalization to a dual string. We apply this method to examine in detail two sets of examples, $3+1$-dimensional theories with an RG flow ending in an IR fixed point; and theories that are confining in the IR, in particular, the Witten QCD and Klebanov-Strassler models. In both cases, we find corrections with a universal dependence on the inter-quark separation. When there is an IR fixed point, that correction decays as a power $\sim 1/L^4$. We explain that dependence in terms of a double-trace deformation in a one-dimensional defect theory. For a confining theory, the decay is exponential $\sim e^{-ML}$, with M a scale of the order of the glueball mass. We interpret this correction using an effective flux tube description as produced by a background internal mode excitation induced by sources localized at the endpoints of the flux tube. We discuss how these results could be confronted with lattice QCD data to test whether the description of confinement via the gauge/gravity is qualitatively correct.

Keywords: Gauge-gravity correspondence, Wilson, 't Hooft and Polyakov loops, Effective Field Theories

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Contents

1 Introduction

The gauge/gravity correspondence [1–3], or holographic duality, has been used extensively as a phenomenological tool to describe properties of strongly coupled systems in QCD and condensed matter (see [4–9] for reviews on the topic). In most cases the gravity dual corresponds to a theory which is microscopically different from the actual system of interest, but whose properties at low energy/large distance compared to some characteristic scale can be qualitatively similar. Since many relevant observables are mostly sensitive to the long distance physics, dissimilarities at high energy/small distances are frequently inconsequential. In the holographic dual description this means that only some part of the geometry is of relevance for those observables. More precisely, while the geometry has an asymptotic boundary that is identified with the ultraviolet (UV) of the field theory, infrared (IR) dynamics are controlled by the deep interior of the geometry and long distance observables are mostly sensitive to this region. The problem of restricting the holographic description to the low energy effective theory has been approached multiple times in different contexts, e.g. [10–19].

In the holographic "Wilsonian" renormalization group (RG) flow approach of [16, 17] the effective IR description is obtained by introducing a cutoff in the holographic radial coordinate and "integrating out" degrees of freedom between the asymptotic boundary and the cutoff. This results in a description consisting of the dual theory below the cutoff plus a boundary action that determines the boundary conditions of the fields at the cutoff. The RG flow equations are obtained from the condition that the full on-shell action has to be independent of the cutoff. The boundary action is a functional of the values of the field at the cutoff, and, by mapping these values to operators in the field theory dual, it is interpreted as introducing multitrace deformations in the effective theory at the cutoff scale. The RG flow equations then become equations for the multi-trace couplings.

A particularly significant set of observables in gauge theories for which an IR effective description would be useful are Wilson loops. Their holographic dual description is a Nambu-Goto string with endpoints attached to the asymptotic boundary of the dual geometry [20, 21]. Even though this is a fairly simple setup, its holographic Wilsonian RG flow has not been worked out.[1] We will partially fill in the blank by studying the expectation value of a Wilson loop corresponding to two static sources separated a fixed distance, much larger than the characteristic scale of the theory. This is equivalent to computing the quark-antiquark potential. In this configuration, most of the profile of the dual string remains in the deep interior of the geometry and is mostly sensitive to IR physics. This observation will allow us to use the inter-quark distance L as an expansion parameter to approximate the string profile and its energy. Applying the holographic Wilsonian RG flow approach, we will derive the effective IR behavior of the potential and determine its most relevant long distance corrections.

We will use this method to analyze two different types of holographic constructions with well understood IR geometries. The first type is dual to a strongly coupled field theory that possesses an IR fixed point, such that at higher energy scales the theory flows away from the IR fixed point in a way determined by an irrelevant scalar deformation. The second type is dual to a confining theory. In the first case, the characteristic scale appears in the coefficient of the irrelevant operator. At energies much lower than this scale, the theory is very close to being conformal, with only small corrections induced by the flow. We will introduce a cutoff much below this scale, in such a way that the dual geometry is close to an AdS dual to the IR fixed point. In the case of a confining theory, the cutoff will be introduced at a scale much larger than the mass gap of the theory. For the confining theory we will use duals with explicit string theory constructions, the Witten QCD (WQCD) [24] and Klebanov-Strassler (KS) [25] models. For the case of an IR fixed point there are several examples in five-dimensional supergravity [26–31]. In principle these can be lifted to ten dimensions using general reduction formulas [28, 32–39], but the examples where this has been carried out explicitly (e.g. [40]) are even scarcer and the geometry turns out to be more complicated than what we will be considering in this work. In both those two families of models, the effective field theory approach we develop will allow us to determine generic properties of the potential, independent of the details of the UV behavior of the dual field theories.

For a flow with an IR fixed point, we observe two types of corrections. One is produced by the non-trivial RG flow, and depends on the dimension of the leading irrelevant operator

[1] In [22] it was suggested that the holographic RG flow will be given by a mean-curvature RG flow of the type described in [23]. Although we do not discard that a map to a description of this form might exist, our results seem to correspond to a different type of flow.

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that drives the flow away from the long distance conformal field theory (CFT). The other correction has a universal form, in the sense that it is independent of the dimensions of the leading irrelevant operator. We propose an interpretation in terms of an effective IR defect theory localized on the Wilson loop. In addition to the bulk RG flow, there is an RG flow on the defect triggered by a double trace deformation, that we identify from the boundary conditions of the string at the cutoff. The L-dependence of both kinds of corrections depends solely on the properties of the IR field theory, and all the information about its UV structure is restricted to the value of a single coefficient, which determines the universal contribution.

For a flow in a confining theory we observe exponentially suppressed corrections to the potential beyond the leading linear dependence on the quark-antiquark separation. The exponent is proportional to the mass scale of glueballs, and it coincides with the mass of some internal excitations of a flux tube in the dual field theory. We interpret this result in terms of the effective IR theory as a flux tube with sources at the endpoints for the internal modes, that have a non-vanishing profile in the ground state of the quark-antiquark pair. Since these excitations correspond to fluctuations of the string along the holographic direction, these type of correction are a generic property of confinement as described by the gauge/gravity duality.

The structure of the paper is as follows: we start reviewing some basic facts about Wilson loops and their calculation using gauge/gravity duality in § 2. In § 3 we present general formulas for the holographic RG flow of a Wilson loop. The case with an IR fixed point is studied in § 4 and confining theories in § 5. A summary of the results and their interpretation from the point of view of the field theory dual is gathered in § 6.

2 Wilson loops in holography

The study of Wilson loops in holographic duals was initiated in [20, 21]. In $\mathcal{N}=4$ super Yang-Mills (SYM), a locally BPS Wilson loop in the fundamental representation is given by the path-ordered exponential

$$\mathcal{W}_{BPS}(\mathcal{C}) = \frac{1}{N}\mathrm{Tr}\,\mathcal{P}\left(e^{i\oint_{\mathcal{C}} d\tau\left(\dot{x}^\mu A_\mu + |\dot{x}|\theta_I \Phi^I\right)}\right), \quad \theta^2 = 1. \tag{2.1}$$

Where $x^\mu(\tau)$ parametrizes the closed curve $\mathcal{C}$ on which the loop is defined, A_μ is the gauge field and Φ^I, $I=1,\cdots,6$ are the adjoint scalar fields of the $\mathcal{N}=4$ SYM theory. In the large-N limit the expectation value of the Wilson loop can be determined at strong 't Hooft coupling $\lambda_{YM}\gg 1$ by the classical Nambu-Goto action of an open string whose boundary is along the curve $\mathcal{C}$ at the asymptotic boundary of the holographic dual geometry. The position of the string endpoints in the internal space is determined by identifying the functions $\theta^I(\tau)$ with coordinates on the S^5 of the dual $AdS_5\times S^5$ geometry.

The identification of BPS loops with a dual string is based on the weakly coupled D-brane construction. The $\mathcal{N}=4$ SYM theory is the low energy description of a stack of coincident N D3 branes. When one of the branes is separated from the rest, a string extended between the separated brane and the rest acts as a source in the fundamental

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representation. When the isolated brane is taken to infinity, the string becomes an infinitely heavy source and therefore it is equivalent to the insertion of a Wilson loop. In the near-horizon limit that replaces D3 branes by geometry, the isolated brane can be though as being at the asymptotic boundary, and the string extends from it to the interior. A similar argument can be used for any low energy effective theory on the worldvolume of a stack of branes, so the identification of the string with a Wilson loop extends naturally to more general gauge/gravity duals obtained from a near-horizon limit.

We will see that the holographic Wilsonian RG flow changes the boundary conditions of the string at the cutoff. This could alter the nature of the Wilson loop, for instance it was proposed in [41] that the holographic dual to an ordinary Wilson loop should correspond to a string satisfying Neumann boundary conditions along the S^5 directions. The logic is that an ordinary Wilson loop does not couple to the scalars Φ^I, and therefore it is invariant under the SO(6) R-symmetry that rotates them. More generally, one could define a family of Wilson loops with different couplings to the scalar fields [42]

$$\mathcal{W}_\zeta(\mathcal{C}) = \frac{1}{N}\mathrm{Tr}\,\mathcal{P}\,\left(e^{i\oint_{\mathcal{C}} d\tau\left(\dot{x}^\mu A_\mu + \zeta|\dot{x}|\theta_I\Phi^I\right)}\right), \tag{2.2}$$

with $\zeta = 1$ corresponding to the BPS loop and $\zeta = 0$ to the ordinary Wilson loop. In [42] it was shown that there can be an RG flow between the ordinary and BPS Wilson loops, both at weak and strong coupling. For Wilson loops with intermediate values of ζ one may expect that the dual description is a string with mixed boundary conditions. The case at hand differs from the ζ-deformed loops in that the boundary conditions that are modified are not along the S^5, but along the field theory directions. There are however some similarities, in that we can identify a deformation of the BPS loop and an associated RG flow.

2.1 Calculation of the quark-antiquark potential in an RG flow

The holographic dual to a conformal field theory is an $AdS_5 \times \mathcal{M}_5$ geometry, where $\mathcal{M}_5$ is a compact space. In Gaussian coordinates

$$ds_{10}^2 = G_{MN}dx^M dx^N = dr^2 + e^{2r/R}\eta_{\mu\nu}dx^\mu dx^\nu + d\mathcal{M}_5^2, \tag{2.3}$$

where $x^\mu = (t, x, y, z)$ are coordinates along the field theory directions and $\eta_{\mu\nu}$ is the flat Minkowski metric. The coordinates along the compact space will be denoted by θ^A, $A = 1, \cdots, 5$. The radial coordinate r characterizes the energy/distance scale in the field theory, with $r \to \infty$ the asymptotic boundary associated to the UV. R is the radius of AdS space.

The potential $V_{q\bar{q}}$ between a static quark-antiquark pair separated a distance L can be computed from the expectation value of a time-like Wilson loop along a rectangular contour of sides of length L along space and β along time. When $\beta \to \infty$,

$$\langle \mathcal{W} \rangle \sim e^{-\beta V_{q\bar{q}}(L)}. \tag{2.4}$$

In the large-N limit, and at strong 't Hooft coupling, the potential is determined by the Nambu-Goto action of a string evaluated on-shell

$$\beta\, V_{q\bar{q}}(L) = -S_{\mathrm{NG}}. \tag{2.5}$$

The Nambu-Goto action is

$$S_{\rm NG} = -\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-h}, \quad h_{ab} = G_{MN}\partial_a X^M \partial_b X^N, \quad \sigma^a = (\tau,\sigma), \tag{2.6}$$

where σ^a are the world-sheet coordinates, h is the determinant of the induced metric h_{ab} and $X^M(\tau,\sigma)$ are the embedding functions that describe the string profile in the target space.

The relevant configuration is a solution to the classical equations of motion with appropriate boundary conditions, following previous works (e.g. [43]) we review here the main points of the derivation and properties of the solutions. We can choose the following static gauge

$$X^0 = \tau, \quad X^1 = x(\sigma), \quad X^2 = X^3 = 0, \quad X^r = \sigma, \quad X^A = \theta_0^A, \tag{2.7}$$

where θ_0^A are constant. The boundary conditions are

$$\lim_{\sigma\to\infty} x(\sigma) = 0, \quad \lim_{\sigma\to\sigma_*} x'(\sigma) = \infty, \quad \lim_{\sigma\to\sigma_*} x(\sigma) = \frac{L}{2}. \tag{2.8}$$

where σ_* is a particular value of the world-sheet coordinate σ, whose value depends on L, via the last condition. This solution actually describes a branch of the solution extending from the asymptotic boundary to a point in the interior, there is another symmetric branch returning to the boundary at the point $x = L$.

In general the action evaluated on this class of solutions is divergent, one can regularize it by introducing a cutoff at $\sigma = r_{(\Lambda)}$ and adding a counter-term at the boundary of the string, in such a way that the total action is $S_{\rm string} = S_{\rm NG} + S_{\rm c.t.}$, so that

$$S_{\rm string} = \lim_{r_{(\Lambda)}\to\infty} \; -\frac{1}{2\pi\alpha'}\int_{\sigma\leq r_{(\Lambda)}} d^2\sigma\sqrt{-h} + \frac{c_\Lambda}{2\pi\alpha'}\int d\tau\,\sqrt{-\gamma}, \tag{2.9}$$

where $\gamma = h_{\tau\tau}\big|_{\sigma=r_{(\Lambda)}}$. For an asymptotically AdS space like (2.3), the value of the coefficient of the counter-term is $c_\Lambda = R$.

The holographic dual of a generic RG flow can be a relatively complicated geometry with various warping factors depending on the radial coordinate and coordinates along the internal space

$$ds_{10}^2 = \Delta(\theta,r)dr^2 + \Sigma(\theta,r)\eta_{\mu\nu}dx^\mu dx^\nu + d\widetilde{\mathcal{M}}_5^2. \tag{2.10}$$

In this work, we will concentrate in simpler examples, in which the ten-dimensional metric can be put in the domain wall form

$$ds_{10}^2 = \frac{dr^2}{f(r)} + e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + d\widetilde{\mathcal{M}}_5^2. \tag{2.11}$$

Where, as the asymptotic boundary at $r\to\infty$ is approached, $A(r)\to\infty$ and $f(r)\to 1$.

In this case (2.7) is a consistent ansatz and the induced metric is

$$ds_2^2 = -e^{2A(\sigma)}d\tau^2 + \left[\frac{1}{f(\sigma)} + e^{2A(\sigma)}(x')^2\right]d\sigma^2. \tag{2.12}$$

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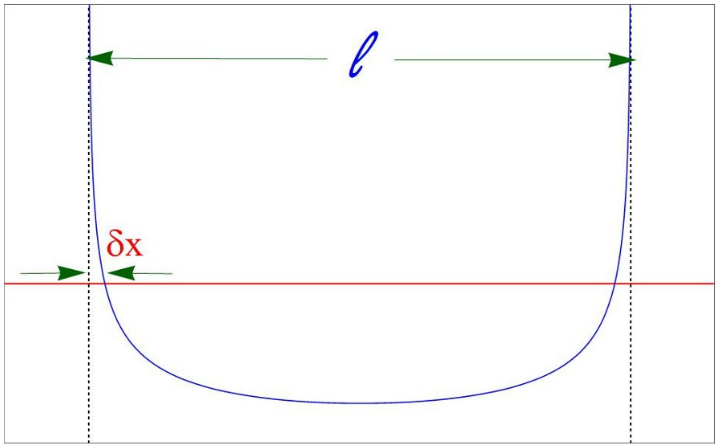

Figure 1. Profile of a string dual to a $q\bar{q}$ pair separated a distance ℓ (blue line). The vertical direction corresponds to the holographic radial coordinate, with the asymptotic boundary (UV) at the top. A cutoff is introduced at an IR scale (horizontal red line) and degrees of freedom above the cutoff are integrated out. The separation in the field theory directions between the endpoint of the string at the boundary and at the cutoff is denoted by δx, it corresponds to $x_{(\mu)}$ in the text.

The action becomes

$$S_{\rm NG} = -\frac{\beta}{2\pi\alpha'}\int d\sigma\,\frac{e^A}{\sqrt{f}}\sqrt{1+fe^{2A}(x')^2}. \tag{2.13}$$

Again, one needs to add a counter-term of the form given in (2.9) to render it finite.

Since it depends only on the derivative of the embedding function x', the conjugate momentum is constant

$$\pi_x = \frac{\delta S_{\rm NG}}{\delta x'} = \frac{\beta}{2\pi\alpha'}p = \text{constant}. \tag{2.14}$$

This leads to the equation

$$\frac{\sqrt{f}e^{3A}x'}{\sqrt{1+fe^{2A}(x')^2}} = -p, \tag{2.15}$$

or, solving for x', we obtain the equation of motion for the embedding

$$x' = -p\frac{e^{-3A}}{\sqrt{f}\sqrt{1-e^{-4A}p^2}}. \tag{2.16}$$

If we picture the string as hanging from the asymptotic boundary, as in figure 1, the conditions (2.8) fix the relation between the lowest point of the string profile σ_*, the separation L of the pair and p

$$p = e^{2A(\sigma_*)}, \quad \frac{L}{2} = p\int_{\sigma_*}^{\infty}\frac{e^{-3A}}{\sqrt{f}\sqrt{1-e^{-4A}p^2}}. \tag{2.17}$$

Note that under a change of the boundary condition $\delta x(\infty) = \delta L$, the solution changes $x \to x+\delta x$ and the change of the Nambu-Goto action is proportional to the conjugate momentum

$$\delta S_{\rm NG} = \int^{\infty} d\sigma\,\pi_x\delta x' = \frac{\beta}{2\pi\alpha'}p\,\delta L. \tag{2.18}$$

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The change of the potential is

$$\delta V_{q\bar{q}} = -\frac{1}{2\pi\alpha'} p\delta L. \tag{2.19}$$

Therefore, p should be identified with the force that the (anti)quark feels

$$\mathcal{F}_x = -\frac{\delta V_{q\bar{q}}}{\delta L} = \frac{1}{2\pi\alpha'} p. \tag{2.20}$$

3 Holographic RG flow of the Wilson loop

The analysis of the previous section demanded to have full knowledge of the dual geometry all the way to the boundary to determine the classical string profile that controls the potential. However, for large enough separation it is expected that it is enough to know the geometry below certain cutoff $r_{(\mu)}$. To materialize this expectation, we note that for sufficiently large values of L, the profile of the string is typically mostly below some finite value of the radial cutoff $r_{(\mu)}$. In the dual geometry the region between the boundary and the cutoff corresponds to length scales much smaller than the separation L. At these scales, the quark and antiquark do not feel much the presence of each other, and the profile of the string around each of the endpoint positions is close to that of a single isolated quark, remaining close to the endpoint position in the parallel directions to the boundary and extending almost completely straight into the interior. The straight shape of the profile persists from the cutoff to the interior, until, far from the cutoff, the profile changes and extends in the directions parallel to the boundary in such a way that the two endpoints are joined (see figure 1). This characteristic behavior implies that the information about the UV properties of the theory is confined to the position of the string profile at the cutoff, which is close to the position of the endpoints at the boundary and introduces a length scale much smaller than the separation between the endpoints. The ratio between these two length scales works as a perturbative parameter that we will use to find the first corrections to the leading order dependence of the potential on the quark-antiquark separation.

Based on the considerations above, we separate the contribution of the straight segments from the rest by introducing a cutoff at $\sigma = r_{(\mu)}$ such that

$$x_{(\mu)} = x(r_{(\mu)}) \ll L. \tag{3.1}$$

In the region closer to the asymptotic boundary $\sigma > r_{(\mu)}$, the string does not wander far from its initial position $0 < x < x_{(\mu)}$, and x' is a small quantity, as shown in figure 1. Taking advantage of this fact, we will split the string action in two parts, an upper part $S^{>}_{\rm string}$, where we integrate for values $\sigma > r_{(\mu)}$ and add the boundary counterterms, and the lower part where we integrate below the cutoff $S^{<}_{\rm NG}$.

$$S_{\rm string} = S^{>}_{\rm string} + S^{<}_{\rm NG} = S^{>}_{\rm NG} + S_{\rm c.t.} + S^{<}_{\rm NG}. \tag{3.2}$$

Expanding the upper action to quadratic order one finds

$$S^{>}_{\rm NG} \simeq -\frac{\beta}{2\pi\alpha'} \int_{\sigma > r_{(\mu)}} d\sigma \frac{e^A}{\sqrt{f}} \left(1 + \frac{1}{2} f e^{2A} (x')^2\right). \tag{3.3}$$

The solution to the equations of motion is

$$x' \simeq -p\frac{e^{-3A}}{\sqrt{f}}. \tag{3.4}$$

Comparing with the exact solution (2.16), this is a valid approximation as long as $p^2 e^{-4A(r_{(\mu)})} \ll 1$. Defining a function

$$a(\sigma) = \int_\sigma^\infty \frac{e^{-3A}}{\sqrt{f}}, \quad a_{(\mu)} = a(r_{(\mu)}), \tag{3.5}$$

the displacement at the cutoff is $x_{(\mu)} = a_{(\mu)}p$. The profile of the string is then

$$x \simeq x_{(\mu)}\frac{a(\sigma)}{a_{(\mu)}}. \tag{3.6}$$

The on-shell action (regularized by an UV cutoff) is

$$S_{\rm NG}^{>} \simeq -\frac{\beta}{2\pi\alpha'}\left[\int_{r_{(\mu)}}^{r_{(\Lambda)}} d\sigma\frac{e^A}{\sqrt{f}} + \frac{\sqrt{f}e^{3A}}{2}x'x\Big|_{r_{(\mu)}}^{r_{(\Lambda)}}\right]. \tag{3.7}$$

The string action, including counterterms, is thus

$$S_{\rm string}^{>} \simeq -\frac{\beta}{2\pi\alpha'}\left[M_{(\mu)} + \frac{1}{2a_{(\mu)}}x_{(\mu)}^2\right], \tag{3.8}$$

where

$$M_{(\mu)} = \lim_{r_{(\Lambda)}\to\infty}\int_{r_{(\mu)}}^{r_{(\Lambda)}} d\sigma\frac{e^A}{\sqrt{f}} - e^{A(r_{(\Lambda)})}, \tag{3.9}$$

We can then replace our original string action by the NG action below the cutoff plus a boundary term that appears as a double-trace deformation, the x_μ^2 term appearing in (3.8), its effect is to modify the boundary conditions at the cutoff. In order to see this, consider a string with slightly perturbed profile, but keeping the endpoints at the boundary fixed $x \to x+\delta x$, $\delta x(r_{(\Lambda)}) = 0$. The variation of the on-shell string action under this perturbation has a bulk contribution and a contribution localized at the cutoff

$$\delta S_{\rm string} = -\frac{\beta}{2\pi\alpha'}\left[\frac{x_{(\mu)}}{a_{(\mu)}}\delta x_{(\mu)} + \int_{\sigma<r_{(\mu)}} d\sigma\,\frac{\sqrt{f}e^{3A}x'}{\sqrt{1+fe^{2A}(x')^2}}\delta x'\right]. \tag{3.10}$$

Integrating the bulk term by parts and using the equations of motion (2.15), one is left with only cutoff contributions

$$\delta S_{\rm string} = -\frac{\beta}{2\pi\alpha'}\left[\frac{x_{(\mu)}}{a_{(\mu)}} - p\right]\delta x_{(\mu)}. \tag{3.11}$$

Since the on-shell action should be stationary for small perturbations of the profile that do not change the boundary conditions, the variation above should vanish for any $\delta x_{(\mu)}$. This condition fixes the conjugate momentum for the solution below the cutoff to the right value

$$p = \frac{x_{(\mu)}}{a_{(\mu)}}. \tag{3.12}$$

Therefore, the string dual to the Wilson loop that determines the quark-antiquark potential can be replaced by a string with endpoints at a cutoff satisfying mixed boundary conditions.

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3.1 IR description of the Wilson loop

The analysis above has made precise the expectation that the long distance potential only depends on the IR physics. By replacing the full string action from the cut-off to the boundary by the quadratic approximation, (3.8), we have managed to express the problem in terms of quantities evaluated at the cut-off. All the information about the UV part of the geometry, and its manifestation in the string embedding, is condensed into the (cut-off dependent) values of the parameters $M_{(\mu)}$ and $a_{(\mu)}$. Starting from this action, in this section we will show how to use the independence of physical quantities on the cutoff to constraint the long distance behavior of the heavy quark potential.

Suppose we are given a geometry that will be used as a holographic dual description of the IR physics of some strongly coupled theory. We introduce a cutoff in this geometry and consider the string action with the additional boundary terms we have derived

$$S_{\rm IR} = S_{\rm NG}^{<} - \frac{\beta}{2\pi\alpha'}\left[M_{(\mu)} + \frac{1}{2a_{(\mu)}}x_{(\mu)}^2\right]. \tag{3.13}$$

Physical quantities computed using the holographic dual should be independent of the cutoff we have introduced. However, the string action has an explicit dependence on the cutoff, that is apparent from the definition of the coefficients of the cutoff terms $a_{(\mu)}$ (3.5) and $M_{(\mu)}$ (3.9). If we regard $r_{(\mu)}$ as corresponding to an RG scale similar to the ones used in perturbative renormalization schemes, the dependence on the cutoff can be encoded in the RG flow equations ($A_{(\mu)} = A(r_{(\mu)})$, $f_{(\mu)} = f(r_{(\mu)})$)

$$\partial_{r_{(\mu)}} a_{(\mu)} = -\frac{e^{-3A_{(\mu)}}}{\sqrt{f_{(\mu)}}}, \quad \partial_{r_{(\mu)}} M_\mu = -\frac{e^{A_{(\mu)}}}{\sqrt{f_{(\mu)}}}. \tag{3.14}$$

Integrating these equations one would obtain $M_{(\mu)}$ and $a_{(\mu)}$ up to indeterminate integration constants. It should be noted that the RG equations involve terms that are evaluated at the cutoff position, so they only depend on the local geometry close to the cutoff. In the language of the field theory dual, the RG equations only depend on the physics of the scale close to the cutoff. All the information about UV physics is hidden in the integration constants. This fits with the usual Wilsonian paradigm of renormalisation, the terms that can appear in the effective action are determined by the IR degrees of freedom, but with coefficients that have to be fixed by experiments or by matching with UV physics.

Using the equations (3.14) it is easy to show that physical quantities are independent of the cutoff. The equations of motion for the embedding below the cutoff are given by (2.16), and we have to impose the conditions

$$p = e^{2A(\sigma_*)}, \quad x_{(\mu)} = a_{(\mu)}p, \quad \frac{L}{2} = x_{(\mu)} + p\int_{\sigma_*}^{r_{(\mu)}} \frac{e^{-3A}}{\sqrt{f}\sqrt{1-e^{-4A}p^2}}. \tag{3.15}$$

The force, which is proportional to p, is independent of the cutoff by construction. The separation between the quark-antiquark pair is invariant under changes of the cutoff at leading order

$$\partial_{r_{(\mu)}} L = 2p\left(\partial_{r_{(\mu)}} a_{(\mu)} + \frac{e^{-3A_{(\mu)}}}{\sqrt{f_{(\mu)}}\sqrt{1-e^{-4A_{(\mu)}}p^2}}\right) = O\left(p^3 e^{-7A_{(\mu)}}\right). \tag{3.16}$$

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Then, the dependence of the force (2.20) with the length is also invariant to leading order. In principle one could systematically add higher order corrections by including further multi-trace terms in the boundary action, we show how to proceed using the Wilsonian RG flow equations for the boundary action and show explicitly that the length is invariant at the next order in appendix A. As we already mentioned, the value of $a_{(\mu)}$ cannot be determined by the IR theory, rather it would have to be fixed by matching with the UV theory if this one is known, or by measuring the force at a separation L and doing a fit.

4 Theory with an IR fixed point

In this section we will use the formalism developed in the previous section to study a particularly simple example, that of a strongly coupled field theory with an IR fixed point. Because it is a fixed point, the long distance dynamics are controlled by a CFT. As a consequence of conformal symmetry, the dual geometry must approach AdS space in the interior, meaning it takes the form in (2.3) as $r \to -\infty$. We will assume that the flow away from the fixed point is driven an irrelevant scalar operator of conformal dimension $\Delta > d$. That is, to the action of the IR CFT we add a coupling to the irrelevant operator

$$S_{\rm IR} = S_{\rm CFT} + \int d^d x\, \tilde{\alpha} \mathcal{O}_\Delta. \tag{4.1}$$

In the dual theory, this is realized by turning on a scalar field ϕ of mass $m^2 R^2 = \Delta(\Delta - d)$. The scalar field back-reacts on the geometry and takes it away from AdS to a different geometry in the asymptotic region, which in principle could be another AdS space of different radius $R_{\rm UV} > R$, corresponding to a UV fixed point.

In order to simplify the discussion we will use the 'fake' supergravity formalism [44–48],[2] such that the equations of motion for the metric and the scalar reduce to a system of first order equations. It should be noted that for any given potential in supergravity, a superpotential describing the solution (fake or not) always exists locally, so the analysis presented here for the leading corrections is quite general.

The metric has a slightly simpler form than the general case

$$ds_5^2 = dr^2 + e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu. \tag{4.2}$$

Note, in particular, that the function $f(r) = 1$, as defined in (2.11). The equations of motion for the metric and scalar are

$$\phi' = -\partial_\phi W, \quad A' = \frac{W}{d-1}. \tag{4.3}$$

There is a critical point at a value $\phi = \phi_{\rm IR}$ such that

$$\partial_\phi W(\phi_{\rm IR}) = 0, \quad W(\phi_{\rm IR}) = \frac{d-1}{R}. \tag{4.4}$$

[2] The formalism can be derived from a Hamilton-Jacobi formulation of the radial evolution of the solutions, which has also been interpreted in terms of the RG flow in the holographic dual, see e.g. [49–51] and [52] for an earlier work in the context of cosmology.

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This corresponds to the IR AdS solution where the scalar is constant

$$\phi(r) = \phi_{\rm IR}, \quad A(r) = \frac{r}{R}. \tag{4.5}$$

Close to the critical point, the fake superpotential can be approximated by

$$W \simeq \frac{d-1}{R} + \frac{d-\Delta}{2R}(\phi - \phi_{\rm IR})^2. \tag{4.6}$$

The solution to leading order away from conformality is

$$\phi \simeq \phi_{\rm IR} + \alpha e^{(\Delta-d)(r-r_{(M)})/R}, \quad A \simeq \frac{r}{R} - \frac{\alpha^2}{4} e^{2(\Delta-d)(r-r_{(M)})/R}, \tag{4.7}$$

where α is of order one and proportional to the irrelevant coupling, $\tilde{\alpha}$, and $r_{(M)}$ determines the region where the geometry deviates significantly from AdS. The type of expansion we are doing is valid for $r \ll r_{(M)}$. In particular, the cutoff should be in the near-AdS region $r_{(\mu)} \ll r_{(M)}$.

From the first condition in (3.15), we obtain

$$p \simeq \exp\left(\frac{2\sigma_*}{R} - \frac{\alpha^2}{2} e^{2(\Delta-d)(\sigma_*-r_{(M)})/R}\right). \tag{4.8}$$

We can solve this condition for σ_* expanding to leading order in α

$$\sigma_* = \sigma_*(0) + \delta\sigma \simeq \frac{R}{4}\log p^2 + R\frac{\alpha^2}{4} e^{-2(\Delta-d)r_{(M)}/R} p^{\Delta-d}. \tag{4.9}$$

From (3.15), the quark-antiquark separation is

$$L = 2a_{(\mu)}p + 2p\int_{\sigma_*}^{r_{(\mu)}} I(\sigma), \tag{4.10}$$

where

$$I(\sigma) = \frac{e^{-3A}}{\sqrt{1-e^{-4A}p^2}}. \tag{4.11}$$

Let us split the integral in a region close to σ_* and the rest

$$\int_{\sigma_*}^{r_{(\mu)}} I(\sigma) = \int_{\sigma_*}^{\sigma_*+\Delta\sigma} I(\sigma) + \int_{\sigma_*+\Delta\sigma}^{r_{(\mu)}} I(\sigma). \tag{4.12}$$

The first integral is approximated expanding around σ_*. Expanding the result in α, one finds

$$\int_{\sigma_*}^{\sigma_*+\Delta\sigma} I(\sigma) \simeq \frac{R^{1/2}}{p^{3/2}}\left(1 + (\Delta-d)\frac{\alpha^2}{4} e^{-2(\Delta-d)r_{(M)}/R} p^{\Delta-d}\right)\sqrt{\Delta\sigma}. \tag{4.13}$$

In the second integral we expand first in α

$$\begin{aligned}\int_{\sigma_*+\Delta\sigma}^{r_{(\mu)}} I(\sigma) \simeq & -\left.\frac{e^{-3\sigma/R}}{\sqrt{1-e^{-4\sigma/R}p^2}}\right|_{\sigma_*(0)+\Delta\sigma}\delta\sigma + \int_{\sigma_*(0)+\Delta\sigma}^{r_{(\mu)}}\left[\frac{e^{-3\sigma/R}}{\sqrt{1-e^{-4\sigma/R}p^2}}\right. \\ & \left. + \frac{\alpha^2}{4} e^{-2(\Delta-d)r_{(M)}/R}\frac{e^{(2(\Delta-d)-3)\sigma/R}\left(3-e^{-4\sigma/R}p^2\right)}{\left(1-e^{-4\sigma/R}p^2\right)^{3/2}}\right].\end{aligned} \tag{4.14}$$

The integrals can be done analytically, with a result that can be expressed in terms of hypergeometric functions, but that is not very illuminating. Expanding for $r_{(\mu)} \gg \sigma_*(0)$, and $\Delta\sigma \to 0$,

$$\begin{aligned}
&\frac{e^{-3\sigma/R}}{\sqrt{1-e^{-4\sigma/R}p^2}}\Big|_{\sigma_*(0)+\Delta\sigma}\delta\sigma \\
&\quad\simeq \frac{\alpha^2}{4}e^{-2(\Delta-d)r_{(M)}/R}\frac{R^{3/2}}{2\sqrt{\Delta\sigma}}p^{\Delta-d-\frac{3}{2}}, \qquad (4.15)\\
&\int_{\sigma_*(0)+\Delta\sigma}^{r_{(\mu)}} \frac{e^{-3\sigma/R}}{\sqrt{1-e^{-4\sigma/R}p^2}} \\
&\quad\simeq -\frac{R^{1/2}}{p^{3/2}}\sqrt{\Delta\sigma}+\frac{R}{p^{3/2}}\frac{\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}-\frac{R}{3}e^{-3r_{(\mu)}/R}, \qquad (4.16)\\
&\int_{\sigma_*(0)+\Delta\sigma}^{r_{(\mu)}} \frac{e^{(2(\Delta-d)-3)\sigma/R}\left(3-e^{-4\sigma/R}p^2\right)}{\left(1-e^{-4\sigma/R}p^2\right)^{3/2}} \\
&\quad\simeq \frac{R^{3/2}}{2\sqrt{\Delta\sigma}}p^{\Delta-d-\frac{3}{2}}-(\Delta-d+1)R^{1/2}p^{\Delta-d-\frac{3}{2}}\sqrt{\Delta\sigma} \qquad (4.17)\\
&\qquad+\frac{3R}{2(\Delta-d)-3}e^{(2(\Delta-d)-3)r_{(\mu)}/R}+\frac{(\Delta-d)\sqrt{\pi}\Gamma\left(\frac{3}{4}-\frac{\Delta-d}{2}\right)}{2\Gamma\left(\frac{5}{4}-\frac{\Delta-d}{2}\right)}Rp^{\Delta-d-\frac{3}{2}}. \qquad (4.18)
\end{aligned}$$

The $\sim 1/\sqrt{\Delta\sigma}$ terms cancel out when we sum over all contributions, so the limit $\Delta\sigma \to 0$ is finite, giving

$$\begin{aligned}
\int_{\sigma_*}^{r_{(\mu)}} I(\sigma) \simeq R\Bigg[&\frac{\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}p^{-3/2}-\frac{1}{3}e^{-3r_{(\mu)}/R}\left(1-\frac{9}{4}\frac{\alpha^2}{2(\Delta-d)-3}e^{-2(\Delta-d)(r_{(M)}-r_{(\mu)})/R}\right)\\
&+\frac{\alpha^2}{4}e^{-2(\Delta-d)r_{(M)}/R}\frac{(\Delta-d)\sqrt{\pi}\Gamma\left(\frac{3}{4}-\frac{\Delta-d}{2}\right)}{2\Gamma\left(\frac{5}{4}-\frac{\Delta-d}{2}\right)}p^{\Delta-d-\frac{3}{2}}\Bigg]. \qquad (4.19)
\end{aligned}$$

The expression is valid for $\Delta - d \neq 3/2$. In order for the correction proportional to α^2 to be small, the cutoff should be at a position in the radial direction $r_{(\mu)} \ll r_{(M)}$. In the field theory dual this means that we are considering energy scales much below the characteristic scale where the RG flow deviates significantly from the IR CFT. For $\Delta - d = 3/2$ one finds

$$\int_{\sigma_*}^{r_{(\mu)}} I(\sigma) \simeq R\Bigg[\frac{\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}p^{-3/2}-\frac{1}{3}e^{-3r_{(\mu)}/R}+\frac{\alpha^2}{4}e^{-3r_{(M)}/R}\left(3\frac{r_{(\mu)}}{R}-\frac{3}{2}\log\left(\frac{p}{2}\right)-1\right)\Bigg]. \qquad (4.20)$$

The quark-antiquark separation is

$$L = R\left[c_0 p^{-1/2}+a_0 p+a_{\Delta-d}p^{\Delta-d-\frac{1}{2}}\right], \quad \Delta-d\neq\frac{3}{2}, \qquad (4.21)$$

or

$$L = R\left[c_0 p^{-1/2}+\tilde{a}_0 p+a_{3/2}p\log\frac{p}{p_0}\right], \quad \Delta-d=\frac{3}{2}. \qquad (4.22)$$

The coefficients are

$$\begin{aligned}
c_0 &= \frac{2\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}, \\
a_0 &= 2a_{(\mu)} - \frac{2}{3}e^{-3r_{(\mu)}/R}\left(1-\frac{9}{4}\frac{\alpha^2}{2(\Delta-d)-3}e^{-2(\Delta-d)(r_{(M)}-r_{(\mu)})/R}\right), \\
a_{\Delta-d} &= \frac{\alpha^2}{2}e^{-2(\Delta-d)r_{(M)}/R}\frac{(\Delta-d)\sqrt{\pi}\Gamma\left(\frac{3}{4}-\frac{\Delta-d}{2}\right)}{2\Gamma\left(\frac{5}{4}-\frac{\Delta-d}{2}\right)}, \\
\tilde{a}_0 &= 2a_{(\mu)} - \frac{2}{3}e^{-3r_{(\mu)}/R}\left(1-\frac{9\alpha^2}{4}e^{-3(r_{(M)}-r_{(\mu)})/R}\frac{r_{(\mu)}}{R}\right), \\
a_{3/2} &= -\frac{3\alpha^2}{4}e^{-3r_{(M)}/R}, \quad p_0 = 2e^{-2/3}.
\end{aligned} \tag{4.23}$$

Only the coefficients a_0 and $\tilde{a}_0$ have an explicit dependence on the cutoff. However, when taking into account the scale dependence of the double trace coefficient $a_{(\mu)}$, this dependence vanishes. Expanding the RG-flow equations for $a_{(\mu)}$ (3.14) to $O(\alpha^2)$, one can show that

$$\partial_{r_{(\mu)}} a_0 \simeq 0, \quad \partial_{r_{(\mu)}} \tilde{a}_0 \simeq 0, \tag{4.24}$$

where the approximate sign indicates that we have only used an approximate RG evolution to leading order in α.

The force between the quark and antiquark as a function of the separation can be found solving for p in (4.21) and (4.22). When $L \to \infty$, the leading corrections are

$$\mathcal{F}_x \simeq \frac{R^2}{2\pi\alpha'}\frac{c_0^2}{L^2}\left[1+\frac{2a_0}{c_0}\left(\frac{c_0 R}{L}\right)^3+\frac{2a_{\Delta-d}}{c_0}\left(\frac{c_0 R}{L}\right)^{2(\Delta-d)}\right], \quad \Delta-d\neq\frac{3}{2}, \tag{4.25}$$

or

$$\mathcal{F}_x = \frac{R^2}{2\pi\alpha'}\frac{c_0^2}{L^2}\left[1+\frac{2\tilde{a}_0}{c_0}\left(\frac{c_0 R}{L}\right)^3+\frac{2a_{3/2}}{c_0}\left(\frac{c_0 R}{L}\right)^3\log\left(\frac{c_0^2 R^2}{p_0 L^2}\right)\right], \quad \Delta-d=\frac{3}{2}, \tag{4.26}$$

These expressions are valid for any RG flow flowing to an IR fixed point having a domain wall geometry as holographic dual. In both these expressions the first term contains the expected conformal length-dependence of the force of the IR CFT, while the last term encodes the contribution from the irrelevant operator that deforms that CFT. These two contributions are solely determined from infrared physics, once the scaling dimension and coupling of the operator are known. The second term is more interesting, since its L dependence is universal. Independently of the details of the RG-flow, provided that the holographic theory flows to an IR fixed point, there is a contribution to the force between the quark-antiquark pair that behaves as L^{-5}. Since the value of a_0, $\tilde{a}_0$ are determined from the RG flow, at long distances all the information of the UV theory is hidden in the coefficient of this universal contribution.

4.1 Defect theory interpretation

The universal contribution identified above is intriguing since, a priori, it would have been hard to guess from the IR theory alone. In this subsection we will clarify its origin by

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analyzing the possible contributions to the quark-antiquark potential when considering the Wilson line as a defect and studying its fluctuations. In order to help us identify the origin of the universal contribution, let us first consider an arbitrary AdS geometry

$$ds^2 \simeq dr^2 + e^{2r/R}\eta_{\mu\nu}dx^\mu dx^\nu. \tag{4.27}$$

Inside this geometry we introduce a string extending straight along the radial direction and ending at the asymptotic boundary. The induced metric on the string is AdS_2

$$ds_2 = h_{ab}d\sigma^a d\sigma^b \simeq d\sigma^2 - e^{2\sigma/R}d\tau^2. \tag{4.28}$$

The string action for fluctuations $\delta x(\tau,\sigma)$ around the straight profile can be expanded to quadratic order

$$S_{\rm NG} \simeq \frac{1}{2\pi\alpha'}\int d^2\sigma\, e^{\sigma/R}\left(-1+\frac{1}{2}\left((\partial_\tau\delta x)^2 - e^{2\sigma/R}(\partial_\sigma\delta x)^2\right)\right). \tag{4.29}$$

Using the field redefinition

$$\delta x = Re^{-\sigma/R}\varphi, \tag{4.30}$$

we can rewrite this as the action for a scalar field in AdS_2

$$S_{\rm NG} \simeq -\frac{R^2}{2\pi\alpha'}\int d^2\sigma\,\sqrt{-h}\left(1+\frac{1}{2}\left(h^{ab}\partial_a\varphi\partial_b\varphi + m^2\varphi^2\right)\right), \tag{4.31}$$

where the mass is

$$m^2R^2 = 2. \tag{4.32}$$

Following the usual AdS/CFT dictionary, the AdS_2 geometry on the string is dual to a $0+1$ dimensional CFT, and the field φ is dual to an operator $\mathcal{O}_\varphi$ of dimension Δ such that $m^2R^2 = \Delta(\Delta-1)$. The root $\Delta = -1$ corresponds to changes in the position of the Wilson loop at the boundary $\delta x(\infty)\neq 0$, so it can be naturally identified with changes in the quark-antiquark separation along the x direction $\mathcal{O}_\phi = \hat{L}$. The operator corresponding to $\Delta = 2$ can be identified from the variation of the expectation of the Wilson loop with respect to changes in the trajectory. For instance, for a BPS loop in $\mathcal{N}=4$ SYM,

$$\frac{\delta}{\delta x^\mu}\log\langle\mathcal{W}\rangle \propto \frac{1}{\langle\mathcal{W}\rangle}\left\langle {\rm Tr}\left[\left(F_{\mu\nu}[x]\dot{x}^\nu + D_\mu^\perp\Phi^I[x]\theta_I|\dot{x}|\right)e^{i\oint(A+\theta\cdot\Phi)}\right]\right\rangle \equiv E_\mu^\perp, \tag{4.33}$$

where $F_{\mu\nu}$ is the field strength of the gauge fields. Generalizing the expression above, the $\Delta = 2$ operator can be identified as the electric field strength produced in the x direction $\mathcal{O}_\varphi = \hat{E}_x^\perp$. Adding to the straight string a boundary action of the form (3.8) would introduce a double-trace deformation $\sim\mathcal{O}_\varphi^2$ of the 0+1 dimensional CFT that would trigger a flow to a different fixed point in the IR (see e.g. [17]). As usual, the flow would be between the alternative quantization in the UV, for which the double trace deformation is relevant $\Delta_{\mathcal{O}^2} = -2$, to the normal quantization in the IR, where the double-trace deformation is irrelevant $\Delta_{\mathcal{O}^2} = 4$.

Now let us go back to our setup. Even though the geometry deviates from AdS at large values of the radial coordinate $r \gtrsim r_{(M)}$, deep in the interior of the space, $r \ll r_{(M)}$, the geometry again approaches AdS, since we are considering a theory with an IR fixed point. Furthermore, in the long distance limit we have considered, the separation between the endpoints is so large that the string profile is approximately straight close to the cutoff $r_{(\mu)} \ll r_{(M)}$. As a consequence, the AdS_2/CFT_1 map for the string is expected work in this region and the small string fluctuations close to that cut-off are governed by the same action (4.31). Therefore, from the point of view of the IR CFT, those fluctuations posses the same scaling dimensions and the same dual operators that we have identified in the pure AdS geometry. In particular, the boundary action (3.8) corresponds to a double-trace deformation $\sim \mathcal{O}_\varphi^2$ and, as the string below the cutoff lays in the IR region, the dimension of the double-trace deformation must be irrelevant and, as a consequence, $\Delta_{\mathcal{O}^2} = 4$. From the point of view of the IR effective description, the boundary action triggers a flow that at higher energy scales drives away the theory from the IR CFT. As a consequence, the double trace deformation will give a contribution to the potential whose dependence on L will be fixed by conformal invariance

$$\Delta V_{q\bar{q}} \propto c_{E^2} \left\langle E_x^2 \right\rangle \sim \frac{c_{E^2}}{L^4}. \tag{4.34}$$

This will give a contribution to the force $\Delta \mathcal{F}_x = -\partial_L \Delta V_{q\bar{q}} \sim 1/L^5$, whose dependence on L agrees with the universal term in (4.25) and (4.26).

5 Confining theories

The second application of our effective description of Wilson loops in the IR is to study the long distance quark-antiquark potential in confining theories. One of the defining characteristics of that type of theories is that Wilson loops follow an area law.[3] This implies that the heavy quark potential grows linearly at sufficiently large distances $V_{q\bar{q}} = \sigma_s L$, with σ_s the string tension. Going beyond this leading behavior, in this second application we will study the leading correction to this potential at long distance in holographic confining theories, which will be sensitive to characteristic features of the holographic description of confinement.

In the holographic dual description of confinement the string dual to the Wilson loop reaches a region of space where its tension remains fixed as the separation between the endpoints at the asymptotic boundary is increased. Although there are several different realizations of confining theories in holography,[4] in this work we will focus on two model examples. The first example is the WQCD model [24], consisting of the gravity dual of a stack of D4 branes wrapped around a compact direction with supersymmetry-breaking boundary conditions. At weak coupling the theory is expected to flow in the IR to pure Yang-Mills. At strong coupling there is really no separation of scales between the gapped four-dimensional modes and the Kaluza-Klein modes corresponding to excitations along the fifth direction. Nevertheless the model captures some of the properties of a confining

[3] If the theory contains degrees of freedom in the fundamental color representation this is only true in the large-N limit.

[4] For the general conditions on the metric under which the Wilson loop will show area law in a given holographic model see [43].

theory, including an area law for the Wilson loop [53]. The second example of a holographic dual to a confining theory is the KS model [25], which is dual to a non-conformal $\mathcal{N}=1$ supersymmetric $\mathrm{SU}(M_c)\times\mathrm{SU}(M_c+N_c)$ gauge theory. In the IR the theory flows to $\mathrm{SU}(N_c)$ $\mathcal{N}=1$ super Yang-Mills and also exhibits an area law for the Wilson loop [54].

In both examples, the area law behavior of the Wilson loop can be traced to the properties of the dual geometry in the interior. Rather than having an infinite throat, as for an AdS space, or a horizon, the geometry ends smoothly at a fixed position in the radial direction when a cycle in the internal space of the geometry collapses to zero size. For the WQCD model, the cycle is the circle corresponding to the fifth direction on the dual D4 branes, while for the KS model, it is a two-cycle in the internal geometry transverse to the field theory directions. For strings with a small separation between its endpoints, the string is hanging far from the point where the space ends. Increasing the separation will make the string go deeper in the geometry, until it reaches the end of space. Since it cannot penetrate further, a larger separation of the endpoints will result in having an increasing stretch of the string lying at the bottom of space. As the tension of the string is finite in this region, the action of the string increases linearly with the separation and produces the area law. As we will see, corrections to the area law emerge as a consequence of the piece of the dual string extended along the radial direction throughout all the geometry.

5.1 WQCD model

In the WQCD model the string-frame metric is usually given as

$$ds^2=\left(\frac{U}{R}\right)^{3/2}\eta_{\mu\nu}dx^\mu dx^\nu+\left(\frac{U}{R}\right)^{3/2}\left(1-\frac{U_M^3}{U^3}\right)d\varphi^2+\left(\frac{R}{U}\right)^{3/2}\frac{dU^2}{1-\frac{U_M^3}{U^3}}. \tag{5.1}$$

The map to field theory quantities is

$$U_M=\frac{2}{9}\lambda_{YM}M\alpha',\quad R^3=\frac{1}{2M}\lambda_{YM}\alpha' \tag{5.2}$$

Where λ_{YM} is the 't Hooft coupling and M is the scale of KK modes along the compact direction and determines the scale of glueball masses. The change of variables $U=r^4/(2^8R^3)$ puts the metric in "domain wall" form (2.11) with

$$e^{2A(r)}=\left(\frac{r}{4R}\right)^6,\quad f(r)=1-\frac{{r_{(M)}}^{12}}{r^{12}}, \tag{5.3}$$

where $r_{(M)}=4R^{3/4}U_M^{1/4}$ is the position at which the geometry ends. Having expressed the metric in this way, we can find the relation between the force, p, and the quark-antiquark separation L, using the relations in (3.15)

From the first condition in (3.15), the relation between the force and the minimum string position, σ_*, is

$$p=\left(\frac{\sigma_*}{4R}\right)^6. \tag{5.4}$$

As we have already explained, confinement implies that for a large enough separation between the string endpoints, σ_* is close to the end of the geometry $\sigma_*=r_{(M)}+\delta r$,

$$\delta r\simeq\frac{p-p_M}{6p_M}r_{(M)},\quad p_M=\left(\frac{r_{(M)}}{4R}\right)^6=\left(\frac{U_M}{R}\right)^{3/2}. \tag{5.5}$$

From (3.15), the quark-antiquark separation is

$$L = 2a_{(\mu)}p + 2p\int_{\sigma_*}^{r_{(\mu)}} d\sigma\, I(\sigma), \tag{5.6}$$

where

$$I(\sigma) = \frac{e^{-3A}}{\sqrt{f}\sqrt{1-e^{-4A}p^2}}. \tag{5.7}$$

The integral can be calculated analytically. Expanding for $r_{(\mu)} \gg r_{(M)}, \sigma_*$ and $\delta r/r_{(M)} \ll 1$ one finds

$$\int_{\sigma_*}^{r_{(\mu)}} d\sigma\, I(\sigma) = -\frac{R}{2}\left[\left(\frac{4R}{r_{(\mu)}}\right)^8 + \frac{2}{3}\left(\frac{4R}{r_{(M)}}\right)^8 \log\left(e^{\frac{\pi}{2\sqrt{3}}}\frac{\delta r}{\sqrt{3}r_{(M)}}\right)\right]. \tag{5.8}$$

The quark-antiquark separation in terms of p is

$$L = R\left[-c_0 p\log\left(\frac{p-p_M}{p_0}\right) + a_0 p\right], \tag{5.9}$$

where the coefficients are

$$\begin{aligned} c_0 &= \frac{2}{3}\left(\frac{4R}{r_{(M)}}\right)^8, \\ a_0 &= 2a_{(\mu)} - \left(\frac{4R}{r_{(\mu)}}\right)^8, \\ p_0 &= 6\sqrt{3}p_M e^{-\frac{\pi}{2\sqrt{3}}}. \end{aligned} \tag{5.10}$$

As in our first application, only one of these coefficients. a_0, depend explicitly on the cut-off. This reveals that its value is sensitive to UV physics, unlike the other two coefficients, that depend on quantities defined at the bottom of the geometry. However, taking into account the RG-flow equation $a_{(\mu)}$ (3.14), the coefficient a_0 is, as expected, independent of the cut-off. Indeed, expanding in $r_{(M)}/r_{(\mu)} \ll 1$, one can show that

$$\partial_{r_{(\mu)}} a_0 \simeq 0. \tag{5.11}$$

The force between the quark and antiquark as a function of the separation can be found solving for p in (5.9). When $L \to \infty$, the leading correction is

$$p \simeq p_M + p_0 e^{-a_0/c_0} e^{-\frac{L}{c_0 p_M R}}. \tag{5.12}$$

Then, the force is

$$\mathcal{F}_x = \sigma_s\left(1 + q_M e^{-ML}\right), \tag{5.13}$$

where the string tension and the coefficient of the exponential term are

$$\sigma_s = \frac{p_M}{2\pi\alpha'} = \frac{2}{27\pi}\lambda_{YM}M^2, \quad q_M = 6\sqrt{3}e^{-\frac{\pi}{2\sqrt{3}}}e^{-a_0/c_0}. \tag{5.14}$$

The slope of the exponential decay M is determined by the IR theory, but the amplitude q_M depends on a_0, which contains information about the RG flow above the cutoff. It should be noted that exponential corrections to the leading order behavior were already found e.g. [43, 55], but the connection with the scale of glueball masses was not made. The appearance of these corrections, that are non-perturbative in the $1/L$ expansion, are not just a feature of this particular model. As we will see they also appear in the KS model, which we analyze next.

5.2 KS model

In the KS model the metric is slightly more complicated. Separating the field theory directions from the internal ones it takes the form

$$ds^2 = h^{-1/2}(\tau)\eta_{\mu\nu}dx^\mu dx^\nu + h^{1/2}(\tau)ds_6^2. \tag{5.15}$$

The metric along the internal directions is the deformed conifold. It can be given in terms of the following basis of one-forms for the angular directions

$$g^1 = \frac{e^1 - e^3}{\sqrt{2}},\ g^2 = \frac{e^2 - e^4}{\sqrt{2}},\ g^3 = \frac{e^1 + e^3}{\sqrt{2}}\ g^4 = \frac{e^2 + e^4}{\sqrt{2}},\ g^5 = e^5, \tag{5.16}$$

where

$$\begin{aligned} &e^1 \equiv -\sin\theta_1 d\phi_1, \qquad e^2 \equiv d\theta_1, \qquad e^3 \equiv \cos\psi \sin\theta_2 d\phi_2 - \sin\psi d\theta_2, \\ &e^4 \equiv \sin\psi\sin\theta_2 d\phi_2 + \cos\psi d\theta_2, \qquad e^5 \equiv d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2. \end{aligned} \tag{5.17}$$

Then, the internal metric is

$$\begin{aligned} ds_6^2 = \frac{\varepsilon^{4/3}K(\tau)}{2}\Bigg[&\frac{1}{3(K(\tau))^3}\left(d\tau^2 + (g^5)^2\right) \\ &+ \cosh^2\frac{\tau}{2}\left((g^3)^2 + (g^4)^2\right) + \sinh^2\frac{\tau}{2}\left((g^1)^2 + (g^2)^2\right)\Bigg] \end{aligned} \tag{5.18}$$

$\varepsilon^{2/3}$ has dimensions of length and sets the scale of the conifold deformation. The warp factors are

$$K(\tau) = \frac{(\sinh 2\tau - 2\tau)^{1/3}}{2^{1/3}\sinh\tau}, \quad h(\tau) = \alpha\int_\tau^\infty dx \frac{x\coth x - 1}{\sinh x}K(x), \tag{5.19}$$

where $\alpha = 2(g_s N_c \alpha')^2 \varepsilon^{-8/3}$, with g_s is the string coupling constant. The asymptotic boundary is at $\tau \to \infty$, although in this case the metric deviates from AdS by logarithmic corrections. When $\tau \to 0$, a two-cycle in the internal metric collapses to zero size and the space terminates. The expansion of the warp factors in that limit is

$$K(\tau) \simeq \left(\frac{2}{3}\right)^{1/3}\left(1 - \frac{\tau^2}{10}\right), \quad h(\tau) = \left(\frac{2}{3}\right)^{1/3}\alpha\left(\hat{h}_M - \frac{\tau^2}{6}\right), \tag{5.20}$$

where $\hat{h}_M \simeq 0.65$.

The map to field theory quantities is

$$\varepsilon^{2/3} \propto (g_s N_c \alpha')M, \quad g_s N_c = \lambda_{YM}. \tag{5.21}$$

Where λ_{YM} is the 't Hooft coupling and M sets the scale of glueball masses [25, 56].

Identifying $r = \varepsilon^{2/3}\tau$ as the domain wall coordinate, the warp factors are

$$e^{2A(r)} = h^{-1/2}(\varepsilon^{-2/3}r), \quad f(r) = 6h^{-1/2}(\varepsilon^{-2/3}r)(K(\varepsilon^{-2/3}r))^2. \tag{5.22}$$

From the first condition in (3.15), we obtain

$$p = h^{-1/2}(\varepsilon^{-2/3}\sigma_*). \tag{5.23}$$

For a large enough separation between the string endpoints, σ_* should be close to the end of the geometry $\sigma_* = \delta r \ll \varepsilon^{2/3}$,

$$\delta r^2 \simeq \frac{12}{\hat{h}_M}\frac{p - p_M}{p_M}\varepsilon^{4/3}, \quad p_M = h^{-1/2}(0) = \left(\frac{2}{3}\right)^{-1/6}\alpha^{-1/2}\hat{h}_M^{-1/2}. \tag{5.24}$$

From (3.15), the quark-antiquark separation is

$$L = 2a_{(\mu)}p + 2p\int_{\sigma_*}^{r_{(\mu)}} d\sigma\, I(\sigma,\sigma_*), \tag{5.25}$$

where

$$I(\sigma,\sigma_*) = \frac{e^{-3A}}{\sqrt{f}\sqrt{1 - e^{-4A}p^2}} = \frac{1}{\sqrt{6}p}\frac{h(\varepsilon^{-2/3}\sigma)}{K(\varepsilon^{-2/3}\sigma)\sqrt{h(\varepsilon^{-2/3}\sigma_*) - h(\varepsilon^{-2/3}\sigma)}}. \tag{5.26}$$

For values of σ close to σ_*,

$$I(\sigma,\sigma_*) \simeq I_0(\sigma,\sigma_*) = \frac{p_M}{p}\frac{\alpha\hat{h}_M^{3/2}\varepsilon^{2/3}}{\sqrt{\sigma^2 - \sigma_*^2}}. \tag{5.27}$$

For $\sigma_* = 0$ the integral has a logarithmic divergence. In order to expand for small values of $\sigma_* = \delta r$, we subtract the divergent contribution and define

$$\int_{\delta r}^{r_{(\mu)}} d\sigma\, I(\sigma) = \int_{\delta r}^{r_{(\mu)}} d\sigma\, [I(\sigma,\delta r) - I_0(\sigma,\delta r)] + \frac{p_M}{p}\alpha\hat{h}_M^{3/2}\varepsilon^{2/3}\log\left(\frac{r_{(\mu)} + \sqrt{r_{(\mu)}^2 - \delta r^2}}{\delta r}\right). \tag{5.28}$$

Expanding for small δr we find, to leading order

$$\begin{aligned}\int_{\delta r}^{r_{(\mu)}} d\sigma\, I(\sigma) \simeq& \int_0^{r_{(\mu)}} d\sigma\, [I(\sigma,0) - I_0(\sigma,0)]_{p=p_M} \\ &+ \alpha\hat{h}_M^{3/2}\varepsilon^{2/3}\left(\log(\varepsilon^{-2/3}r_{(\mu)}) - \log\frac{\varepsilon^{-2/3}\delta r}{2}\right).\end{aligned} \tag{5.29}$$

The quark-antiquark separation in terms of p is

$$L = \varepsilon^{2/3}\left[-c_0 p\log\left(\frac{p - p_M}{p_0}\right) + a_0 p\right], \tag{5.30}$$

where the coefficients are

$$\begin{aligned}
c_0 &= \alpha \hat{h}_M^{3/2}, \\
a_0 &= 2\varepsilon^{-2/3} a_{(\mu)} + 2\alpha \hat{h}_M^{3/2} \log(\varepsilon^{-2/3} r_{(\mu)}) \\
&\quad + 2\sqrt{h(0)} \int_0^{r_{(\mu)}} d\sigma \left[\frac{\varepsilon^{-2/3}}{\sqrt{6}} \frac{h(\varepsilon^{-2/3}\sigma)}{K(\varepsilon^{-2/3}\sigma)\sqrt{h(0) - h(\varepsilon^{-2/3}\sigma)}} - \frac{\alpha \hat{h}_M^{3/2}}{\sigma} \right], \\
p_0 &= p_M \hat{h}_M / 3.
\end{aligned} \tag{5.31}$$

Only the coefficient a_0 has a dependence on the cutoff. Expanding the RG-flow equations for $a_{(\mu)}$ (3.14) for $\varepsilon^{-2/3} r_{(\mu)} \gg 1$, one can show that

$$\partial_{r_{(\mu)}} a_0 \simeq 0. \tag{5.32}$$

The force between the quark and antiquark as a function of the separation can be found solving for p in (5.30). When $L \to \infty$, the leading corrections are

$$p \simeq p_M + p_0 e^{-a_0/c_0} e^{-\frac{\varepsilon^{-2/3} L}{c_0 p_M}}. \tag{5.33}$$

We define the glueball mass scale M as

$$M = \frac{\varepsilon^{-2/3}}{c_0 p_M} = \frac{1}{(12)^{1/6} \hat{h}_M} \frac{\varepsilon^{2/3}}{g_s N_c \alpha'}. \tag{5.34}$$

Then, the force takes the same form as in (5.13), where the string tension and the coefficient of the exponential term are

$$\sigma_s = \frac{p_M}{2\pi\alpha'} = \frac{3^{1/6}}{2\pi} \hat{h}_M^{3/2} \lambda_{YM} M^2, \quad q_M = \frac{\hat{h}_M}{3} e^{-a_0/c_0}. \tag{5.35}$$

5.3 Interpretation of the Wilson loop as a flux tube in the IR

A popular and successful description of the physics of confined matter is formulated in terms of effective flux tubes, whose dynamics are determined by massless transverse fluctuations in the field theory directions. These objects describe the configuration of the color gauge fields sourced by the heavy quark and antiquark that run along the Wilson loop. According to this picture, in a confining theory, and for a large enough separation compared to the confining scale, the color flux between the two sources is concentrated in a tube that extends from one to the other. In principle, flux tubes in the absence of sources also describe dynamical excitations of the gauge theory, but they have to be either closed or ending on dynamical quarks (or other colored particles). The Wilson loop that determines the quark-antiquark potential is then determined by the properties of the flux tube in the IR theory. We will now interpret the result obtained for the force (5.13) using the holographic Wilsonian renormalisation in terms of a modification of the effective theory of the flux tube.

This picture emerges naturally in holography. A long classical string at the bottom of the dual geometry realizes a flux tube in the field theory. These elongated objects posses their own excitations, which are determined by the vibrational modes of the string in the different direction of the gravity theory. Fluctuations in the transverse field theory directions are massless and are determined by the four-dimensional Nambu-Goto action with string tension σ_s. In addition, there are massive fluctuations corresponding to perturbations away from the bottom, which from the point of view of the field theory dual should be interpreted as internal excitations of the flux tube. The piece of the string that connects the parallel paths of the rectangular Wilson loop and is close to the bottom of the geometry has the same properties as a flux tube.

To avoid subtleties with boundary conditions, let us characterize those fluctuations for a flux tube wrapping a compact spacial direction x of length L. This sourceless flux-tube excitation is realized by a closed string at the bottom of the geometry ($U = U_M$ in WQCD or $\tau = 0$ in KS). To better characterize the fluctuation dynamics of the flux-tube, we will use a series of changes of coordinates in each of the two models we have considered that provide a simple Lagrangian for small (quadratic) fluctuation around the classical string.

In the WQCD model, we will use the following change of coordinates

$$U = U_M \left(1 + c_\rho \rho^2 + c_\rho^{(2)} \rho^4\right), \quad \varphi = c_\theta \theta. \tag{5.36}$$

For $\rho \to 0$, if one fixes

$$c_\rho = \frac{3}{4R^{3/2} U_M^{1/2}}, \quad c_\rho^{(2)} = -\frac{3}{32 R^2 U_M}, \quad c_\theta = \frac{2}{3}\frac{R^{3/2}}{U_M^{1/2}}, \tag{5.37}$$

the metric is, to leading order,

$$ds^2 = p_M \left(1 + \frac{\rho^2}{\rho_M^2}\right) \eta_{\mu\nu} dx^\mu dx^\nu + d\rho^2 + \rho^2 d\theta^2, \tag{5.38}$$

where

$$\rho_M^2 = \frac{8}{9} R^{3/2} U_M^{1/2}. \tag{5.39}$$

The periodicity of θ is

$$\theta \sim \theta + 2\pi \;\Rightarrow\; \varphi \sim \varphi + \beta_\varphi, \quad \beta_\varphi = \frac{2\pi}{c_\theta}. \tag{5.40}$$

We now change to Cartesian coordinates

$$d\rho^2 + \rho^2 d\theta^2 = dV^2 + dW^2, \tag{5.41}$$

such that the metric is

$$ds^2 = p_M \left(1 + \frac{V^2 + W^2}{\rho_M^2}\right) \eta_{\mu\nu} dx^\mu dx^\nu + dV^2 + dW^2. \tag{5.42}$$

Similarly, in the KS model we expand the metric for small values of τ and pick the directions transverse to the three-cycle

$$\theta_1 = \theta_2 = \theta, \quad \phi_1 = -\phi_2 = \phi, \quad \psi = \pi. \tag{5.43}$$

The metric takes the form

$$ds^2 = p_M\left(1+\frac{\tau^2}{12\hat{h}_M}\right)\eta_{\mu\nu}dx^\mu dx^\nu + \frac{\varepsilon^{4/3}}{2(12)^{1/3}p_M}\left(d\tau^2+\tau^2(d\theta^2+\sin^2\theta d\phi^2)\right). \tag{5.44}$$

We see explicitly that the two-cycle collapses smoothly to zero size at $\tau = 0$. We now introduce Cartesian coordinates to describe the region around $\tau = 0$

$$d\tau^2+\tau^2(d\theta^2+\sin^2\theta d\phi^2) = 2(12)^{1/3}p_M\varepsilon^{-4/3}\left(dZ^2+dV^2+dW^2\right), \tag{5.45}$$

where we have rescaled the τ coordinate in such a way that the metric takes the form

$$ds^2 = p_M\left(1+\frac{Z^2+V^2+W^2}{\rho_M^2}\right)\eta_{\mu\nu}dx^\mu dx^\nu + dZ^2+dV^2+dW^2. \tag{5.46}$$

Where, in this case

$$\rho_M^2 = (12)^{2/3}\hat{h}_M\frac{\varepsilon^{4/3}}{2p_M}. \tag{5.47}$$

Denoting the field theory directions as $x^\mu = (t,x,y,z)$, we allow the string to fluctuate in the transverse directions y and W, keeping $z = 0$ and $V = 0$ (and $Z = 0$ for KS) fixed. In the static gauge, the embedding functions of the string are

$$X^0 = \tau,\quad X^1 = \sigma,\quad X^2 = y(\tau,\sigma),\quad X^3 = 0,\quad (Z=0),\quad V=0,\quad W = W(\tau,\sigma). \tag{5.48}$$

The induced metric is

$$\begin{aligned} g_{\tau\tau} &= -\left[p_M\left(1+\frac{W^2}{\rho_M^2}\right)(1-\dot{y}^2)-\dot{W}^2\right],\\ g_{\sigma\sigma} &= \left[p_M\left(1+\frac{W^2}{\rho_M^2}\right)(1+(y')^2)+(W')^2\right],\\ g_{\tau\sigma} &= p_M\left(1+\frac{W^2}{\rho_M^2}\right)\dot{y}y'+\dot{W}W', \end{aligned} \tag{5.49}$$

where $\dot{y} = \partial_\tau y$, $y' = \partial_\sigma y$, etc. Expanding to second order in the fluctuations, the determinant is

$$\sqrt{-g} = p_M\left(1-\frac{1}{2}\dot{y}^2+\frac{1}{2}(y')^2+\frac{1}{\rho_M^2}W^2\right)-\frac{1}{2}\dot{W}^2+\frac{1}{2}(W')^2. \tag{5.50}$$

Changing the normalization

$$W = \sqrt{p_M}\chi, \tag{5.51}$$

we find

$$\sqrt{-g} = p_M\left(1-\frac{1}{2}\dot{y}^2+\frac{1}{2}(y')^2-\frac{1}{2}\dot{\chi}^2+\frac{1}{2}(\chi')^2+\frac{m^2}{2}\chi^2\right). \tag{5.52}$$

Where

$$m^2 = \frac{2p_M}{\rho_M^2},\quad m^2_{WQCD} = \frac{9}{4}\frac{U_M}{R^3},\quad m^2_{KS} = \frac{1}{(12)^{1/3}\hat{h}_M^2}\frac{\varepsilon^{4/3}}{(g_sN_c\alpha')^2}. \tag{5.53}$$

In both cases the mass of the string mode is of the same order as the mass of the glueballs $m^2 = M^2$. The action for the fluctuations to quadratic order is

$$S_{\rm NG} = -\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-g} \simeq \sigma_s \int d^2\sigma \left(-1 + \frac{1}{2}\dot{y}^2 - \frac{1}{2}(y')^2 + \frac{1}{2}\dot{\chi}^2 - \frac{1}{2}(\chi')^2 - \frac{M^2}{2}\chi^2\right). \tag{5.54}$$

Therefore, M is the mass of the internal excitation of a flux tube in the IR theory. For the WQCD model, the same quadratic action was derived in [57], where one can also find the quadratic action for world-sheet fermions. The fluctuations of the flux tube between the quark-antiquark pair are also governed by the same quadratic Lagrangian.

We now consider an open flux tube of length L. This is described in the holographic dual by a string satisfying Neumann boundary conditions at the endpoints. In this particular case, a string extended along the x direction at the bottom of the geometry $W = 0$ is a solution to the classical action satisfying the boundary conditions. This excitation does not describe a Wilson loop, since the endpoints of the flux do not connect to the boundary. However, for large inter-quark separation, it approximates well the dual string configuration. In order to describe a Wilson loop, we must modify this string such that it bends towards the boundary. Far from the edges, where the inter-quark string profile lies close to the bottom of the geometry, this can be described by a small perturbation of the open flux tube string induced by additional static sources for the massive mode at the edges of the string,

$$S = S_{\rm NG} - \frac{1}{2\pi\alpha'}\int d^2\sigma W \mathcal{J}_W, \tag{5.55}$$

where

$$\mathcal{J}_W = -\sqrt{p_M}\, q_\chi \left(\delta(\sigma) + \delta(\sigma - L)\right). \tag{5.56}$$

The equation of motion for the massive mode is, for a static configuration

$$-\chi'' + M^2\chi = q_\chi \left(\delta(\sigma) + \delta(\sigma - L)\right). \tag{5.57}$$

This can be solved by using a Green's function, which for Neumann boundary conditions is

$$G(\sigma, \sigma') = \frac{\Theta(\sigma - \sigma')\chi_2(\sigma)\chi_1(\sigma') + \Theta(\sigma' - \sigma)\chi_1(\sigma)\chi_2(\sigma')}{M\sinh(ML)}. \tag{5.58}$$

Where

$$\chi_1 = \cosh(M\sigma), \quad \chi_2 = \cosh(M(\sigma - L)) \tag{5.59}$$

The solution is

$$\chi = q_\chi \int_0^L d\sigma' G(\sigma, \sigma') \big(\delta(\sigma') + \delta(\sigma' - L)\big) = q_\chi \frac{\cosh(M\sigma) + \cosh(M(L - \sigma))}{M\sinh(ML)}. \tag{5.60}$$

Far from the string edges, the quark-antiquark string is well described by the classical open flux tube configuration plus the massive mode perturbation χ, since outside the endpoints the solution satisfies the classical string equations. We can then use the massive mode profile to determine the radial position of the lowest point of the string, in this case at

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$\sigma = L/2$, which, in turn, via (2.17) determines the force between the quark and antiquark to be

$$p = p_M \left(1 + 2q_\chi^2 e^{-ML}\right). \tag{5.61}$$

This reproduces (5.13) provided the strength of the source is tuned to the right value

$$q_\chi = \sqrt{\frac{q_M}{2}}. \tag{5.62}$$

The analysis we have just performed shows that the exponential corrections we have found in the quark-aniquark potential are due to non-vanishing excitations of an internal massive mode of the flux tube that connects those two sources. In holography, this mode corresponds to the fluctuations of the string along the holographic direction. For all holographic models in which confinement is associated with the closing of some cycle, as the two models we have considered, this excitation is massive and its mass is related to the glueball mass scale. Therefore, this type of exponential correction in the heavy quark potential is a generic expectation of the holographic description of confinement.

6 Discussion

In this paper we have given the first steps towards a Wilsonian RG flow analysis of Wilson loops in strongly coupled field theories which enjoy a holographic dual. We have employed this method to analyze the expectation value of the rectangular Wilson loop in those theories, which encodes the heavy quark-antiquark potential. In the long distance limit, the effective theory approach we have developed has allowed us to determine the heavy quark potential from IR physics, without detailed information of the UV properties of the corresponding gauge theory. The leading correction to the long distance potential in a $1/L$-expansion is introduced via a double trace deformation of the effective one-dimensional theory localized on the world-line of the loop. The coefficient of that deformation depends explicitly on the effective theory cut-off and satisfies a renormalisation group equation which ensures the independence of observables on the renormalisation scale. All the information on UV physics is reduced to the fixed values of the coefficients at some given scale. The procedure we have developed is systematically improvable, and higher order corrections can be in principle introduced by adding further multitrace terms to the IR effective action, following the standard Wilsonian procedure. By applying this method to two concrete holographic examples, we have been able to determine analytically the leading corrections to the heavy quark potential, which provides us with new understanding of the long distance dynamics in those set-ups.

In our first example we have analyzed a strongly coupled theory that flows to a (conformal) fixed point in the IR. As dictated by conformal symmetry in the IR, the long distance heavy quark potential behaves as $1/L$, with a coefficient determined by the IR physics, which is easy to determine. Employing our Wilsonian approach, we have found that the double trace term introduces a universal correction, independent of how the IR CFT is deformed, that contributes to the potential as $\sim 1/L^4$ or, equivalently, as $\sim 1/L^5$ in the force (4.25), (4.26). This is consistent with the expected flow in a one-dimensional defect theory between a double trace term of scaling dimension $\Delta = -2$ in the UV to a scaling dimension $\Delta = 4$ in the IR. In the UV the associated single trace operator is a variation in the trajectory of the Wilson loop, while in the IR it is the transverse electric field as defined in (4.33).

The second example that we have studied is a set of confining holographic gauge theories. For these theories, the analytic access to the long distance properties of the potential has allowed us to better understand the properties of the potential in terms of an effective theory of flux tubes. This description appears naturally in holography, where large Wilson loops are dual to long strings that stretch along a large distance at the bottom of the geometry and lead to the characteristic linear confining potential. Holographic models predict that the effective action of a flux tube in a confining theory is given by the four-dimensional Nambu-Goto action plus a set of additional internal massive modes, which correspond to fluctuations along the holographic direction. Using our Wilsonian approach, we have identified the contribution of those modes to the heavy quark potential, that appear as an exponentially decaying therm $\sim e^{-ML}$, where M is of the same order as the glueball masses. The information about the UV physics in this case is hidden in the factor multiplying the exponential correction.

The exponential term can be understood as originating from the profile of a massive mode induced by sources localized at the endpoints. The profile of the massive mode maps to the profile of the string along the holographic radial coordinate. This is a classical contribution (from the point of view of the effective action) to the quark-antiquark potential.[5] Note that while the exponential correction looks somewhat similar to the contributions from a rigidity term [61–63] (see [64, 65] for recent lattice calculations in $2+1$ dimensions), those would be suppressed at strong 't Hooft coupling, and the coefficient of the exponential would decrease at long separations as $\sim 1/L^{1/2}$, unlike the contribution we have identified. Therefore, this new contribution cannot be explained in terms of an effective action of the flux tube with massless transverse modes alone, it involves massive modes that have not been yet observed but should be present in any confining gauge/gravity dual qualitatively similar to the known examples. This opens the door to test if our understanding of confinement in gauge/gravity duality is qualitatively correct.

Consistently with this understanding, it has been now established by lattice calculations that the Nambu-Goto action is a very good effective description of a flux tube in $2+1$ [66–68] and $3+1$ dimensions [69] (see [70] for reviews). At long distances, this observation may be partially expected from just effective theory arguments applied to a derivative expansion of transverse fluctuations of the flux tube, without invoking holography, since the energy of a long flux tube and its excitations have an expansion in odd powers of the length L, and corrections that deviate from four-dimensional Nambu-Goto can only start at $O(1/L^7)$ (or $O(1/L^5)$ for open strings) [59, 60, 71–73]. However, lattice calcu-

[5] Note that there can also be quantum corrections to the potential from integrating both massive and massless modes, the Luscher term being the most significant, see for instance [57–60]. At very strong coupling they are suppressed by inverse powers of the 't Hooft coupling, this is the reason our classical string calculation does not capture them. At finite coupling the classical term is not necessarily dominant since it is exponentially suppressed at long distances

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lations of the spectrum of fluctuations of flux-tubes agree with that of a four-dimensional Nambu-Goto string even at distances of order of the string length $\sigma_s L^2 \sim 1$, except in some parity odd channels, where an additional mode has been observed [69]. While the success of the four-dimensional Nambu-Goto description even in this regime is still an open problem,[6] the existence of new modes beyond those of the four-dimensional Nambu-Goto string is expected from holography. However, the observed new mode does not correspond to the massive mode responsible for corrections to the potential, since this latter mode is parity even. It would be interesting to test whether other massive bosonic and fermionic modes of the holographic string are consistent with the additional excitations observed in the lattice.

The new mode we have identified may be directly observed by a detailed comparison of the quark-antiquark potential and the energy of a flux tube. In confining holographic models, a flux tube of length L has an energy which is the same as the quark-antiquark potential except without the classical contributions induced by sources at the endpoints. Comparing the two one may be able to isolate the exponential correction, and get rid of other possible contributions such as finite size effects, which are also expected to be suppressed in the large-N limit [69]. Since this new massive mode is a generic expectation from holography, this analysis could provide an interesting check of gauge/gravity duality for confining theories. To further test holographic expectations, it would be interesting to develop further the effective action of a flux tube including internal massive modes, as well as to study the effect on meson Regge trajectories if a background profile of those modes is turned on. A comparison could be made with other holography-inspired models, such as a flux tubes with massive endpoints [79, 80]. We would also like to mention that finite quark effects introduce corrections to the relation between the potential and the expectation value of the Wilson loop that depend on the chromoelectric field squared. In the context of the effective flux tube picture, these corrections where analyzed in [81]. Nevertheless, these have a different nature than the ones we have discussed, since the latter remain in the strictly infinite quark limit. It would be interesting to study the interplay of these two sets of corrections in the analysis of heavy meson properties. We leave these questions for future work.

Looking ahead, we would like to stress that our current analysis is restricted to static heavy quark sources. It would be very interesting to develop a similar Wilsonian RG approach for Wilson loops along arbitrary curves, which would allow us to address a whole new suit of physical processes, such as the acceleration and radiation of heavy quarks. It would also be interesting to include finite temperature and density effects in the holographic description. In this way, we could address the energy loss of partons in strongly coupled plasma focussing in the IR properties of the plasma. This analysis could provide new theory input to the already existing holographically inspired analyses of jet quenching

[6]The success of the four-dimensional Nambu-Goto description even in this regime has been attributed to the approximate integrability of the effective action [74–77], explaining also the deviations in parity odd channels from the appearance of an internal axion-like mode. It has been conjectured that in the large-N limit the theory could become exactly integrable [76], although lattice calculations seem to disfavor this possibility [78].

data in heavy ion collisions [82–86]. In connection with this and other phenomenological applications, and as final remark, we would like to mention that the Wilsonian approach we have pursued could be exploited in a semi-holographic approach, where only IR physics are described using the gauge/gravity dual and the coefficients of the double trace deformations sensitive to UV physics is used to match with a given microscopic theory.

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A RG flow evolution of the boundary action

We will follow the procedure introduced in [17], where the evolution of the coefficients in the boundary action is derived from a Hamilton-Jacobi type of equation. We will not consider the most general type of string profiles, but we will restrict to work in the static gauge $X^r = \sigma$, $X^0 = \tau$ and consider only static profiles $X^i = x^i(\sigma)$.

The induced string metric for these configurations is

$$ds_2^2 = -e^{2A} d\tau^2 + \frac{1}{f}\left(1 + f e^{2A}(\partial_\sigma \vec{x})^2\right) d\sigma^2. \tag{A.1}$$

Then, the determinant of the induced metric has the form

$$\sqrt{-h} = \frac{e^A}{\sqrt{f}}\sqrt{1 + f e^{2A}(\partial_\sigma \vec{x})^2}. \tag{A.2}$$

We introduce the conveniently normalized conjugate momenta that determine the force acting on the quarks in the dual theory

$$p_i = -\frac{\delta\sqrt{-h}}{\delta\partial_\sigma x^i} = -\frac{e^{4A}}{\sqrt{-h}}\partial_\sigma x_i. \tag{A.3}$$

Therefore, the derivative of the profile is

$$\partial_\sigma x^i = -p^i e^{-4A}\sqrt{-h} = -p^i \frac{e^{-3A}}{\sqrt{f}\sqrt{1 - e^{-4A}\vec{p}^2}}, \tag{A.4}$$

and the Nambu-Goto action is going to be proportional to

$$\sqrt{-h} = \frac{e^A}{\sqrt{f}\sqrt{1 - e^{-4A}\vec{p}^2}}. \tag{A.5}$$

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We write string action as the Nambu-Goto action plus a boundary action that depends on the position of the cutoff in the radial direction $r_{(\mu)}$ and the position of the string profile at the cutoff $\vec{x}_{(\mu)}$, with $\vec{x}=0$ being the position of the quark in the dual field theory:

$$S_{\text{string}} = -\frac{1}{2\pi\alpha'}\int_{\sigma<r_{(\mu)}} d^2\sigma\sqrt{-h} - \frac{1}{2\pi\alpha'}\int d\tau\, L_B[\vec{x}_{(\mu)}, r_{(\mu)}]. \tag{A.6}$$

Let us write down the condition that the action is stationary under changes of the position of the string at the cutoff

$$\delta S_{\text{string}} = \frac{1}{2\pi\alpha'}\int_{\sigma<r_{(\mu)}} d^2\sigma\, p_i \partial_\sigma \delta x^i - \frac{1}{2\pi\alpha'}\int d\tau\, \frac{\delta L_B}{\delta x^i_{(\mu)}}\delta x^i_{(\mu)} = 0. \tag{A.7}$$

Since $\partial_\sigma p_i = 0$, we get the condition

$$\frac{\delta L_B}{\delta x^i_{(\mu)}} = p_i. \tag{A.8}$$

On the other hand, the string action and the solution for the string profile should be independent of the position of the cutoff, which gives a second condition

$$\frac{d}{dr_{(\mu)}}S_{\text{string}} = -\frac{1}{2\pi\alpha'}\int d\tau\sqrt{-h}\Big|_{\sigma=r_{(\mu)}} - \frac{1}{2\pi\alpha'}\int d\tau\left[\partial_{r_{(\mu)}}L_B + \frac{\delta L_B}{\delta x^i_{(\mu)}}\partial_{r_{(\mu)}}x^i_{(\mu)}\right] = 0. \tag{A.9}$$

Solving for the radial derivative of the boundary action, we find the following RG flow evolution equation

$$\partial_{r_{(\mu)}}L_B = -\sqrt{-h}\Big|_{\sigma=r_{(\mu)}} - \frac{\delta L_B}{\delta x^i_{(\mu)}}\partial_{r_{(\mu)}}x^i_{(\mu)}. \tag{A.10}$$

This can be cast in the form of a Hamilton-Jacobi type of equation for the boundary action using (A.4), (A.5) and (A.8),

$$\partial_{r_{(\mu)}}L_B = -\frac{e^{A_{(\mu)}}}{\sqrt{f_{(\mu)}}}\sqrt{1 - e^{-4A_{(\mu)}}\delta_{ij}\frac{\delta L_B}{\delta x^i_{(\mu)}}\frac{\delta L_B}{\delta x^j_{(\mu)}}}. \tag{A.11}$$

Now, to solve this equation, we try an ansatz where the boundary action admits a power expansion for small values of $x^i_{(\mu)}$

$$L_B = M_{(\mu)} + \frac{1}{2a_{(\mu)}}(\vec{x}_{(\mu)})^2 + \cdots \tag{A.12}$$

Plugging this in (A.11), expanding for small $x^i_{(\mu)}$ and equating terms with equal powers of $x_{(\mu)}$, this gives the RG flow equations for the coefficients

$$\partial_{r_{(\mu)}}M_{(\mu)} = -\frac{e^{A_{(\mu)}}}{\sqrt{f_{(\mu)}}}, \quad \partial_{r_{(\mu)}}a_{(\mu)} = -\frac{e^{-3A_{(\mu)}}}{\sqrt{f_{(\mu)}}}. \tag{A.13}$$

These formulas coincide with the ones we derived before by direct inspection of the action integrated above the cutoff (3.14).

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The advantage of this formalism is that it allows a systematic computation of the next order corrections by adding multitrace terms to the boundary action. For instance, the first subleading correction would correspond to a quartic term

$$L_B = M_{(\mu)} + \frac{1}{2a_{(\mu)}}(\vec{x}_{(\mu)})^2 + \frac{1}{4b_{(\mu)}}((\vec{x}_{(\mu)})^2)^2 + \cdots . \tag{A.14}$$

The condition (A.8) gives the relation between the conjugate momenta and the positions at the cutoff, that depends on the coefficients of the multitrace terms

$$p^i \simeq \frac{x^i_{(\mu)}}{a_{(\mu)}} + \frac{(\vec{x}_{(\mu)})^2 x^i_{(\mu)}}{b_{(\mu)}}. \tag{A.15}$$

The RG flow equation for the coefficient of the quartic term can be derived by expanding to quartic order in $x^i_{(\mu)}$ the equation (A.11) and collecting terms with the same factors of the string profile. A straightforward calculation shows that

$$\partial_{r_{(\mu)}} b_{(\mu)} = -\frac{1}{2}\frac{e^{-7A_{(\mu)}}}{\sqrt{f_{(\mu)}}}\frac{b^2_{(\mu)}}{a^4_{(\mu)}} - 4\frac{e^{-3A_{(\mu)}}}{\sqrt{f_{(\mu)}}}\frac{b_{(\mu)}}{a_{(\mu)}}. \tag{A.16}$$

For the particular case we are studying, where the string is extended along one direction, (A.15) becomes

$$p \simeq \frac{x_{(\mu)}}{a_{(\mu)}} + \frac{x^3_{(\mu)}}{b_{(\mu)}}. \tag{A.17}$$

The, solving for $x_{(\mu)}$ to $O(p^3)$,

$$x_{(\mu)} \simeq a_{(\mu)} p - \frac{a^4_{(\mu)}}{b_{(\mu)}} p^3. \tag{A.18}$$

Therefore, the total length L, as defined in (3.15), is, to $O(p^3)$

$$\frac{L}{2} \simeq a_{(\mu)} p - \frac{a^4_{(\mu)}}{b_{(\mu)}} p^3 + p\int_{\sigma_*}^{r_{(\mu)}} \frac{e^{-3A}}{\sqrt{f}\sqrt{1-e^{-4A}p^2}}. \tag{A.19}$$

We can now show that L is RG invariant at this order, using first (A.13) and then (A.16)

$$\begin{aligned}
\partial_{r_{(\mu)}} L &= 2p\Bigg(\partial_{r_{(\mu)}} a_{(\mu)} - 4p^2 \frac{\partial_{r_{(\mu)}} a_{(\mu)} a^3_{(\mu)}}{b_{(\mu)}} + p^2 \partial_{r_{(\mu)}} b_{(\mu)} \frac{a^4_{(\mu)}}{b^2_{(\mu)}} \\
&\qquad\qquad + \frac{e^{-3A_{(\mu)}}}{\sqrt{f_{(\mu)}}}\left(1 + \frac{1}{2}e^{-4A_{(\mu)}}p^2 + O\left(p^4 e^{-8A_{(\mu)}}\right)\right)\Bigg) \\
&= 2p^3\left(4\frac{e^{-3A_{(\mu)}}}{\sqrt{f_{(\mu)}}}\frac{a^3_{(\mu)}}{b_{(\mu)}} + \partial_{r_{(\mu)}} b_{(\mu)} \frac{a^4_{(\mu)}}{b^2_{(\mu)}} + \frac{1}{2}\frac{e^{-7A_{(\mu)}}}{\sqrt{f_{(\mu)}}} + O\left(p^2 e^{-11A_{(\mu)}}\right)\right) \\
&= O\left(p^5 e^{-11A_{(\mu)}}\right).
\end{aligned} \tag{A.20}$$

This procedure can in principle be extended to an arbitrary order of the multitrace corrections, until the desired precision is achieved.

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Chapter 5

Holographic Wilsonian renormalization of a heavy quark moving through a strongly coupled plasma

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Holographic Wilsonian renormalization of a heavy quark moving through a strongly coupled plasma

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Abstract: A heavy quark moving through a strongly coupled deconfined plasma has a holographic dual description as a string moving in a black brane geometry. We apply the holographic Wilsonian renormalization method to derive a holographic effective string action dual to the heavy quark. The effective action only depends on the geometry between the black brane horizon and a cutoff localized in the radial direction, corresponding to the IR of the dual theory. We derive RG flow equations for the coefficients in the effective action and show that the force acting on the heavy quark is independent of the position of the cutoff. All the information about the UV is hidden in integration constants of the RG flow equations. This type of approach could be used to improve semi-holographic models where the UV is described by perturbative QCD and the IR by a holographic model.

Keywords: Gauge-gravity correspondence, AdS-CFT Correspondence, Holography and quark-gluon plasmas, Wilson, 't Hooft and Polyakov loops

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Contents

1 Introduction

The gauge/gravity duality [1–3], aka holography, has been successful in describing some qualitative properties of the quark-gluon plasma, in particular predicting an almost perfect fluid behavior and producing the famous KSS formula for the shear viscosity over entropy density ratio [4, 5], that captures the right order of magnitude deduced from hydrodynamic simulations of heavy ion collisions [6–10].

One of the most indicative signals of the formation of a deconfined quark gluon plasma is jet quenching (see [11] for a comprehensive review). If a highly energetic parton collision

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takes place close to the surface of the plasma ball, some of the particles produced may escape almost immediately, producing an observable jet, while the path of particles moving in the opposite direction might have to cross a significant portion of the plasma. In this case energy dissipation produced by the interaction with the plasma components weakens or prevents the formation of a back jet. The observation of jet quenching in heavy ion collisions is one of the most convincing evidences in favor of the formation of a deconfined plasma, and one of the most important probes into its properties.

Early on, the duality has been used to model the energy loss of heavy quarks moving through the plasma [12–15] in order to give an estimation of jet quenching.[1] The energy loss can be determined from the drag force the quark experiences, which in turn can be obtained from the expectation value of a Wilson line along the quark trajectory. Following [12, 13], the heavy quark maps to a dragging string moving through a black brane geometry in the holographic dual, from which the expectation value of the Wilson line can be extracted. The string ends at the asymptotic boundary of space and extends all the way to the black brane horizon. According to the usual holographic map, these two regions correspond to energy scales of the UV and IR of the field theory, respectively. This means that the heavy quark motion is sensitive to all the energy scales of the theory, in contrast for instance to hydrodynamic evolution, which is limited to the IR. The sensitivity to multiple energy scales is a challenge for the holographic description. In QCD the gauge coupling becomes weak at high energies, and if this feature was introduced in the holographic model, then the curvature in the dual geometry would become large and stringy corrections to the classical gravity approximation would not be negligible anymore. Introducing stringy corrections may be doable in principle by adding higher derivative terms in the gravitational action. In this fashion, the first corrections away from the strong coupling limit of some properties of the plasma have been studied for conformal theories [19–24]. However, for a theory with a running coupling like QCD, finding these corrections has not been attempted yet.

As a full string theory description of holographic duals is out of reach at the moment, it is desirable to tackle this issue in a way that avoids dealing with large curvature corrections. A possibility is to adopt a purely phenomenological approach and model the weakly coupled region neglecting possible higher curvature corrections. A second possibility is to follow an effective theory approach and use the holographic model to describe a limited range of energy scales where the theory is strongly coupled. In this category fall the hybrid or semi-holograhic models studied for instance in [25–32]. A drawback of hybrid models is that the theory changes abruptly when the holographic model is used, with no obvious systematic way to improve the method. We can compare this situation with the usual low energy effective field theory approach. In this case the couplings in the effective action are free parameters that have to be fixed by experiments or by matching to the microscopic theory, and a systematic improvement of the low energy effective theory is possible. This is one of the main properties that makes the effective field theory so successful, and hybrid models could be significantly improved if a similar procedure could be implemented for the holographic model. We will take the first steps in this direction by applying the method

[1]Different estimations of jet quenching involve particles moving at the speed of light [16] or light quarks or gluons [17, 18].

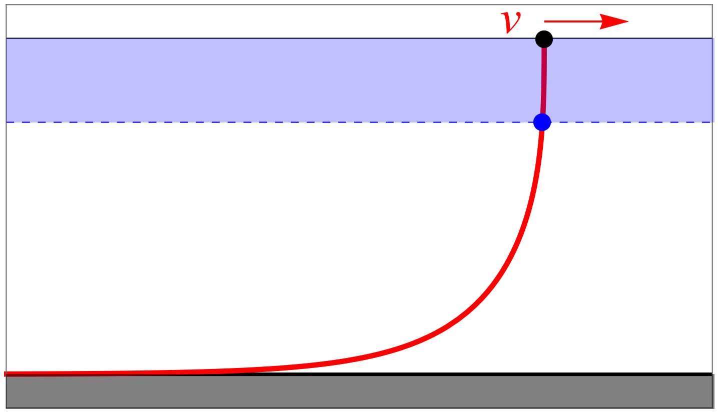

Figure 1. The holographic dual of a heavy quark moving at speed v is a string (red curve) ending at the asymptotic boundary at the position of the quark (black dot). The strings extends from the asymptotic boundary at the top to the black brane horizon at the bottom of the figure. A cutoff (dashed blue line) is introduced and the shaded region between the boundary and the cutoff is "integrated out". One is left with the string in the region between the cutoff and the horizon and determined boundary conditions for the endpoint of the string at the cutoff (blue dot).

of holographic "Wilsonian" renormalization [33, 34] to a moving string. Our analysis is an extension of [35], where we applied this method to static strings in order to extract the quark-antiquark potential.

In the Wilsonian method we introduce a cutoff in the dual geometry localized at a fixed distance from the asymptotic boundary. The region between the cutoff and the boundary is replaced by an action for the string endpoints at the cutoff, as sketched in figure 1. The cutoff action ensures that the string satisfies the right boundary conditions, in such a way that the force felt by the heavy quark is independent of the position of the cutoff. The coefficients of the cutoff action satisfy RG flow equations that are determined by the local geometry around the cutoff, in such a way that all the knowledge about the region between the cutoff and the asymptotic boundary is hidden in integration constants of the RG flow equations. Then, for a given IR effective theory, with a holographic dual corresponding to the geometry between the horizon and the cutoff, it is possible to match to multiple UV theories by appropriately tuning the integration constants of the RG flow.

We will derive the cutoff action and RG flow equations of the coefficients for a heavy quark moving at approximately constant velocity. We will consider a general black brane geometry, but we will also present the results for a IR geometry approximated by an AdS_5 black brane, as a simple example to illustrate the method. We will use these results to compute the drag force, including a couple of contributions proportional to the acceleration of the quark and the jerk, or acceleration rate. The first can be interpreted as originating from a thermal correction to the mass of the quark, while the second can be thought of as a combination of the Abraham-Lorentz force [36, 37], due to Larmor radiation, and a viscous contribution produced by the surrounding fluid [38]. The two contributions add up giving the total value found in [39].

The paper is organized as follows. In section 2 we introduce the holographic description of a heavy quark moving through a strongly coupled plasma as a string in a general black brane geometry. We simplify the analysis by considering a homogeneous and isotropic state for the plasma, and slow variations of the quark trajectory, compared to the inverse temperature. In section 3 we introduce a cutoff and derive the cutoff action for the string and the RG flow equations for the coefficients. In section 4, we apply the general formalism to a specific case where the theory has an IR fixed point and consequently the IR geometry is an AdS_5 black brane. We compute the drag force and the corrections proportional to the acceleration and the jerk. In section 5, we explain how the cutoff action and RG flow equations can be obtained more generally using the cutoff independence of the string action, and show that they agree with the previous results obtained by direct integration. We end with a discussion of the results in section 6. Some technical details of the calculations have been collected in the appendices.

2 Holographic description of a heavy quark in a plasma

We will start by reviewing the holographic dual to a heavy quark in a high temperature deconfined phase. We will simplify the analysis by imposing that the heavy quark is moving along one spatial direction in a trajectory of almost constant velocity, with changes in the trajectory that are slow compared to the time scale set by the inverse temperature. This last approximation is necessary in order to apply a low energy effective description. Note that the trajectory is fixed, so we are not considering the dynamics of the quark, only the force of the plasma acting over it.

The holographic dual of the plasma is a five-dimensional geometry with an event horizon extended along four directions, that are identified with the dual field theory directions. A homogeneous and isotropic black brane geometry dual to a strongly coupled 3+1 plasma can be cast in the general form

$$ds^2 = G_{MN}dx^M dx^N = G_{zz}(z)dz^2 + G_{tt}(z)dt^2 + G_{xx}(z)\delta_{ij}dx^i dx^j. \tag{2.1}$$

where (t, x^i), $i = 1, 2, 3$ are coordinates along the field theory directions and z is the holographic radial coordinate. The holographic radial direction is identified with energy or length scales in the dual field theory. Low energies, or long wavelengths (IR) correspond to the region close to the horizon, and high energies or short wavelengths (UV) to an asymptotic region far away from the horizon. If the space is asymptotically AdS_5, the region approaching the conformal boundary is the far UV. In addition to the metric shown in (2.1), there can be additional internal directions, but they will not play any role in the following.

We pick coordinates in such a way that there is a horizon at $G_{tt}(z_h) = 0$. For the AdS_5 black brane metric

$$G_{tt}(z) = -\frac{R^2}{z^2}h(z), \quad G_{zz}(z) = \frac{R^2}{z^2 h(z)}, \quad G_{xx} = \frac{R^2}{z^2}, \quad h(z) = 1 - \frac{z^4}{z_h^4}. \tag{2.2}$$

Therefore, R is the AdS radius and the boundary is at $z \to 0$. The Hawking temperature of the black brane maps to the temperature of the field theory and, as the temperature is increased, the horizon moves towards the asymptotic region, indicating that there are degrees of freedom of higher energy contributing to the plasma. The temperature for the AdS_5 black brane geometry is $T = 1/(\pi z_h)$. For a general metric we will define the functions

$$h(z) \equiv \left|\frac{G_{tt}}{G_{zz}}\right|^{1/2}, \quad f(z) \equiv G_{xx} h(z), \quad g(z) \equiv \frac{G_{xx}}{h(z)}. \tag{2.3}$$

In AdS_5 the function h coincides with its usual definition, while $f = |G_{tt}|$ and $g = G_{zz}$. In general, f and g are not equal to metric components, but we will still assume that there is an asymptotic boundary at $z \to 0$ even if the space is not asymptotically AdS_5.

We now introduce a heavy quark in the plasma. Considering the mass of the heavy quark to be effectively infinite, the heavy quark maps to a Wilson line and the holographic dual is a string ending at the quark trajectory on the asymptotic boundary. This identification was done originally for static quarks [40–42] and later on generalized to quarks in motion [12–15]. So we do not solve for the quark motion but rather find the force with which the plasma acts when the quark follows a fixed path.

To start with, we consider the simplest case of a heavy quark in $\mathcal{N} = 4$ SYM. The dual geometry is $AdS_5 \times S^5$, with a string attached to the AdS_5 boundary at the location of the Wilson line and localized at a point in the internal space, in this case the S^5. To be more precise, this corresponds to a 1/2 BPS loop that also couples to $\mathcal{N} = 4$ SYM scalars, but for simplicity we will restrict to this case. We will assume in the following that this setup can be generalized to other holographic duals, i.e. we will use a string to describe a Wilson loop in different geometries, implicitly taking the metric in the string frame and neglecting any motion along internal space directions.

The dynamics of the string are given by the Nambu-Goto action for the embedding functions X^M

$$S_{\text{NG}} = -T_s \int d^2\sigma\, \mathcal{L}_{\text{NG}} = -T_s \int d^2\sigma\, \sqrt{-\det g_{ab}(X)}, \tag{2.4}$$

where $T_s = 1/(2\pi\alpha')$ is the string tension, σ^a, $a = 0, 1$ are the worldsheet coordinates and g_{ab} is the induced metric

$$g_{ab} = G_{MN}(X)\partial_a X^M \partial_b X^N. \tag{2.5}$$

2.1 Slowly moving quarks

In the first place we will consider a heavy quark that is almost at rest, but can move with a small varying velocity. Let us assume that the quark moves along one spatial direction. An ansatz for the embedding functions in the static gauge is

$$t = X^0 = \sigma^0,\ X^1 = X(t, z),\ X^2 = X^3 = 0,\ z = X^z = \sigma^1. \tag{2.6}$$

The Nambu-Goto action simplifies to

$$\mathcal{L}_{\text{NG}} = \sqrt{|G_{tt}|G_{zz}} \sqrt{1 + G_{xx}\left(\frac{1}{G_{zz}}(X')^2 - \frac{1}{|G_{tt}|}(\dot{X})^2\right)}. \tag{2.7}$$

Where we have defined the derivatives $X' = \partial_z X$, $\dot{X} = \partial_t X$.

A slowly moving string $\dot{X} \ll 1$ will have a profile that is almost a straight line $X' \ll 1$, as long as we are not too close to the horizon. Then, the string action can be approximated away from the horizon by the quadratic terms

$$\mathcal{L}_{\rm NG} \simeq \sqrt{|G_{tt}|G_{zz}} - \frac{g(z)}{2}(\dot{X})^2 + \frac{f(z)}{2}(X')^2 \,. \tag{2.8}$$

Within this approximation the equations of motion take the simple form

$$(fX')' - g\ddot{X} = 0. \tag{2.9}$$

Assuming that the quark follows a trajectory $x^\mu = (t, x(t), 0, 0)$, we should fix the position of the string at the boundary to $X(z=0,t) = x(t)$. Since we are considering slow motion we can use a derivative expansion to find the solutions, at least away from the horizon where the function $g(z)$ remains bounded. We expand the profile of the string according to the order in time derivatives

$$X = X^{(0)} + X^{(1)} + X^{(2)} + \cdots \tag{2.10}$$

Order by order we have the set of equations

$$\left(fX^{(0)\prime}\right)' = \left(fX^{(1)\prime}\right)' = 0, \quad \left(fX^{(n)\prime}\right)' = g\ddot{X}^{(n-2)}, n \geq 2. \tag{2.11}$$

The equations can be solved recursively. The lowest order solutions are

$$X^{(0)}(t,z) = x(t) + p^{(0)}(t)a(z), \quad X^{(1)}(t,z) = p^{(1)}(t)a(z), \quad a(z) = \int_0^z \frac{du}{f(u)}. \tag{2.12}$$

where $p^{(0)}(t)$, $p^{(1)}(t)$ are integration constants. The solutions at higher orders take the general form

$$X^{(n)}(t,z) = p^{(n)}(t)a(z) + \int_0^z \frac{du}{f(u)} \int_{z_c}^u dv g(v)\ddot{X}^{(n-2)}(t,v). \tag{2.13}$$

We have fixed the limits of the integrals in such a way that

$$X(t,z=0) = x(t), \quad \partial_z X\Big|_{z=z_c} = \frac{1}{f(z_c)}\left(p^{(0)}(t) + p^{(1)}(t) + p^{(2)}(t) + \cdots\right) \equiv \frac{p}{f(z_c)}. \tag{2.14}$$

2.2 Fast moving quarks

The previous analysis is a perturbation around a quark at rest. We can generalize it by taking as the unperturbed solution a quark moving at constant velocity. The background profile is the "dragging string" found in the original calculations of the drag force [12, 13].

The ansatz for the embedding functions is a modification of the ansatz used in the static gauge. Taking $\sigma^0 = t$, $\sigma^1 = z$,

$$X^0 = t + t_0(z), \; X^1 = vt + x_0(z) + X(t,z), \; X^2 = X^3 = 0, \; X^z = z. \tag{2.15}$$

The background profile has constant velocity v. The functions $t_0(z)$, $x_0(z)$ determine the profile of the dragging string. t_0 determines the gauge and can be conveniently chosen, and x_0 is obtained by solving the equations of motion of the background profile. They are fixed to

$$t_0' = \frac{G_{xx}}{|G_{tt}|} v x_0', \quad x_0' = p_0 \sqrt{\frac{|G_{tt}| G_{zz}}{G_{xx} \left(|G_{tt}| G_{xx} - p_0^2\right) \left(|G_{tt}| - G_{xx} v^2\right)}} \tag{2.16}$$

Where p_0 is a constant. Note that our gauge choice differs from [12, 13], where $t_0 = 0$.

The condition that the solution is real everywhere fixes

$$p_0^2 = |G_{tt}| G_{xx} \Big|_{z=z_*}, \quad v^2 = \frac{|G_{tt}|}{G_{xx}} \Big|_{z=z_*}. \tag{2.17}$$

The point z_* is where the speed of the string equals the speed of light on a radial slice, it corresponds to an effective horizon on the string worldsheet outside the black hole horizon. If the geometry is the AdS_5 black brane, the solution to these equations is

$$z_* = \gamma^{1/4} z_h, \quad p_0 = \frac{R^2}{z_h^2} \gamma v, \quad \gamma = \frac{1}{\sqrt{1-v^2}}. \tag{2.18}$$

The Nambu-Goto action expanded to quadratic order in the perturbation is

$$\begin{aligned} \mathcal{L}_{\rm NG} \simeq & \sqrt{|G_{tt}| G_{zz} G_{xx}} \sqrt{\frac{|G_{tt}| - G_{xx} v^2}{|G_{tt}| G_{xx} - p_0^2}} + p_0 X' - v \sqrt{\frac{|G_{tt}| G_{xx}^3 G_{zz}}{\left(|G_{tt}| G_{xx} - p_0^2\right) \left(G_{tt} - G_{xx} v^2\right)}} \dot{X} \\ & - \frac{1}{2} g_v(z) (\dot{X})^2 + \frac{1}{2} f_v(z) (X')^2. \end{aligned} \tag{2.19}$$

Where we have defined the functions

$$\begin{aligned} g_v(z) &= \left(|G_{tt}| G_{xx} G_{zz}\right)^{1/2} \frac{\left(|G_{tt}| G_{xx} - p_0^2\right)^{1/2}}{\left(|G_{tt}| - G_{xx} v^2\right)^{3/2}}, \\ f_v(z) &= \left(|G_{tt}| G_{xx} G_{zz}\right)^{-1/2} \frac{\left(G_{tt} G_{xx} - p_0^2\right)^{3/2}}{\left(|G_{tt}| - G_{xx} v^2\right)^{1/2}}. \end{aligned} \tag{2.20}$$

The terms linear in the perturbation are total derivatives, as expected in an expansion around a solution of the equations of motion. Therefore, they do not affect to the equations of motion of the perturbation. The quadratic terms take the same form in this gauge as those for a slowly moving quark (2.8), replacing the functions f, g by f_v, g_v. Then, we can apply the same type of derivative expansion as for the slowly moving quark and all the results have a straightforward generalization to the case of a fast moving quark.

3 Effective string action with a radial cutoff

We now proceed to the derivation of the holographic Wilsonian effective action for the string. First we will use holographic renormalization on the string action to identify the quantity that maps to the force acting on the heavy quark in the dual field theory. For quarks moving along varying trajectories it is not exactly the canonical momentum conjugate to the position of the quark, as it was for quarks moving at constant velocities in [12, 13], because the canonical momentum depends on the holographic radial coordinate

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in this case. Next, we will introduce a cutoff localized in the radial direction and "integrate out" the region between the asymptotic boundary and the cutoff. As a result, the remaining string action is defined in the region between the horizon and the cutoff, and there is an additional boundary contribution at the cutoff. Finally, we will derive the RG flow equations obtained from varying the position of the cutoff and use them to show that physical observables obtained from the effective action are independent of the cutoff. As the action for fast moving quarks can be mapped straightforwardly to the action for slowly moving quarks, we will restrict the analysis in this section only to the latter, and show results for both cases in the next section.

3.1 Force acting on the quark

The action (2.7) determines the string profile for a given quark trajectory $x(t)$ by fixing the position of the string at the asymptotic boundary $z = 0$. We will now identify the relevant physical observable, the force acting on the quark. It is convenient to momentarily assume that the metric is asymptotically AdS_5. Then, when $z \to 0$

$$f(z) \simeq g(z) \simeq \frac{R^2}{z^2}, \tag{3.1}$$

Then, the equation for the profile can be approximated by

$$X'' - \frac{2}{z}X' - \ddot{X} = 0. \tag{3.2}$$

Solutions to this equation have a boundary expansion

$$X(t,z) = x(t) - \frac{1}{2}\ddot{x}(t)z^2 + \frac{F(t)}{3R^2}z^3 + \cdots . \tag{3.3}$$

The string action evaluated on shell is

$$\mathcal{L}_{\rm NG} \simeq \frac{R^2}{z^2} + \partial_z\left(\frac{f}{2}X'X\right). \tag{3.4}$$

The action is divergent, we will regularize it by introducing a UV cutoff z_Λ and adding a boundary counterterm that cancels the divergence when $z_\Lambda \to 0$. The boundary counterterm is determined by the induced metric at the boundary

$$S_{c.t} = T_s R\int dt\sqrt{-g^b}, \quad g^b = g^b_{00} = g^b_{00}\Big|_{z_\Lambda} \simeq -\frac{R^2}{z_\Lambda^2}(1-\dot{X}^2(z_\Lambda)). \tag{3.5}$$

Since we are approximating the action to quadratic order,

$$\sqrt{-g^b} \simeq \frac{R^2}{z_\Lambda}\left(1-\frac{1}{2}\dot{X}(z_\Lambda)^2\right). \tag{3.6}$$

The regularized action is then

$$S_{\rm string} = \lim_{z_\Lambda\to 0} T_s\int dt\left[-\int_{z_\Lambda}^{z_h} dz\mathcal{L}_{\rm NG} + R\sqrt{-g^b}\right]. \tag{3.7}$$

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The string action determines the effective potential felt by the quark $S_{\rm string} = -\int dt V_q$, so its variation respect to the position of the quark at the boundary determines the force

$$\delta S_{\rm string} \simeq \lim_{z_\Lambda \to 0} T_s \int dt \left[f X' \delta X \Big|_{z_\Lambda} + \frac{R^2}{z_\Lambda} \ddot{X}(z_\Lambda) \delta X(z_\Lambda) \right] = T_s \int dt F(t) \delta x. \tag{3.8}$$

Where we have used (3.3). The force acting in the x direction is

$$\mathcal{F}_x = -\frac{\delta V_q}{\delta x} = T_s F(t). \tag{3.9}$$

For a fast moving quark there is a small modification of this result, as the background profile also gives a contribution to the force. The background contribution comes from the linear terms in the action (2.19). Doing a variation, the term with a time derivative drops, and the term with a radial derivative gives the boundary contribution

$$\delta S_{\rm back} \simeq \lim_{z_\Lambda \to 0} T_s \int dt \left(-p_0 \delta X \Big|_{z_\Lambda} \right) = -T_s \int dt\, p_0 \delta x. \tag{3.10}$$

Therefore, the contribution of the unperturbed string profile to the force is

$$\mathcal{F}_x^v = -T_s p_0. \tag{3.11}$$

Let us now find an expression for the slow moving quark, using the solutions (2.12), (2.13). The derivatives give

$$f \partial_z X^{(0)} = p^{(0)}, \quad f \partial_z X^{(1)} = p^{(1)}, \quad f \partial_z X^{(n)} = p^{(n)} + \int_{z_c}^{z} g(v) \ddot{X}^{(n-2)}(v). \tag{3.12}$$

Summing over all orders in the expansion, the radial derivative is

$$f X' = p + \ddot{x} \int_{z_c}^{z} g(v) + \ddot{p} \int_{z_c}^{z} g(v) a(v) + O(\partial_t^4 x, \partial_t^4 p). \tag{3.13}$$

We can do a Taylor expansion if we extract the divergent contribution from the term with a factor $\ddot{x}$:

$$f X' = p + \ddot{x} \left[A(z) - \frac{R^2}{z} \right] + \ddot{p} B(z) + O(\partial_t^4 x, \partial_t^4 p). \tag{3.14}$$

Where we have defined

$$A(z) = \frac{R^2}{z_c} + \int_{z_c}^{z} \left(g(v) - \frac{R^2}{v^2} \right), B(z) = \int_{z_c}^{z} g(v) a(v). \tag{3.15}$$

Then,

$$X' \simeq -\ddot{x} z + F(t) \frac{z^2}{R^2} + \cdots. \tag{3.16}$$

Where the coefficient that determines the force is

$$F(t) = p + \ddot{x} A(0) + \ddot{p} B(0) + O(\partial_t^4 x, \partial_t^4 p). \tag{3.17}$$

Although it may not look like it at first sight, this expression is independent of z_c. Using (2.14) and the equations of motion (2.9),

$$\partial_{z_c} p = \partial_{z_c}(f(z_c)X'(z_c)) = g(z_c)\ddot{X}(z_c). \tag{3.18}$$

Then, the derivative of the force is

$$\partial_{z_c} F = g(z_c)\ddot{X}(z_c) - g(z_c)\ddot{x} - g(z_c)a(z_c)\ddot{p} + O(\partial_t^4 x, \partial_t^4 p) = 0 + O(\partial_t^4 x, \partial_t^4 p). \tag{3.19}$$

where we have used that $X(z_c) = x + pa(z_c) + O(\partial_t^2 x, \partial_t^2 p)$.

3.2 Introducing a cutoff

Once we have determined that the force is (3.9), we will formulate a prescription to compute it when there is a cutoff at $z = z_c$ in the geometry such that the string does not reach the AdS boundary, but it is extended between the horizon and the cutoff.

First, we will split the string action in two parts, corresponding to the integration along the radial coordinate in the UV region $z_c > z > z_\Lambda$ and the IR region $z_h > z > z_c$.

$$S_{\text{string}} = S_{\text{IR}} + S_{\text{UV}}. \tag{3.20}$$

Where

$$\begin{aligned} S_{\text{IR}} &= -T_s \int dt \int_{z_c}^{z_h} dz \mathcal{L}_{\text{NG}}, \\ S_{\text{UV}} &= \lim_{z_\Lambda \to 0} T_s \int dt \left[-\int_{z_\Lambda}^{z_c} dz \mathcal{L}_{\text{NG}} + R\sqrt{-g^b} \right]. \end{aligned} \tag{3.21}$$

We will use the on-shell expression (3.4) and the counterterm (3.6) to evaluate the UV action expanded to quadratic order in the perturbation

$$S_{\text{UV}} \simeq T_s \int dt \left[\frac{R^2}{z_c} - \frac{f(z_c)}{2} X(z_c)X'(z_c) + \frac{1}{2}\lim_{z_\Lambda \to 0}\left(f(z_\Lambda)X(z_\Lambda)X'(z_\Lambda) + \frac{R^2}{z_\Lambda}\dot{X}(z_\Lambda)^2 \right) \right]. \tag{3.22}$$

From now on we will denote the position of the string at the cutoff as $X(z_c) \equiv x_c$. As we show in appendix A, for a slowly moving quark the UV action can be approximated by

$$\begin{aligned} S_{\text{UV}} \simeq T_s \int dt \Big[& M_c - \frac{1}{2}K_c \dot{x}^2 - \frac{1}{2a_c}(x_c - x)^2 + \frac{1}{2}m_c(\dot{x}_c - \dot{x})^2 \\ & - \kappa_c \ddot{x}(x_c - x) + O(\partial_t^4 x, \partial_t^4 x_c) \Big]. \end{aligned} \tag{3.23}$$

Where we have defined the coefficients as

$$M_c = \frac{R^2}{z_c},\ K_c = A(0),\ a_c = a(z_c),\quad m_c = \frac{1}{a_c^2}\int_0^{z_c} dv g(v)a(v)^2,\quad \kappa_c = \frac{1}{a_c}\int_0^{z_c} dv g(v)a(v). \tag{3.24}$$

The string action has now the desired form, it is the Nambu-Goto action in the region of the space between the horizon and the cutoff plus a boundary action defined at the cutoff, that we will use to determine boundary conditions for the string profile in the IR region. Note that the cutoff action depends on both the trajectory of the Wilson loop and the position of the string at the cutoff, and their derivatives. The coefficients in the cutoff action are determined by integrations over the region between the cutoff and the asymptotic boundary, but the dependence on the cutoff is sensitive only to the region close to its radial position. The cutoff action takes the form of the action for a particle at position x_c, defined as a scalar field on the wordline. The particle is subject to a potential that pins it at the position of the Wilson loop, x, which is a fixed source in the field theory. From this point of view, a_c determines the strength of the effective potential at leading (quadratic) order and m_c renormalizes the two-derivative kinetic term. A combination of m_c and κ_c multiplies a term linear in x_c, which is a source term, and K_c, m_c and κ_c enter in the coefficient of a background contribution $\sim \dot{x}^2$ to the energy.

The total action is stationary respect to changes in the profile of the string that change the position at the cutoff but keep the position at the asymptotic boundary fixed

$$\delta S_{\rm string} = \delta S_{\rm IR} + \delta S_{\rm UV} = 0. \tag{3.25}$$

The variation of the on-shell action in the IR region is

$$\delta S_{\rm IR} = T_s \int dt f X' \delta X \Big|_{z=z_c} = T_s \int dt\, p \delta x_c. \tag{3.26}$$

The variation of the cutoff action (3.23) is

$$\delta S_{\rm UV} \simeq T_s \int dt \left[-\frac{1}{a_c}(x_c - x) - m_c(\ddot{x}_c - \ddot{x}) - \kappa_c \ddot{x} + O(\partial_t^4 x, \partial_t^4 x_c) \right] \delta x_c. \tag{3.27}$$

This results in the boundary condition

$$p \simeq \frac{1}{a_c}(x_c - x) + m_c(\ddot{x}_c - \ddot{x}) + \kappa_c \ddot{x} + O(\partial_t^4 x, \partial_t^4 x_c), \tag{3.28}$$

which is a mixed boundary condition between the radial derivative $p = f(z_c)X'(z_c)$ and the value of the profile $x_c = X(z_c)$ at the cutoff.

Substituting in the expression for the force (3.17) and using $B(0) = -a_c\kappa_c$, $A(0) = K_c$ produces

$$F(t) \simeq \frac{1}{a_c}(x_c - x) + (m_c - \kappa_c)\ddot{x}_c + (K_c - m_c + 2\kappa_c)\ddot{x} + O(\partial_t^4 x, \partial_t^4 x_c). \tag{3.29}$$

We have managed to write the force in terms of the coefficients in the cutoff action, the position of the string at the cutoff and the trajectory of the Wilson line. At this point, all the information about the geometry in the UV region is hidden in the value of the coefficients.

3.3 RG flow equations

The cutoff action (3.23) could be interpreted as an effective description of the Wilson line after UV degrees of freedom have been integrated out up to the energy scale defined by the cutoff. It has a similar form as a putative Wilsonian action, although it may not be exactly the same as the outcome of an actual field theory calculation. Nevertheless, this line of thought can be pursued further in the context of holographic RG flows. In particular we can define RG flow equations for the coefficients in the cutoff action from the dependence on the position of the cutoff in the radial direction. The set of equations we obtain are

$$\begin{aligned}
\partial_{z_c} M_c &= -\frac{R^2}{z_c^2}, \\
\partial_{z_c} K_c &= -g(z_c), \\
\partial_{z_c} a_c &= \frac{1}{f(z_c)}, \\
\partial_{z_c} m_c &= -\frac{2}{f(z_c)}\frac{m_c}{a_c} + g(z_c), \\
\partial_{z_c} \kappa_c &= -\frac{1}{f(z_c)}\frac{\kappa_c}{a_c} + g(z_c).
\end{aligned} \tag{3.30}$$

In addition, the position of the string at the cutoff obeys an RG flow equation. Taking into account (3.28),

$$\partial_{z_c} x_c = \frac{p}{f(z_c)} = \frac{1}{f(z_c)}\left[\frac{1}{a_c}(x_c - x) + m_c(\ddot{x}_c - \ddot{x}) + \kappa_c \ddot{x}\right] + O(\partial_t^4 x, \partial_t^4 x_c), \tag{3.31}$$

Using the RG flow equations it is straightforward to show that the force (3.29) is an RG-flow invariant quantity

$$\partial_{z_c} F(t) = 0 + O(\partial_t^4 x, \partial_t^4 x_c). \tag{3.32}$$

Although we used the AdS boundary expansion to help us identify (3.9) as the force, in fact we can generalize (3.29) and also the cutoff action (3.23) to any geometry in the UV region, since it only depends on quantities evaluated at the cutoff and the Wilson line trajectory. Note that RG flow equations determine the coefficients of the cutoff action, that only depend on the local geometry close to the cutoff up to integration constants, where all the information about the UV is hidden.

4 Force in a heated IR fixed point

We will use the results of the previous section to find the first terms that appear in the force when we do a derivative expansion of the quark trajectory in a specific example. Consider a strongly coupled theory that has an IR fixed point. The theory is at finite temperature, but low enough such that the physics is still dominated by the IR conformal theory. The holographic dual for the geometry in the IR region can then be approximated by the AdS_5 black brane (2.2). The geometry in the UV region is in principle unknown, but all the information about the UV will be hidden in integration constants of the RG flow equations.

4.1 Profile of the string perturbation in the AdS_5 black brane

The equations of motion for the string profile below the cutoff are (2.9). We will use an expansion in plane waves to find solutions:

$$X(t, z) = \int \frac{d\omega}{2\pi} X_\omega(z) e^{-i\omega t}, \tag{4.1}$$

and we will use a similar expansion for the position of the string at the cutoff and the quark trajectory

$$x_c(t) = \int \frac{d\omega}{2\pi} \widetilde{x}_c(\omega) e^{-i\omega t}, \quad x(t) = \int \frac{d\omega}{2\pi} \widetilde{x}(\omega) e^{-i\omega t}. \tag{4.2}$$

In addition, in order to remove the explicit dependence on the position of the horizon z_h, we do the change of variables

$$z = z_h u, \quad w = z_h \omega. \tag{4.3}$$

The position of the cutoff in the new coordinate is $u_c = z_c/z_h$. The equation for the string profile becomes

$$X''_\omega - \left(\frac{2}{u} + \frac{4u^3}{1-u^4}\right) X'_\omega + \frac{w^2}{(1-u^4)^2} X_\omega = 0. \tag{4.4}$$

We must impose ingoing boundary conditions at the horizon $X_\omega(u) \sim (1-u^4)^{-iw/4}$ as $u \to 1$. It is possible to do an expansion of the solutions in powers of the frequency, in such a way that they take the form

$$X_\omega(u) \simeq \widetilde{x}_c\,(1-u^4)^{-iw/4}(1 - iw\chi_1(u) - w^2\chi_2(u) + iw^3\chi_3(u) + \ldots\cdots). \tag{4.5}$$

The functions χ_i must be regular at the horizon and we will identify the overall coefficient with the value of the profile solution at the cutoff $X_\omega(u_c) = \widetilde{x}_c$. This fixes the values of the functions χ_i at the cutoff. Finally, we have to impose the boundary condition (3.28).

4.2 Cutoff action and boundary conditions

In the following it will be convenient to define a rescaled version of the metric functions and coefficients of the cutoff action, in terms of the coordinate u. First we introduce the rescaled f and g functions

$$\widehat{f}(u) = \frac{1-u^4}{u^2}, \quad \widehat{g}(u) = \frac{1}{u^2(1-u^4)}. \tag{4.6}$$

And, with these definitions, the rescaled functions that determine the coefficients of the cutoff action

$$\begin{aligned} \widehat{a}(u) &= \int_0^u \frac{du_1}{\widehat{f}(u)}, & \widehat{K}(u) &= \frac{1}{u} - \int_0^u du_1 \left(\widehat{g}(u_1) - \frac{1}{u_1^2}\right), \\ \widehat{m}(u) &= \frac{1}{\widehat{a}(u_c)^2} \int_0^u du_1 \widehat{g}(u_1) \widehat{a}(u_1)^2, & \widehat{\kappa}(u) &= \frac{1}{\widehat{a}(u_c)} \int_0^u du_1 \widehat{g}(u_1) \widehat{a}(u_1), \end{aligned} \tag{4.7}$$

As before, we define $\widehat{a}_c = \widehat{a}(u_c)$, $\widehat{K}_c = \widehat{K}(u_c)$, $\widehat{m}_c = \widehat{m}(u_c)$ and $\widehat{\kappa}_c = \widehat{\kappa}(u_c)$. In addition, we introduce $\widehat{f}_c = \widehat{f}(u_c)$ and $\widehat{g}_c = \widehat{g}(u_c)$ to ease notation. The relation to the original functions is

$$f(z_c) = \frac{R^2}{z_h^2}\widehat{f}_c, \quad g(z_c) = \frac{R^2}{z_h^2}\widehat{g}_c, \quad a_c = \frac{z_h^3}{R^2}\widehat{a}_c, \quad K_c = \frac{R^2}{z_h}\widehat{K}_c, \quad m_c = \frac{R^2}{z_h}\widehat{m}_c, \quad \kappa_c = \frac{R^2}{z_h}\widehat{\kappa}_c. \tag{4.8}$$

Using the plane wave expansion and rescaling by the position of the horizon, the boundary condition (3.28) becomes

$$\widehat{f}_c X'_\omega(u_c) = \left(\frac{1}{\widehat{a}_c} - w^2\widehat{m}_c\right)\widetilde{x}_c - \left(\frac{1}{\widehat{a}_c} - w^2(\widehat{m}_c - \widehat{\kappa}_c)\right)\widetilde{x} + O(w^4\widetilde{x}_c, w^4\widetilde{x}). \tag{4.9}$$

This will determine the value of $\widetilde{x}_c$, that in turn we will use to compute the force (3.29).

The details of the calculation of the profile solutions and the derivatives can be found in appendix B. The result, to the order in derivatives we are considering, is

$$\widetilde{x}_c = \widetilde{x} + \sum_{i=1}^{3} s_i(-iw)^i\widetilde{x} + O(w^4\widetilde{x}). \tag{4.10}$$

The coefficients in the expansion are

$$\begin{aligned}
s_1 &= -\widehat{a}_c,\\
s_2 &= \widehat{a}_c\left(\widehat{a}_c - \widehat{\kappa}_c + H_2(u_c)\right),\\
s_3 &= \widehat{a}_c\left[\widehat{a}_c(\widehat{m}_c + \widehat{\kappa}_c - \widehat{a}_c) - (2\widehat{a}_c + c_1(u_c))H_2(u_c) + H_3(u_c)\right].
\end{aligned} \tag{4.11}$$

The explicit expression for c_1 is given in (B.7)

$$c_1(u_c) = -\frac{1}{4}\log(1-u_c^4). \tag{4.12}$$

The definitions of the functions $H_2(u)$ and $H_3(u)$ are in (B.11), but we will not use the explicit expressions. Instead, we compute directly the cutoff values $H_2(u_c)$ and $H_3(u_c)$.

4.3 Force acting on a slowly moving quark

Using (4.10) to obtain the position of the string at the cutoff $x_c(t)$ and plugging the result in the force (3.29), one finds the following terms to leading order in derivatives of the quark trajectory

$$F(t) = \frac{R^2}{z_h^3}\sum_{i=1}^{3} F_i\,(z_h\partial_t)^i x + O(\partial_t^4 x). \tag{4.13}$$

With coefficients

$$\begin{aligned}
F_1 &= \frac{s_1}{\widehat{a}_c} = -1,\\
F_2 &= \widehat{K}_c + \widehat{\kappa}_c + \frac{s_2}{\widehat{a}_c} = \widehat{a}_c + \widehat{K}_c + H_2(u_c),\\
F_3 &= (\widehat{m}_c - \widehat{\kappa}_c)s_1 + \frac{s_3}{\widehat{a}_c} = \widehat{a}_c(2\widehat{\kappa}_c - \widehat{a}_c) - (c_1(u_c) + 2\widehat{a}_c)H_2(u_c) + H_3(u_c).
\end{aligned} \tag{4.14}$$

We can find the explicit values of F_2 and F_3 by solving RG flow equations for $\widehat{a}_c$, $\widehat{\kappa}_c$ and $H_2(u_c)$, $H_3(u_c)$. This is done in appendix C, the results are

$$\begin{aligned}
\widehat{a}_c &= \frac{1}{4}\log\frac{1+u_c}{1-u_c} - \frac{1}{2}\tan^{-1}u_c = \frac{1}{2}\left(\tanh^{-1}u_c - \tan^{-1}u_c\right) + a_{\mathrm{UV}},\\
\widehat{K}_c &= \frac{1}{u_c} - \widehat{a}_c + K_{\mathrm{UV}},
\end{aligned}$$

$$
\begin{aligned}
\widehat{\kappa}_c &= -\frac{1}{u_c} + \frac{\widehat{a}_c}{2} + \frac{1}{2\widehat{a}_c}\tanh^{-1}(u_c^2) + \frac{\kappa_{\rm UV}}{\widehat{a}_c}, \\
H_2(u_c) &= -\frac{1}{u_c} + 1, \\
H_3(u_c) &= \frac{1}{4}(\pi - \log 4) - \frac{c_1(u_c)}{u_c} + \widehat{a}_c - \frac{1}{2}\tan^{-1} u_c + \frac{1}{4}\left(2\log(1+u_c) - 3\log\left(1+u_c^2\right)\right).
\end{aligned}
\tag{4.15}
$$

Where $a_{\rm UV}$, $K_{\rm UV}$ and $\kappa_{\rm UV}$ are integration constants that depend on the UV region. In the case where the geometry is just the AdS_5 black brane everywhere between the boundary and the horizon, these integration constants vanish $a_{\rm UV} = K_{\rm UV} = \kappa_{\rm UV} = 0$. They will be generically non-zero in geometries that approach an AdS_5 geometry close to the horizon but deviate from it in other regions. A class of models that could be interesting to study are holographic duals to RG flows between two fixed points, consisting of a domain wall geometry between two AdS spaces of different radius, e.g. [43, 44]. In these models we expect that the RG flow integration constants introduce an additional dependence on the ratio between the temperature and the scale of the deformation that triggers the RG flow.

Plugging the solutions to the RG flow equations (4.15) in (4.14) produces the cutoff-independent values

$$
F_2 = 1 + K_{\rm UV}, \quad F_3 = \frac{1}{4}(\pi - \log 4) + 2(\kappa_{\rm UV} - a_{\rm UV}). \tag{4.16}
$$

Let us now interpret the final result in field theory language. First we multiply by the string tension as in (3.9). The holographic dictionary maps the AdS radius and the position of the horizon to the 't Hooft coupling λ and temperature T of the dual field theory

$$
T_s R^2 = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}, \quad z_h = \frac{1}{\pi T}. \tag{4.17}
$$

The force acting on the heavy quark is, to third order in derivatives of the trajectory,

$$
\mathcal{F}_x \simeq \frac{\sqrt{\lambda}}{2\pi}\left(-(\pi T)^2 \partial_t x + \pi T\, F_2\, \partial_t^2 x + F_3\, \partial_t^3 x\right) + O(\partial_t^4 x). \tag{4.18}
$$

In the first place we observe that the coefficient of the term proportional to the velocity of the quark, $\partial_t x$, agrees with the drag force of [12, 13] and is insensitive to the UV physics, at least in the approximation we are doing of fixing the IR geometry to the AdS black brane solution.

The coefficient proportional to the acceleration, $\partial_t^2 x$, agrees with the expected thermal correction to the quark mass in pure AdS when $K_{\rm UV} = 0$. The quark mass can be determined from the length of a straight string extended between the horizon and a "flavor brane" at $z = z_m$

$$
M_q = T_s R^2 \int_{z_m}^{z_h} \frac{dz}{z^2} = M_0 - \frac{T_s R^2}{z_h} = M_0 - \frac{\sqrt{\lambda} T}{2}. \tag{4.19}
$$

In the formula above M_0 is interpreted as the quark mass at zero temperature. This term modifies the inertial mass of the quark. Indeed if we allowed a very large, but not infinite, mass for the quark, the Newton equation for the quark would be

$$M_0 \partial_t^2 x = \mathcal{F}_x. \tag{4.20}$$

Moving the acceleration term in the force to the left side of the equation results in replacing M_0 by the thermal corrected mass M_q. Therefore, we can interpret $K_{\rm UV}$ as a modification of the thermal mass due to UV physics.

Finally, the coefficient of the jerk or acceleration rate, $\partial_t^3 x$, computed for the AdS_5 black brane in [39], can be interpreted as a combination of the Abraham-Lorentz force produced by the emission of Larmor radiation (see [36, 37]) and a viscous contribution from the surrounding plasma that has been computed in [38] following the method developed in [45]. In a conformal theory in vacuum the viscous part is absent and the coefficient of the jerk term is $\sqrt{\lambda}/(2\pi)$. At finite temperature the viscous correction is obtained by subtracting the vacuum contribution from our result. Since the coefficient of the viscous contribution does not depend on temperature, the $T \to 0$ limit of the acceleration rate contribution does not coincide with the $T = 0$ value, as noted in [39].

4.4 Force acting on a fast moving quark

As we showed in section 2.2, the quadratic action for the fast moving quark takes the same form as for the slowly moving quark, replacing the functions f, g by the functions f_v, g_v given in (2.20). Let us introduce the rescaled coordinates and embedding perturbation

$$s = \gamma^{-1/2} \frac{t}{z_h}, \quad u = \gamma^{1/2} \frac{z}{z_h}, \quad Y = \gamma^{3/2} X \tag{4.21}$$

Then, in the AdS_5 black brane geometry, the quadratic terms in string action become

$$S_{\rm NG} \sim \frac{T_s R^2}{z_h^2} \int ds du \frac{1}{2} \left(\widehat{g} (\partial_s Y)^2 - \widehat{f} (\partial_u Y)^2 \right). \tag{4.22}$$

Where we have used the expression for p_0 in (2.18) and $\widehat{f}$, $\widehat{g}$ have the same definition as in (4.6). This allows us to translate directly the results for the slowly moving quark to this case.

The variation of the action gives a boundary term

$$\delta S_{\rm NG} \sim \lim_{u \to 0} \frac{T_s R^2}{z_h^2} \int ds \, \frac{1}{u^2} \partial_u Y \delta Y = \lim_{z \to 0} T_s \gamma \int dt \, \frac{R^2}{z^2} \partial_z X \delta X. \tag{4.23}$$

So we should add a factor of γ to the expression we found for the force of the slowly moving quark.

A solution close to the AdS_5 boundary has same form as (3.3) in the rescaled variables,

$$Y \simeq y(s) - \frac{1}{2} \partial_s^2 y(s) u^2 + \frac{\widehat{F}(s)}{3} u^3 + \cdots. \tag{4.24}$$

Then, from (4.21) and comparing to (3.3), we have that

$$F(t) = \frac{1}{z_h^3} \widehat{F}. \tag{4.25}$$

The coefficient $\widehat{F}$ is the same as (4.13) taking into account the rescaling

$$\widehat{F}(s) = \sum_{i=1}^{3} F_i\,(\partial_s)^i y + O(\partial_s^4 y) = \gamma^{3/2}\sum_{i=1}^{3} F_i\,(\gamma^{1/2} z_h \partial_t)^i x + O(\partial_t^4 x). \tag{4.26}$$

Then, taking into account the background contribution to the force (3.11),

$$\mathcal{F}_x \simeq \frac{\sqrt{\lambda}}{2\pi}\left(-(\pi T)^2\gamma v - (\pi T)^2\gamma^3\partial_t x + \pi T\, F_2\,\gamma^{7/2}\partial_t^2 x + F_3\,\gamma^4\partial_t^3 x\right) + O(\partial_t^4 x). \tag{4.27}$$

Note that the term proportional to $\partial_t x$ could have been obtained by replacing $v \to v+\partial_t x$ in the first term and expanding to linear order. The γ factors appearing in higher derivative terms imply that this expansion requires time derivatives to be much smaller than the temperature for very fast quarks $\partial_t \ll \gamma^{-1/2}\pi T$.

5 General RG flow equations

We have presented an explicit derivation of the cutoff action and RG flow equations from a direct integration of the string action in the UV region using the approximation that changes in the quark trajectory are slow in time compared to the time scale given by the inverse temperature. It is possible to rederive and generalize these results by introducing an ansatz for the cutoff action and using the conditions that the total action should be invariant under changes in the position of the cutoff. In the following we will derive the general RG flow equation for a quark moving in a straight trajectory and compare to the previous results. The interest of this method is that it simplifies somewhat the derivation and allows a systematic extension to non-linear and higher derivative terms, as well as possibly curved trajectories of the quark.

5.1 String action and momentum

We will work in the static gauge (2.6) and assume that the background metric takes the diagonal form

$$ds^2 = G_{zz}dz^2 + G_{tt}dt^2 + G_{xx}\delta_{ij}dx^i dx^j. \tag{5.1}$$

We will not make other assumptions about the dependence of the metric components on the coordinates. It will be useful to define a metric for the quark at rest γ_{ab}, with non-zero components $\gamma_{00} = G_{tt}$ and $\gamma_{11} = G_{zz}$. The induced metric on the string worldsheet is

$$g_{ab} = \gamma_{ab} + G_{xx}\partial_a X\partial_b X. \tag{5.2}$$

The effective string action consists of the Nambu-Goto action in the IR region of the geometry plus a boundary action defined at the cutoff

$$S_{\text{string}} = S_{\text{IR}} + S_c, \tag{5.3}$$

where

$$S_{\text{IR}} = -T_s\int dt\int_{z_c}^{z_h}\mathcal{L}_{\text{NG}}, \quad S_c = T_s\int dt\mathcal{L}_c[x_c, \dot{x}_c; z_c]. \tag{5.4}$$

The Nambu-Goto Lagrangian density in the case we are considering is given by

$$\mathcal{L}_{\rm NG} = \sqrt{-g} = \sqrt{-\gamma}\Delta^{1/2}, \tag{5.5}$$

where for later convenience we have defined

$$\Delta = 1 + G_{xx}\gamma^{ab}\partial_a X\partial_b X. \tag{5.6}$$

There is a conserved worldsheet current $\partial_a p_x^a = 0$ corresponding to the shift symmetry $X \to X + \delta X$. It can also be identified as the conjugate momentum to the position X. It can be obtained from the variation of the string action

$$p_x^a = -\frac{\delta \mathcal{L}_{\rm NG}}{\delta \partial_a X} = -\frac{\sqrt{-\gamma}}{\Delta^{1/2}} G_{xx}\gamma^{ab}\partial_b X. \tag{5.7}$$

Using this expression we can solve for p_x^0 and X' in terms of p_x^1 and $\dot{X}$.

$$p_x^0 = -\Sigma^{1/2} G_{xx}\gamma^{00}\dot{X}, \quad X' = -\Sigma^{-1/2} G^{xx}\gamma_{11} p_x^1, \tag{5.8}$$

where we have defined

$$\Sigma = \frac{|\gamma| - G^{xx}\gamma_{11}(p_x^1)^2}{1 + G_{xx}\gamma^{00}(\dot{X})^2}. \tag{5.9}$$

As we have seen, the cutoff action can be obtained from integrating along the radial coordinate the string action in the UV region. The total action should then satisfy the condition that it is stationary under changes of the string profile that preserve the boundary conditions, in particular when the position of the string at the cutoff is displaced keeping the string at the boundary and the horizon fixed

$$\delta S_{\rm string} = \delta S_{\rm IR} + \delta S_c = 0. \tag{5.10}$$

The variation of the IR part is, using the conservation of the momentum $\partial_a p_x^a = 0$,

$$\delta S_{\rm IR} = T_s \int dt \int_{z_c}^{z_h} dz \, (p_x^a \partial_a \delta X) = T_s \int dt \int_{z_c}^{z_h} dz \partial_a \, (p_x^a \delta X) = -T_s \int dt \, p_x^1 \delta x_c. \tag{5.11}$$

The variation of the cutoff action is proportional to the Euler-Lagrange equations of $\mathcal{L}_c$

$$\delta S_c = T_s \int dt \left[\frac{\delta \mathcal{L}_c}{\delta x_c} - \partial_t \left(\frac{\delta \mathcal{L}_c}{\delta \dot{x}_c}\right)\right] \delta x_c. \tag{5.12}$$

Then, we find the condition

$$p_x^1\Big|_{z=z_c} = \frac{\delta \mathcal{L}_c}{\delta x_c} - \partial_t \left(\frac{\delta \mathcal{L}_c}{\delta \dot{x}_c}\right) \equiv \delta_{x_c}\mathcal{L}_c. \tag{5.13}$$

5.2 RG flow of the cutoff action

The RG flow equations for the cutoff action can be derived from the requirement that the total action should be independent of the position of the cutoff, as it would be the case if we had obtained it by integrating over the UV region. The condition is

$$\frac{d}{dz_c} S_{\rm string} = \frac{d}{dz_c} S_{\rm IR} + \frac{d}{dz_c} S_c = 0. \tag{5.14}$$

The IR term depends on z_c just through the limits of integration

$$\frac{d}{dz_c} S_{\rm IR} = T_s \int dt\, \mathcal{L}_{\rm NG}\Big|_{z=z_c}. \tag{5.15}$$

The cutoff action can have an explicit dependence and an implicit dependence in the position of the string at the cutoff

$$\frac{d}{dz_c} S_c = T_s \int dt\, \left[\partial_{z_c}\mathcal{L}_c + (\delta_{x_c}\mathcal{L}_c)\,\partial_{z_c}x_c\right]. \tag{5.16}$$

Note that there is an integration over time, so the RG flow equation for the cutoff action will be defined up to a total derivative

$$\partial_{z_c}\mathcal{L}_c = -(\delta_{x_c}\mathcal{L}_c)\,\partial_{z_c}x_c - \mathcal{L}_{\rm NG}\Big|_{z=z_c} + \partial_t V^t. \tag{5.17}$$

From (5.8) and (5.13) we can derive the RG flow equation for x_c

$$\partial_{z_c}x_c = -\Sigma_c^{-1/2} G^{xx}\gamma_{11}(\delta_{x_c}\mathcal{L}_c)\Big|_{z=z_c}, \tag{5.18}$$

where now

$$\Sigma_c = \frac{|\gamma| - G^{xx}\gamma_{11}(\delta_{x_c}\mathcal{L}_c)^2}{1+G_{xx}\gamma^{00}(\dot{x}_c)^2}\Big|_{z=z_c}. \tag{5.19}$$

Using the same formulas, the evaluation of the Nambu-Goto action at the cutoff will be

$$\mathcal{L}_{\rm NG}\Big|_{z=z_c} = \sqrt{-\gamma}\,\Delta_c^{1/2}\Big|_{z=z_c} = |\gamma|\Sigma_c^{-1/2}\Big|_{z=z_c}. \tag{5.20}$$

Adding all the contributions results in the RG flow equation

$$\partial_{z_c}\mathcal{L}_c = -\sqrt{-\gamma}\left(1+G_{xx}\gamma^{00}(\dot{x}_c)^2\right)^{1/2}\left(1-G^{xx}\gamma_{11}\frac{(\delta_{x_c}\mathcal{L}_c)^2}{|\gamma|}\right)^{1/2}\Bigg|_{z=z_c} + \partial_t V^t. \tag{5.21}$$

This is our final result, it takes the form of a functional equation for the cutoff action $\mathcal{L}_c$. We do not have a complete solution, but as we will see this equation admits an expansion in derivatives of x_c, in such a way that at each order the RG flow equation for the action reduces to RG flow equations for the coefficients in the expansion.

5.3 Slowly moving quark

We proceed to solve (5.21) in the case we have studied before, a slowly moving quark. An obvious ansatz for the cutoff action is to adapt (3.23) to the more general formulas we have derived, in particular the form of the non-derivative term. We will use[2]

$$\mathcal{L}_c = \sqrt{-\gamma}\left[\Lambda - \frac{k_0}{2}(x_c - x)^2 + \frac{k_1}{2}(\dot{x}_c - \dot{x})^2 + \frac{k_2}{2}(\dot{x}_c)^2 + \frac{k_3}{2}(\dot{x})^2 + O(\partial_t^4 x, \partial_t^4 x_c)\right]. \tag{5.22}$$

It is implicit in the formula above and the ones that will follow that all the functions depending on the radial coordinate are evaluated at the cutoff. The overall factor is convenient to

[2]Note that there is invariance under translations in the x direction, so terms that would break this invariance, such as x^2 are forbidden.

cancel out similar factors in (5.21). Comparing with (3.23) requires some reshuffling and an integration by parts, the map between the two sets of coefficients is

$$\begin{aligned} M_c &= \sqrt{-\gamma}\Lambda, & \frac{1}{a_c} &= \sqrt{-\gamma}k_0, & m_c &= \sqrt{-\gamma}(k_1+k_2), \\ \kappa_c &= \sqrt{-\gamma}k_2, & K_c &= -\sqrt{-\gamma}(k_2+k_3). \end{aligned} \tag{5.23}$$

The derivative with respect to the cutoff position is

$$\begin{aligned} \partial_{z_c}\mathcal{L}_c = \sqrt{-\gamma}\Big[\nabla_{z_c}\Lambda - \frac{1}{2}\nabla_{z_c}k_0(x_c-x)^2 + \frac{1}{2}\nabla_{z_c}k_1(\dot{x}_c-\dot{x})^2 \\ + \frac{1}{2}\nabla_{z_c}k_2(\dot{x}_c)^2 + \frac{1}{2}\nabla_{z_c}k_3(\dot{x})^2\Big], \end{aligned} \tag{5.24}$$

where we have defined, for any coefficient C,

$$\nabla_{z_c}C = \frac{1}{\sqrt{-\gamma}}\partial_{z_c}\left(\sqrt{-\gamma}C\right) = \left(\partial_{z_c} + \frac{\partial_{z_c}\sqrt{-\gamma}}{\sqrt{-\gamma}}\right)C. \tag{5.25}$$

Introducing (**??**) and (5.24) in (5.21), and expanding to quadratic order in x, x_c and derivatives, one finds that terms with derivatives of the cutoff do not completely match with other terms. While terms in (5.24) only involve first time derivatives, terms from (**??**) will include mixed contributions where one factor has two time derivatives and the other none. This is fixed by an appropriate choice of the total time derivative term. In this case all the terms can be matched for

$$V^t = -\sqrt{|\gamma|}G^{xx}\gamma_{11}k_0(x_c-x)\left[k_1(\dot{x}_c-\dot{x}) + k_2\dot{x}_c\right]. \tag{5.26}$$

Demanding that the coefficients of terms with different factors of x, x_c and their time derivatives vanish independently of each other leads to the RG flow equations for the coefficients:

$$\begin{aligned} \nabla_{z_c}\Lambda &= -1, \\ \nabla_{z_c}k_0 &= -G^{xx}\gamma_{11}k_0^2, \\ \nabla_{z_c}k_1 &= -G^{xx}\gamma_{11}k_0(2k_1+k_2), \\ \nabla_{z_c}k_2 &= -G_{xx}\gamma^{00} - G^{xx}\gamma_{11}k_0k_2, \\ \nabla_{z_c}k_3 &= G^{xx}\gamma_{11}k_0k_2. \end{aligned} \tag{5.27}$$

If we use the AdS_5 black brane solution (2.1) and (2.2), the RG flow equations simplify to

$$\begin{aligned} \partial_{z_c}(\sqrt{-\gamma}\Lambda) &= -\frac{R^2}{z_c^2}, \\ \partial_{z_c}(\sqrt{-\gamma}k_0) &= -g(z_c)k_0^2, \\ \partial_{z_c}(\sqrt{-\gamma}k_1) &= -g(z_c)k_0(2k_1+k_2), \\ \partial_{z_c}(\sqrt{-\gamma}k_2) &= g(z_c)(1-k_0k_2), \\ \partial_{z_c}(\sqrt{-\gamma}k_3) &= g(z_c)k_0k_2. \end{aligned} \tag{5.28}$$

Using the identifications (5.23) and the definitions of f and g (2.3), it is straightforward to recover the RG flow equations (3.30). We thus arrive to the same results both by doing a direct integration of the string action between the boundary and the cutoff and by deriving the RG flow equation for the cutoff action. However, this last method admits in principle a simpler generalization to more complicated cases.

6 Discussion

The effective IR string action we have derived is valid in the region close to an arbitrary static black brane geometry, assuming homogeneity and isotropy. In principle these conditions could be relaxed. It would be particularly interesting to study time-dependent geometries emulating the dual to a heavy ion collision, see section VII of [46] for a recent review on the topic. An example of this type is the calculation of the drag force in [47], for a plasma formed by the collision of two infinite sheets with finite energy density in a conformal theory. Another natural extension would be to use the general method presented in section 5 for less constrained quark trajectories, allowing sudden changes in the trajectory and motion in more than one spatial direction. The general method could also be used to compute nonlinear contributions of acceleration to energy and momentum loss, that in vacuum show in Liènard's formula and the Abraham-Lorenz force [36, 48].

One should keep in mind that we are making an assumption by taking the Nambu-Goto action for the string. In many cases the holographic dual is not presented as a ten-dimensional geometry, but has been truncated to five dimensions, and the metric is presented in the Einstein frame. The string action will then be modified by some additional factors. Similarly, if the model is bottom-up, with not known string theory construction behind, then the string action dual to a Wilson line may be chosen in a different way. Nevertheless, in all these cases the method we have presented here can be easily generalized. The derivative expansion of the action will be similar even if the detailed dependence of the coefficients on the geometry can change.

The holographic Wilsonian renormalization method applied here to the string action can be used more generally, for other observables like Wilson lines in different representations or 't Hooft lines as well as for observables obtained from the background geometry, such as the expectation value of local operators. A fully effective description would involve introducing the cutoff and deriving the RG flow equations for the holographic actions dual to all the observables under consideration. It would be interesting to combine the holographic Wilsonian approach with other phenomenological approaches trying to fit QCD lattice data or experiments. Among these, we have the traditional holographic QCD models where the gravitational action is adjusted [49–53] or, more recently, the application of machine learning [54–57] and Monte Carlo techniques [58] to constrain the background geometry. In both cases, the holographic description of UV physics is expected to be problematic due to the asymptotic freedom of QCD. The holographic Wilsonian formalism limits the range of energy scales where the model is applied, so it could be used to avoid this issue without introducing additional assumptions.

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A Derivation of the cutoff action

In this appendix we explain how to derive the cutoff action from direct integration of the string action between the boundary and the cutoff. We can expand the solutions using the derivative expansion. Their form is (2.12), (2.13), and they satisfy the conditions (2.14). This gives the following simplifications,

$$f(z_c)X'(z_c) = p, \quad \lim_{z_\Lambda \to 0} X(z_\Lambda) = x, \quad \lim_{z_\Lambda \to 0} \dot{X}(z_\Lambda) = \dot{x}. \tag{A.1}$$

It remains to evaluate the derivative at the boundary. From (3.14)

$$\lim_{z_\Lambda \to 0} \left(f(z_\Lambda)X(z_\Lambda)X'(z_\Lambda) + \frac{R^2}{z_\Lambda}\dot{X}(z_\Lambda)^2 \right) = xF(t) + \lim_{z_\Lambda \to 0} \frac{R}{z_\Lambda}\left(\ddot{x}x + \dot{x}^2\right). \tag{A.2}$$

The last term is a total derivative and we can drop it, while $F(t)$ is given by the expression in (3.17). We are left with

$$S_{\rm UV} \simeq T_s \int dt \left[\frac{R^2}{z_c} + \frac{1}{2}(F\,x - p\,x_c) \right]. \tag{A.3}$$

Instead of p, we would like the action to depend on the position of the string at the cutoff and the boundary x_c, x and on their derivatives. In order to solve for p, first we integrate (3.14) between the boundary and the cutoff

$$x_c - x = pa(z_c) + \ddot{x}C(z_c) + \ddot{p}D(z_c) + O(\partial_t^4 x, \partial_t^4 p). \tag{A.4}$$

Where we have defined

$$\begin{aligned} C(z) &= \int_0^z \frac{dv}{f(v)}\left(A(v) - \frac{R^2}{v} \right), \\ D(z) &= \int_0^z dv \frac{B(v)}{f(v)}. \end{aligned} \tag{A.5}$$

We can further simplify these expressions using the explicit form of A and B (3.15), the definition of $a(z)$ in (2.12) and integration by parts

$$\begin{aligned} C(z) &= \int_0^z dva'(v) \int_{z_c}^v du\, g(u) = a(z) \int_{z_c}^z dug(u) - \int_0^z dvg(v)a(v), \\ D(z) &= \int_0^z dva'(v) \int_{z_c}^v du\, g(u)a(u) = a(z)B(z) - \int_0^z dvg(v)a(v)^2. \end{aligned} \tag{A.6}$$

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Evaluating at the cutoff we obtain

$$C(z_c) = -\int_0^{z_c} dvg(v)a(v) = B(0), \qquad D(z_c) = -\int_0^{z_c} dvg(v)a(v)^2. \tag{A.7}$$

Then, we can solve for p as

$$p = \frac{1}{a(z_c)}(x_c - x) - \frac{C(z_c)}{a(z_c)}\ddot{x} - \frac{D(z_c)}{a(z_c)^2}(\ddot{x}_c - \ddot{x}) + O(\partial_t^4 x, \partial_t^4 x_c). \tag{A.8}$$

Introducing this in (3.17) to the same order,

$$F = \frac{1}{a(z_c)}(x_c - x) + \frac{a(z_c)C(z_c) - D(z_c)}{a(z_c)^2}(\ddot{x}_c - \ddot{x}) + \left(A(0) - \frac{C(z_c)}{a(z_c)}\right)\ddot{x} + O(\partial_t^4 x, \partial_t^4 x_c). \tag{A.9}$$

Finally, the UV action can be arranged, up to a total derivative in time, to be (3.23) with coefficients

$$M_c = \frac{R^2}{z_c}, \;\; K_c = A(0), \;\; a_c = a(z_c), \quad m_c = -\frac{D(z_c)}{a(z_c)^2}, \quad \kappa_c = -\frac{C(z_c)}{a(z_c)}. \tag{A.10}$$

Using that

$$C(z_c) = -\int_0^{z_c} dvg(v)a(v), \quad D(z_c) = -\int_0^{z_c} dvg(v)a(v)^2, \tag{A.11}$$

leads to the expressions in (3.24).

B Solution for the string profile in the *AdS* black brane

In this appendix we give explicit formulas for the solutions of the perturbation of the string profile in the AdS_5 black brane geometry. Introducing (4.5) into the equation (4.4) and expanding in w, we get the following equations at each order in the expansion

$$\chi_i'' + \frac{\widehat{f}'}{\widehat{f}}\chi_i' - \frac{1}{\widehat{f}}j_i = 0, \tag{B.1}$$

where

$$j_1 = 1, \quad j_2 = \sigma(u) + 2u\chi_1' + \chi_1, \quad j_3 = \sigma(u)\chi_1 + 2u\chi_2' + \chi_2, \tag{B.2}$$

and we have defined

$$\sigma(u) = \frac{1}{u^2} + \frac{u^2}{1+u^2}. \tag{B.3}$$

The general form of the solutions that satisfy regularity at the horizon $u = 1$ is

$$\chi_i(u) = c_i(u_c) + \int_{u_c}^{u} du_1 \frac{J_i(u_1)}{\widehat{f}(u_1)}, \quad J_i(u) = \int_1^u du_2 j_i(u_2). \tag{B.4}$$

where $c_i(u_c)$ are integration constants.

The expansion of (4.5) in powers of w leads to

$$X_\omega(u) \simeq \tilde{x}_c\left(1 - iw\Gamma_1(u) - w^2\Gamma_2(u) + iw^3\Gamma_3(u) + \cdots\right), \tag{B.5}$$

where we have introduced the functions

$$\begin{aligned}
\Gamma_1(u) &= \chi_1(u) + \frac{1}{4}\log(1-u^4),\\
\Gamma_2(u) &= \chi_2(u) - \frac{1}{32}\log^2(1-u^4) + \frac{1}{4}\Gamma_1(u)\log(1-u^4),\\
\Gamma_3(u) &= \chi_3(u) + \frac{1}{4}\Gamma_2(u)\log(1-u^4) - \frac{1}{32}\Gamma_1(u)\log^2(1-u^4) + \frac{1}{384}\log(1-u^4)^3.
\end{aligned} \tag{B.6}$$

The boundary condition at the cutoff is $\Gamma_i(u_c) = 0$, this fixes the integration constants of the solutions to

$$c_1(u_c) = -\frac{1}{4}\log(1-u_c^4), \quad c_2(u_c) = \frac{1}{32}\log^2(1-u_c^4), \quad c_3(u_c) = -\frac{1}{384}\log^3(1-u_c^4). \tag{B.7}$$

In order to determine $\widetilde{x}_c$ we need to compute X'_ω given by (B.5) at the cutoff, introduce it in the boundary condition (4.9) together with (4.10) and solve order by order in w. The result is

$$\begin{aligned}
s_1 &= \widehat{a}_c\widehat{f}_c\Gamma'_1(u_c),\\
s_2 &= \widehat{a}_c\left(\widehat{f}_c\Gamma'_2(u_c) + \widehat{a}_c(\widehat{f}_c\Gamma'_1(u_c))^2 - \widehat{\kappa}_c\right),\\
s_3 &= \widehat{a}_c\left[\widehat{f}_c\Gamma'_3(u_c) + \widehat{a}_c^2(\widehat{f}_c\Gamma'_1(u_c))^3 + \widehat{a}_c\widehat{f}_c\Gamma'_1(u_c)\left(2\widehat{f}_c\Gamma'_2(u_c) - \widehat{m}_c - \widehat{\kappa}_c\right)\right].
\end{aligned} \tag{B.8}$$

These expressions can be further simplified. First, from the definitions (B.6) one can derive the following relations

$$\begin{aligned}
\Gamma'_1(u_c) &= \frac{J_1(u_c)}{\widehat{f}_c} - c'_1(u_c),\\
\Gamma'_2(u_c) &= \frac{J_2(u_c)}{\widehat{f}_c} - c_1(u_c)\Gamma'_1(u_c) - c'_2(u_c),\\
\Gamma'_3(u_c) &= \frac{J_3(u_c)}{\widehat{f}_c} - c_1(u_c)\Gamma'_2(u_c) - c_2(u_c)\Gamma'_1(u_c) - c'_3(u_c).
\end{aligned} \tag{B.9}$$

In addition, one can extract some constant factors

$$J_2(u) = c_1(u_c)J_1(u) + H_2(u), \quad J_3(u) = c_2(u_c)J_1(u) + H_3(u), \tag{B.10}$$

where

$$\begin{aligned}
H_2(u) &= \int_1^u du_1\left(\sigma(u_1) + 2c'_1(u_1)J_1(u_1) + \int_{u_c}^{u_1} du_2\,\frac{J_1(u_2)}{\widehat{f}(u_2)}\right),\\
H_3(u) &= \int_1^u du_1\left(\chi_1(u_1)\sigma(u_1) + 2c'_1(u_1)J_2(u_1) + \int_{u_c}^{u_1} du_2\,\frac{J_2(u_2)}{\widehat{f}(u_2)}\right).
\end{aligned} \tag{B.11}$$

C RG flow in AdS_5 black brane

In this appendix we explain how to obtain the solutions to the RG flow equations in the case where the IR geometry is approximately an AdS_5 black brane. The RG flow equations for the rescaled coefficients of the cutoff action (4.7) $\widehat{a}_c$, $\widehat{K}_c$ and $\widehat{\kappa}_c$ are

$$\partial_{u_c}\widehat{a}_c = \frac{1}{\widehat{f}_c}, \quad \partial_{u_c}\widehat{K}_c = -\widehat{g}_c, \quad \partial_{u_c}\widehat{\kappa}_c = -\frac{1}{\widehat{a}_c\widehat{f}_c}\widehat{\kappa}_c + \widehat{g}_c. \tag{C.1}$$

From (4.6), $\widehat{f}_c = (1-u_c^4)/u_c^2$ and $\widehat{g}_c = 1/(u_c^2(1-u_c^4))$. In pure AdS_5 the coefficients satisfy the conditions

$$\widehat{a}_c\Big|_{u_c} = 0, \quad \widehat{a}_c\widehat{\kappa}_c\Big|_{u_c=0} = 0. \tag{C.2}$$

Otherwise there will be integration constants that depend on the UV geometry in a non-trivial way.

By direct integration of the equations, one finds the following values for the coefficients

$$\begin{aligned}
\widehat{a}_c &= \frac{1}{4}\log\frac{1+u_c}{1-u_c} - \frac{1}{2}\tan^{-1}u_c = \frac{1}{2}\left(\tanh^{-1}u_c - \tan^{-1}u_c\right) + a_{\rm UV},\\
\widehat{K}_c &= \frac{1}{u_c} - \widehat{a}_c + K_{\rm UV},\\
\widehat{\kappa}_c &= -\frac{1}{u_c} + \frac{\widehat{a}_c}{2} + \frac{1}{2\widehat{a}_c}\tanh^{-1}(u_c^2) + \frac{\kappa_{\rm UV}}{\widehat{a}_c}.
\end{aligned} \tag{C.3}$$

We will now derive RG flow equations for $H_2(u_c)$ and $H_3(u_c)$ obtained from evaluating (B.11) at the cutoff. We will be using that $J_1(u_c) = u_c - 1$, the relations

$$c_1'(u) = \frac{u}{f(u)}, \quad c_2'(u) = c_1(u)c_1'(u), \quad c_3'(u) = c_2(u)c_1'(u), \tag{C.4}$$

and, from (B.10),

$$\partial_{u_c}J_2(u) = c_1'(u_c)J_1(u) - \frac{J_1(u_c)J_1(u)}{\widehat{f}_c} = \frac{J_1(u)}{\widehat{f}_c}. \tag{C.5}$$

Then, we derive the following RG flow equations

$$\partial_{u_c}H_2(u_c) = \sigma(u_c) + 2c_1'(u_c)J_1(u_c) - \frac{J_1(u_c)^2}{\widehat{f}_c} = \sigma(u_c) + 2\frac{J_1(u_c)}{\widehat{f}_c} + \frac{J_1(u_c)^2}{\widehat{f}_c}\,, \tag{C.6}$$

$$\begin{aligned}
\partial_{u_c}H_3(u_c) &= c_1(u_c)\sigma(u_c) + 2c_1'(u_c)J_2(u_c) - \frac{J_2(u_c)J_1(u_c)}{\widehat{f}_c} \qquad\qquad (\text{C.7})\\
&\quad + \int_1^{u_c} du\left(\underbrace{\left[c_1'(u_c) - \frac{J_1(u_c)}{\widehat{f}_c}\right]}_{1/\widehat{f}_c}\sigma(u) + 2c_1'(u)\frac{J_1(u)}{\widehat{f}_c} + \frac{1}{\widehat{f}_c}\int_{u_c}^{u} du_1\,\frac{J_1(u_1)}{f(u_1)}\right)\\
&= c_1(u_c)\sigma(u_c) + 2\frac{J_2(u_c)}{\widehat{f}_c} + \frac{J_2(u_c)J_1(u_c)}{\widehat{f}_c}\\
&\quad + \frac{1}{\widehat{f}_c}\underbrace{\int_1^{u_c} du\left(\sigma(u) + 2c_1'(u)J_1(u) + \int_{u_c}^{u} du_1\,\frac{J_1(u_1)}{f(u_1)}\right)}_{H_2(u_c)}\\
&= c_1(u_c)\sigma(u_c) + 2\frac{J_2(u_c)}{\widehat{f}_c} + \frac{J_2(u_c)J_1(u_c)}{\widehat{f}_c} + \frac{H_2(u_c)}{\widehat{f}_c}.
\end{aligned} \tag{C.8}$$

Finally, with the explicit expressions for σ, J_1 and $\widehat{f}$, we get the simple RG flow equation for H_2:

$$\partial_{u_c}H_2(u_c) = \frac{1}{u_c^2}. \tag{C.9}$$

Similarly, if in addition we use that $J_2(u_c) = c_1(u_c)J_1(u_c) + H_2(u_c)$, the RG flow equation for H_3 takes the simpler form

$$\partial_{u_c} H_3(u_c) = \frac{c_1(u_c)}{u_c^2} + \frac{H_2(u_c)(J_1(u_c)+3)}{\widehat{f}_c}. \tag{C.10}$$

We can integrate both equations taking into account that $H_2(1) = 0$, $H_3(1) = 0$, the solutions are

$$\begin{aligned} H_2(u_c) &= -\frac{1}{u_c} + 1, \\ H_3(u_c) &= \frac{1}{4}(\pi - \log 4) - \frac{c_1(u_c)}{u_c} + \widehat{a}_c - \frac{1}{2}\tan^{-1} u_c + \frac{1}{4}\left(2\log(1+u_c) - 3\log\left(1+u_c^2\right)\right). \end{aligned} \tag{C.11}$$

Note that H_2, H_3 and their boundary conditions are defined in the IR region of the geometry, so there are no additional integration constants associated to the RG flow equations of $H_2(u_c)$ and $H_3(u_c)$.

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Chapter 6

Holographic RG flow and reparametrization invariance of Wilson loops

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Holographic RG flow and reparametrization invariance of Wilson loops

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Abstract: We study the fate of reparametrization invariance of Wilson loops, also known as 'zig-zag' symmetry, under the RG flow using some simple cases as guidance. We restrict our analysis to large-N, strongly coupled CFTs and use the holographic dual description of a Wilson loop as a fundamental string embedded in asymptotically AdS spaces, at zero and nonzero temperature. We then introduce a cutoff in the holographic radial direction and integrate out the section of the string closer to the AdS boundary in the spirit of holographic Wilsonian renormalization. We make explicit the map between Wilson loop reparametrizations and conformal transformation of the string worldsheet and show that a cutoff anchored to the worldsheet breaks conformal invariance and induces an effective defect action for reparametrizations at the cutoff scale, in a way similar to nearly-AdS_2 gravity or SYK models. On the other hand, a cutoff in the target space breaks worldsheet diffeomorphisms and Weyl transformations but keeps conformal transformations unbroken and does not generate a non-trivial action for reparametrizations.

Keywords: AdS-CFT Correspondence, Gauge-Gravity Correspondence, Wilson, 't Hooft and Polyakov loops, Random Systems

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Contents

1 Introduction

Gauge/gravity duality has been extensively used as a phenomenological tool to describe strongly coupled systems in particle physics and condensed matter (see e.g. [1–3]). In the search of holographic duals of realistic theories like QCD one has to face the problem that the UV physics cannot be captured in general by a weakly coupled gravity dual, making the problem effectively intractable in this regime. Provided the identification between the energy scale in the field theory side and the position along the holographic radial direction in the dual, a possible way out of this issue is to introduce a radial cutoff in the gravity side, thus dividing the geometry in an "IR" region on one side of the cutoff and a "UV" region on the other side, as in figure 1. Once this is done one can give away with the troublesome UV region and work only with the IR region where the gravitational theory is weakly coupled. An important problem with this approach is that observables in the field theory have to be read from the asymptotic behavior of the fields in the UV region. Then, in order to be able to extract any useful information from the gravity dual, one needs to introduce a prescription that allows to recover the UV information after the UV region has been removed.

Some intuition can be gained from studying this problem in a gravity dual that is weakly coupled everywhere. One can then introduce the radial cutoff and see what is necessary in order to reproduce the same values for the observables one would have been obtained from the full geometry. Since in practice one is solving classical equations of motion for the gravity fields, this amounts in the end to determining the boundary conditions for the fields at the radial cutoff. This can be accomplished by introducing an effective action at the cutoff. The cutoff effective action can be thought of as an effective action in the field theory, obtained after integrating out UV degrees of freedom, and coupled to the strongly

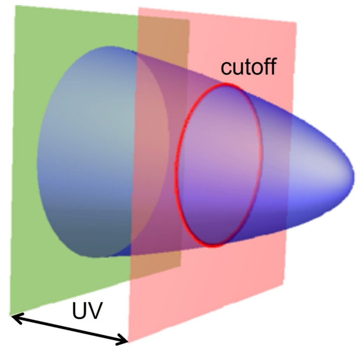

Figure 1. The string dual to a Wilson loop is attached to a circle at the boundary (back green plane) and extends along the holographic radial direction. Introducing a radial cutoff (front red plane) separates the geometry in the UV region between the cutoff plane and the boundary, and the IR region at the other side of the cutoff. Removing the UV region leaves the IR section of the string plus a cutoff action defined at the boundary of the truncated string (red circle).

coupled sector described by the IR region of the gravity dual. The cutoff action depends on the values of the gravitational fields and their derivatives at the cutoff. The cutoff action follows the general rules of effective field theories, it is constrained by symmetries, admits an expansion in small derivatives and all the information about the UV region is hidden in the values of the coefficients appearing in the action. Then, by fitting the coefficients of the cutoff action, one may be able to capture the right UV physics in the holographic dual, while remaining in the region where gravity is weakly coupled. For Wilson loops, which is our topic of interest here, the cutoff action has been studied in [4, 5], following the general approach of holographic Wilsonian renormalization [6, 7].

Wilson loops provide a complete set of gauge invariant operators that could in principle be used to compute any observable in Yang-Mills theory, and determine directly some important phenomenological quantities like the quark-antiquark potential. Their expectation value in the large-N limit can be computed at strong coupling by means of the gauge/gravity duality. For a CFT like $\mathcal{N}=4$ super Yang-Mills, the holographic dual of the Wilson loop is a fundamental string anchored at the asymptotic AdS boundary of the gravity dual geometry [8].

Wilson loops enjoy a quite large reparametrization invariance. By their definition, they are determined by the holonomy of the gauge field along a closed curve $\mathcal{C}$.[1] The curve itself can be parametrized as the trajectory of a particle $x^\mu(\tau)$, where τ parametrizes the wordline, in such a way that

$$\oint_{\mathcal{C}} A = \int d\tau\, \dot{x}^\mu A_\mu[x(\tau)]. \tag{1.1}$$

Any parametrization whose image is said curve should lead to the same value for the Wilson loop, even if it traces the curve back and forth, this is the so-called "zig-zag" symmetry [9].

[1]We will consider a Wilson loop extending to spatial infinity as "closed", from the point of view that gauge transformations should go to a constant at infinity in flat space.

This is true at strong coupling even for 1/2 BPS Wilson loops as noted in [10], even though the coupling to the scalar fields in principle breaks the zig-zag symmetry. The difference in the holographic dual between ordinary and BPS Wilson loops are boundary conditions for the string along the internal space, but the configurations we will study are valid for both sets of boundary conditions, so the symmetries turn out to be the same in this case.

It is unclear whether introducing a cutoff as described above preserves or breaks reparametrization invariance of the Wilson loop, in the same way that other symmetries such as conformal invariance are broken. If that were the case, the cutoff action should include additional terms that compensate the non-invariance of the string in the IR region, in a way analogous to the anomaly inflow mechanism of gauge theories. Clarifying this point will be our goal in this work. In order to proceed, it will be much more illuminating to work with the Polyakov action for the string dual to the Wilson loop, rather than the Nambu-Goto action that it is usually employed. As we will see, in the gravity dual description the reparametrization invariance of the Wilson loop corresponds to the conformal invariance of the string. This is the analog of the usual map between isometries of the geometry in the gravity side and global symmetries of the field theory dual, except in this case we are treating with a group with an infinite number of generators.

It turns out that the fate of reparametrization invariance when a radial cutoff is introduced depends on how this is done. If the radial cutoff is on the worldsheet the situation is similar to the nearly-AdS_2 physics described in [11], connected to Jackiw-Teitelboim (JT) gravity [12–14] and the Sachdev-Ye-Kitaev model (SYK) [15–19] (see [20] for a review on these topics). Effectively reparametrization invariance is broken and there is an effective action at the cutoff proportional to a Schwarzian derivative of the Goldstones associated to the broken symmetries. On the other hand, if the radial cutoff is at a fixed position on the target space of the string, reparametrization invariance is unbroken. In this second case both worldsheet diffeomorphisms and Weyl transformations are broken by the cutoff, so there are Schwarzian effective actions for the broken (gauge) symmetries, but they cancel out when a conformal transformation involving both is considered.

The content of the paper is as follows. In section 2 we first study in quite detail a straight Wilson line in a CFT, described by a fundamental string in the holographic dual with Polyakov action. We introduce a radial cutoff and derive the cutoff action by integrating the section of the string in the UV region. We then discuss in detail the reparametrization symmetries and the cutoff action. Next, in section 3, we generalize the results to other simple cases in a CFT: a circular Wilson loop at zero temperature and the straight line and Polyakov loop at nonzero temperature. We discuss the results and conclude in section 4. Some technical details of the calculation have been gathered in appendix A.

2 Straight Wilson line in a CFT

We will start by considering a straight Wilson line in a d-dimensional CFT, extended along a spatial direction. The dual is a string in AdS_{d+1} space (and localized in the internal directions). The metric in Poincaré coordinates is

$$ds^2 = G_{MN}dx^M dx^N = \frac{L^2}{z^2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right). \tag{2.1}$$

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We are using indices $M = \mu, z$ and $\mu = 0, 1, \ldots, d-1$. In the following we will work with dimensionless coordinates by doing the rescaling

$$z \to Lz, \qquad x^\mu \to Lx^\mu. \tag{2.2}$$

To specify the action of the string we introduce $\sigma^a = (\tau, \sigma)$ as the (dimensionless) worldsheet coordinates of the string, $X^M(\tau, \sigma)$ as the embedding functions and h_{ab} as the worldsheet metric. Removing the overall L^2 factor, the induced metric is

$$g_{ab} = \frac{1}{Z^2}\eta_{MN}\partial_a X^M \partial_b X^N. \tag{2.3}$$

Then, the Polyakov action for a string of tension T_s is

$$S_P = \frac{T_s L^2}{2}\int d^2\sigma\sqrt{h}h^{ab}g_{ab} + \phi_0\chi_E. \tag{2.4}$$

Where χ_E is the Euler characteristic of the string surface with a coefficient proportional to the constant dilaton $\phi_0 = \log g_s$, with g_s the string coupling. If boundary terms are properly accounted for, the Euler characteristic is just a constant determined by the string topology.

The Polyakov action is invariant under both worldsheet diffeomorphisms and Weyl transformations, which are gauge symmetries of the string. We will use them to fix the metric to be (Euclidean) AdS_2, in the Poincaré patch

$$h_{ab} = \frac{1}{\sigma^2}\delta_{ab}. \tag{2.5}$$

After the gauge fixing there is a remnant conformal symmetry that leaves the metric invariant and consists of simultaneous worldsheet diffeomorphisms and Weyl transformations. This is a true symmetry of the string that corresponds to the reparametrization invariance of the Wilson loop, as it will be clear later.

For a straight Wilson line along the x^1 direction we need to impose boundary conditions on the string. By our choice of metric $\sigma = 0$ should correspond to the boundary of the worldsheet, so that

$$\lim_{\sigma\to 0} X^1 = x^1, \qquad \lim_{\sigma\to 0} X^M = 0, \quad M \neq 0. \tag{2.6}$$

As we have formulated them, these conditions are invariant under conformal transformations. To ease the notation in the following we will use $X = X^1$ and $Z = X^z$.

We have some restrictions on the embedding functions. In the first place, the induced metric should be compatible with the worldsheet metric (the worldsheet energy-momentum tensor vanishes)

$$g_{ab} - \frac{1}{2}h_{ac}h^{bd}g_{cd} = 0. \tag{2.7}$$

And in the second place the embedding functions have to satisfy the equations of motion

$$\frac{1}{\sqrt{h}}\partial_a\left(\sqrt{h}h^{ab}\frac{\partial_b X^M}{Z^2}\right) + \frac{2}{Z}h^{ab}g_{ab}\delta^M_z = 0. \tag{2.8}$$

The simplest solution is a string extended along the (x^1, z) directions

$$X = \tau, \qquad Z = \sigma, \qquad X^M = 0, \quad M \neq 1, z. \tag{2.9}$$

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The area of the string is divergent, it can be regularized by introducing appropriate local counterterms at a cutoff $z = \epsilon$ that will eventually be sent to the boundary $\epsilon \to 0$. The regularized action is

$$S_\epsilon = \frac{T_s L^2}{2} \int_{\sigma > \epsilon} d^2\sigma \sqrt{h} h^{ab} g_{ab} - T_s L^2 \int_{\sigma=\epsilon} d\tau \, e_\tau + \phi_0 \chi_E. \tag{2.10}$$

Where $e_\tau = \sqrt{h_{\tau\tau}}$ is the einbein. The Euler characteristic is

$$\chi_E = \frac{1}{4\pi} \left[\int_{\sigma > \epsilon} d^2\sigma \sqrt{h} R + 2 \int_{\sigma=\epsilon} d\tau e_\tau K \right] = 1, \tag{2.11}$$

where R is the Ricci scalar of the worldsheet metric and K is the extrinsic curvature.

2.1 String solution with arbitrary boundary reparametrizations

Instead of the simple solution (2.9) we may consider an arbitrary reparametrization of the line at the boundary

$$\lim_{\sigma \to 0} X = x_0(\tau), \tag{2.12}$$

without modifying the shape of the string in the embedding space. This implies modifying the embedding functions $X = X(\tau, \sigma)$, $Z = Z(\tau, \sigma)$ and keeping $X^M = 0$ for $M \neq 1, z$. The constraint (2.7) can be satisfied as long as the induced metric is conformally flat $g_{ab} = \Omega \delta_{ab}$. From now on, let us denote $\partial_\tau = \dot{}$, $\partial_\sigma = {}'$. The induced metric for this more general embedding is

$$g_{ab} = \frac{1}{Z^2} \begin{pmatrix} \dot{X}^2 + \dot{Z}^2 & \dot{X} X' + \dot{Z} Z' \\ \dot{X} X' + \dot{Z} Z' & (X')^2 + (Z')^2 \end{pmatrix}. \tag{2.13}$$

In order to have a conformally flat metric, the conditions we need to impose are

$$\dot{X} X' + \dot{Z} Z' = 0, \qquad \dot{X}^2 + \dot{Z}^2 = (X')^2 + (Z')^2. \tag{2.14}$$

Which can be solved by

$$Z' = \dot{X}, \qquad X' = -\dot{Z}, \qquad X'' + \ddot{X} = 0, \qquad Z'' + \ddot{Z} = 0. \tag{2.15}$$

It can be easily checked that solutions to the equations above are also solutions to the equations of motion (2.8).

The solutions are

$$\begin{aligned} X &= \int_{-\infty}^{\infty} d\tau_0 \frac{1}{\pi} \frac{\sigma}{\sigma^2 + (\tau - \tau_0)^2} x_0(\tau_0), \\ Z &= \int_{-\infty}^{\infty} d\tau_0 \frac{1}{\pi} \frac{\tau - \tau_0}{\sigma^2 + (\tau - \tau_0)^2} x_0(\tau_0). \end{aligned} \tag{2.16}$$

However, when the derivatives of x_0 are small compared to $1/\sigma$, it is more interesting to express the solutions as an infinite series expansion (see appendix A)

$$\begin{aligned} X &= \cos\left(\sigma \frac{d}{d\tau}\right) x_0(\tau) = x_0 - \frac{1}{2}\sigma^2 \ddot{x}_0 + \frac{1}{24}\sigma^4 x_0^{(4)} + \cdots, \\ Z &= \sin\left(\sigma \frac{d}{d\tau}\right) x_0(\tau) = \sigma \dot{x}_0 - \frac{1}{6}\sigma^3 \dddot{x}_0 + \frac{1}{120}\sigma^5 x_0^{(5)} + \cdots. \end{aligned} \tag{2.17}$$

In this form it is straightforward to find the conformal factor in the induced metric in a similar expansion

$$\Omega = \frac{1}{\sigma^2} - \frac{2}{3}\{x_0, \tau\} + \sigma^2 \left(\frac{1}{15}\partial_\tau^2 \{x_0, \tau\} + \frac{4}{15}(\{x_0, \tau\})^2 \right) + \cdots . \tag{2.18}$$

Here we have introduced the Schwarzian derivative

$$\{x_0, \tau\} = \frac{\dddot{x}_0}{\dot{x}_0} - \frac{3}{2}\left(\frac{\ddot{x}_0}{\dot{x}_0}\right)^2 . \tag{2.19}$$

Higher order terms can also be written in terms of the Schwarzian and its derivatives. The Schwarzian is invariant under $\mathrm{GL}(2, \mathbb{R})$ reparametrizations of the form

$$x_0(\tau) \longrightarrow \frac{a x_0 + b}{c x_0 + d}, \qquad a, b, c, d \in \mathbb{R}, \qquad ad - bc \neq 0. \tag{2.20}$$

For $x_0(\tau) = \tau$ the Schwarzian vanishes, these are the transformations induced at the boundary by AdS_2 isometries. Defining the complex coordinate $\zeta = \tau + i\sigma$, the AdS_2 metric in these coordinates is

$$ds^2 = -\frac{4 d\zeta d\bar{\zeta}}{\left(\zeta - \bar{\zeta}\right)^2}, \tag{2.21}$$

which is manifestly invariant under the transformation

$$\zeta \longrightarrow \frac{a\zeta + b}{c\zeta + d}. \tag{2.22}$$

When $\sigma \to 0$, $\zeta = \bar{\zeta} = \tau$ leading to the transformations we wrote above. Thus boundary reparametrizations of the form (2.20) with $x_0 = \tau$ do not lead to changes in the induced metric and the conformal factor stays fixed as $\Omega = 1/\sigma^2$.

Let us show now that a conformal transformation trivializes the embedding. First, we perform a worldsheet diffeomorphism

$$\tau = \tau(\bar{\tau}, \bar{\sigma}), \qquad \sigma = \sigma(\bar{\tau}, \bar{\sigma}), \tag{2.23}$$

such that

$$X(\tau, \sigma) = \bar{\tau}, \qquad Z(\tau, \sigma) = \bar{\sigma}. \tag{2.24}$$

In the near boundary expansion the transformed coordinates have expansions similar to X and Z

$$\tau = t(\bar{\tau}) - \frac{1}{2}\ddot{t}(\bar{\tau})\bar{\sigma}^2 + \cdots , \qquad \sigma = \dot{t}(\bar{\tau})\bar{\sigma} - \frac{1}{6}\dddot{t}(\bar{\tau})\bar{\sigma}^3 + \cdots . \tag{2.25}$$

The induced and worldsheet metrics in the new coordinates are

$$\bar{g}_{ab} = \frac{1}{\bar{\sigma}^2}\delta_{ab}, \qquad \bar{h}_{ab} = \bar{\Omega}\delta_{ab}, \tag{2.26}$$

where the conformal factor in the worldsheet metric equals to

$$\bar{\Omega} = \frac{1}{\bar{\sigma}^2} - \frac{2}{3}\left\{t\left(\bar{\tau}\right), \bar{\tau}\right\} + \cdots . \tag{2.27}$$

Finally, to put back the worlsheet metric in its original form we do a Weyl transformation

$$\bar{h}_{ab} \;\longrightarrow\; h_{ab} = \frac{1}{\bar{\sigma}^2\bar{\Omega}}\bar{h}_{ab} = \frac{1}{\bar{\sigma}^2}\delta_{ab}. \tag{2.28}$$

This shows that conformal transformations on the worldsheet correspond to reparametrizations of the Wilson loop, as we could in principle follow these steps backwards to produce an arbitrary reparametrization from the trivial embedding.

2.2 Induced anomalies in the cutoff action

The expectation value of the Wilson line in the dual field theory is determined by the string action on-shell. We can introduce a cutoff in the radial direction that splits the dual geometry in two parts. The region between the AdS boundary and the radial cutoff is identified with UV degrees of freedom of the field theory dual and the region beyond the cutoff captures the IR degrees of freedom.

The IR region of the geometry still describes the dual of a strongly coupled theory, with a line defect that has a holographic description as a string ending at the cutoff along a line in the x^1 direction. In addition to the string, that captures the dynamics of the IR degrees of freedom of the strongly coupled field theory dual, there is an effective action at the cutoff for the defect that is obtained integrating over the radial direction the string action between the boundary and the cutoff. The natural interpretation of the cutoff action is that it captures the effect of the UV degrees of freedom close to the Wilson line after they have been integrated out.

There are two possible natural choices for the cutoff, we could introduce a cutoff in the worldsheet coordinate $\sigma = 1/(L\Lambda)$, or we could introduce a cutoff in the radial coordinate of the geometry $z = 1/(L\Lambda)$, with Λ an energy scale. If the cutoff is taken in the worldsheet, the cutoff action is

$$\begin{aligned} S_\Lambda = T_s L^2 \int d\tau \left(-L\Lambda - \frac{2}{3}\frac{1}{L\Lambda}\{x_0,\tau\} + \frac{1}{3}\frac{1}{(L\Lambda)^3}\left(\frac{1}{15}\partial_\tau^2\{x_0,\tau\} + \frac{2}{5}(\{x_0,\tau\})^2\right) + \cdots \right) \\ + \frac{\phi_0}{2\pi}\int d\tau L\Lambda. \end{aligned} \tag{2.29}$$

This shows that the cutoff action is not invariant under reparametrizations. From the bulk perspective, the string extended beyond the cutoff is reparametrization invariant up to boundary terms. This non-invariance is compensated by the action at the cutoff, so that the total action consisting of string plus defect is invariant. This can be seen as analogous to the anomaly inflow between a Chern-Simons action for a gauge field in $2+1$ dimensions and chiral edge modes at a boundary. Thus, we can see the terms depending on the Schwarzian as originating from a reparametrization anomaly in the cutoff action.

However, one might object that a radial cutoff in the geometry is more natural than a cutoff in the worldsheet, since the AdS radial direction is typically more readily identified with energy scales in the field theory dual. If we fix the radial cutoff, then we should integrate the string action up to a value of the worldsheet coordinate determined by the condition

$$Z(\tau, \sigma_\Lambda(\tau)) = 1/(L\Lambda). \tag{2.30}$$

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The solution can be expanded as

$$\sigma_\Lambda(\tau) = \frac{1}{L\Lambda\dot{x}_0}\left[1 + \frac{1}{6}\frac{1}{(L\Lambda)^2}\frac{\dddot{x}_0}{(\dot{x}_0)^3} + \frac{1}{6}\frac{1}{(L\Lambda)^4}\frac{10(\ddot{x}_0)^2 - \dot{x}_0 x_0^{(5)}}{(\dot{x}_0)^6} + \cdots\right]. \tag{2.31}$$

Then, the action integrated up to this value is

$$S_\Lambda = T_s L^2 \int d\tau \left(-L\Lambda\dot{x}_\Lambda\right) + \frac{\phi_0}{2\pi}\int d\tau \frac{1}{\sigma_\Lambda}. \tag{2.32}$$

Where we have defined

$$x_\Lambda = x_0 + \frac{1}{2}\frac{1}{(L\Lambda)^2}\frac{\ddot{x}_0}{\dot{x}_0} + \frac{1}{72}\frac{1}{(L\Lambda)^4}\frac{4\ddot{x}_0\,\dddot{x}_0 - \dot{x}_0 x_0^{(4)}}{(\dot{x}_0)^5} + \cdots. \tag{2.33}$$

Therefore, with this choice of cutoff, the contribution of the induced metric to the defect action is simply a reparametrization of the worldline coordinate $d\tau_\Lambda = d\tau\dot{x}_\Lambda$. This is to be expected because the area of the string between the AdS boundary and the radial cutoff in the geometry should be independent of the reparametrization. However, we must now pay attention to the contribution to the defect action deriving from the Ricci scalar of the worldsheet metric, which now gives a non-trivial contribution

$$S_\Lambda = T_s L^2 \int d\tau_\Lambda \left(-L\Lambda\right) + \frac{\phi_0}{2\pi}\int d\tau_\Lambda \left(L\Lambda + \frac{2}{3}\frac{1}{L\Lambda}\{t(\tau_\Lambda), \tau_\Lambda\} + \cdots\right). \tag{2.34}$$

Where we have defined t as the inverse of x_0: $x_0[t(\theta)] = \theta$ and used that

$$\{t(\theta), \theta\} = -\frac{1}{\dot{x}_0^2}\{x_0(\tau), \tau\}. \tag{2.35}$$

We can recover the same result by performing the worldsheet diffeomorphism (2.24). Once we have trivialized the embedding, the terms proportional to the induced metric that contribute to the cutoff action are trivial. However, the integral over the Ricci scalar introduces a boundary term proportional to the extrinsic curvature

$$\bar{K} = -\frac{1}{2}\frac{\bar{\Omega}'}{\bar{\Omega}^{3/2}} = 1 + \bar{\sigma}^2\{t(\bar{\tau}), \bar{\tau}\} + \cdots. \tag{2.36}$$

Where $\bar{\Omega}$ is given in (2.27). The resulting cutoff action is the same we found before (2.34) identifying $\tau_\Lambda = \bar{\tau}$

$$S_\Lambda = T_s L^2 \int d\bar{\tau} \left(-L\Lambda\right) + \frac{\phi_0}{2\pi}\int d\bar{\tau} \left(L\Lambda + \frac{2}{3}\frac{1}{L\Lambda}\{t(\bar{\tau}), \bar{\tau}\} + \cdots\right). \tag{2.37}$$

The Schwarzian derivative indicates that effectively there is an anomaly at the cutoff, which compensates the non-invariance of the string under worldsheet diffeomorphisms. Note that the Weyl transformation (2.28) would remove this term, so there is another associated anomaly at the cutoff, in such a way that the anomalous terms cancel out for conformal transformations of the worldsheet.

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3 Generalizations

In order to highlight the universality of the reparametrization anomaly we will now study three straightforward generalizations of spatial Wilson loops in a CFT: a straight Wilson line at nonzero temperature, a circular Wilson line at zero temperature and a Polyakov loop at nonzero temperature.

3.1 Straight Wilson line at nonzero temperature

At nonzero temperature the holographic dual of a CFT_d is an AdS_{d+1} black brane solution, that in Poincaré patch coordinates reads

$$ds^2 = \frac{L^2}{z^2}\left(\frac{dz^2}{f(z)} - f(z)(dx^0)^2 + \delta_{ij}dx^i dx^j\right), \qquad f(z) = 1 - \left(\frac{z}{z_H}\right)^d. \tag{3.1}$$

The temperature of the dual CFT is $T = \frac{d}{4\pi z_H}$. It will be convenient for us to do a change of coordinates such that the induced metric on the string becomes conformally flat. This can be achieved by picking a new radial coordinate u such that

$$du = \frac{dz}{\sqrt{f(z)}}. \tag{3.2}$$

The solution is

$$u = \frac{z_H B_{\left(\frac{z}{z_H}\right)^d}\left(\frac{1}{d}, \frac{1}{2}\right)}{d}. \tag{3.3}$$

Where $B_x(a,b)$ is the incomplete Beta function. The horizon in the u coordinate is located at

$$u_H = \frac{B\left(\frac{1}{d}, \frac{1}{2}\right)}{d} z_H \tag{3.4}$$

The relation can be inverted to

$$\left(\frac{z(u)}{z_H}\right)^d = I^{-1}_{\frac{u}{u_H}}\left(\frac{1}{d}, \frac{1}{2}\right), \tag{3.5}$$

where $I_x(a,b) = B_x(a,b)/B(a,b)$ is the regularized incomplete Beta function and $I_x^{-1}(a,b)$ is its inverse. For convenience let us do the following rescaling of the coordinates

$$u \to u_H u, \qquad z \to z_H z, \qquad x^\mu \to u_H x^\mu. \tag{3.6}$$

Then, the metric is

$$ds^2 = \frac{\tilde{L}^2}{z(u)^2}\left(du^2 - f[z(u)]\left(dx^0\right)^2 + \delta_{ij}dx^i dx^j\right), \qquad f(z) = 1 - z^d, \qquad z(u)^d = I_u^{-1}\left(\frac{1}{d}, \frac{1}{2}\right), \tag{3.7}$$

where $\tilde{L} = L u_H / z_H = B\left(\frac{1}{d}, \frac{1}{2}\right) L/d$.

The worldsheet action dual to a Wilson line in the black brane geometry is

$$S_P = \frac{T_s \tilde{L}^2}{2}\int_{u<1} d^2\sigma \sqrt{h} h^{ab} g_{ab} + \phi_0 \hat{\chi}_E. \tag{3.8}$$

If the string reaches the black brane horizon, only the part of the string outside the horizon is taken into account. This introduces a cutoff in the radial direction at $u = 1$. The term $\hat{\chi}_E$ equals (2.11) with the same cutoff at $u = 1$, but without a extrinsic curvature term at the horizon, so it is no longer equal to the Euler characteristic of the string worldsheet and does not take integer values in general.

For a straight spatial Wilson line, we can take as embeddding and worldsheet metric

$$X^1 \equiv X = \tau, \qquad X^u \equiv U = \sigma, \qquad X^M = 0, M \neq 1, u. \qquad h_{ab} = \frac{1}{\sigma^2}\delta_{ab}. \tag{3.9}$$

With this choice the induced metric is the same as the worldsheet metric up an overall factor, which automatically satisfies the constraint (2.7). The equations for the embedding functions, which now have the form

$$\frac{1}{\sqrt{h}}\partial_a\left(\sqrt{h}h^{ab}\frac{\partial_b X^M}{z(U)^2}\right) + \frac{2}{z(U)}z'(U)h^{ab}g_{ab}\delta^M_u = 0, \tag{3.10}$$

are also satisfied.

As in the zero temperature case we consider an arbitrary reparametrization of the Wilson line at the boundary

$$\lim_{\sigma\to 0} X = x_0(\tau). \tag{3.11}$$

The embedding functions will be modified as before $X(\tau, \sigma)$ and $\mathrm{U}(\tau, \sigma)$. The induced string metric is

$$g_{ab} = \frac{1}{z(U)^2}\begin{pmatrix} \dot{X}^2 + \dot{U}^2 & \dot{X}X' + \dot{U}U' \\ \dot{X}X' + \dot{U}U' & (X')^2 + (U')^2 \end{pmatrix}, \tag{3.12}$$

which can be made conformally flat by imposing the same conditions as at zero temperature (2.14) and (2.15), simply replacing Z by U. As happened at zero temperature the embedding equations of motion (3.10) are automatically satisfied even with the new conformal factor.

Following the discussion at zero temperature, we can fix the radial cutoff in the geometry, but now on the u coordinate

$$\mathrm{U}\left(\tau, \sigma_\Lambda(\tau)\right) = 1/(\Lambda L). \tag{3.13}$$

Performing the same worldsheet diffeomorphism as before (2.25), the induced and worldsheet metric become

$$g_{ab} = \frac{1}{z(\bar{\sigma})^2}\delta_{ab}, \qquad h_{ab} = \bar{\Omega}\delta_{ab}, \tag{3.14}$$

where the conformal factor is the same as at zero temperature (2.27).

It follows that the cutoff effective action is the same at zero and nonzero temperature. However, at nonzero temperature there is a physical cutoff that is the black brane horizon, where $U = 1$, implying $\Lambda L = 1$, in our variables. Then, the effective action after integrating all the way to the horizon is

$$S_H = T_s\tilde{L}^2\int d\bar{\tau}\left(\int_\epsilon^1 d\bar{\sigma}\frac{1}{z(\bar{\sigma})^2} - \frac{1}{\epsilon}\right) + \frac{\phi_0}{2\pi}\int d\bar{\tau}\left(1 + \frac{2}{3}\{t(\bar{\tau}), \bar{\tau}\} + \cdots\right). \tag{3.15}$$

Restoring units $x_0 \to x_0/u_H$ and $\bar{\tau} \to \bar{\tau}/u_H$, the Schwarzian term is

$$S_{Sch} = \frac{\phi_0}{12\pi^2} \frac{B\left(\frac{1}{d}, \frac{1}{2}\right)}{T} \int d\bar{\tau} \left\{t(\bar{\tau}), \bar{\tau}\right\} \tag{3.16}$$

3.2 Circular Wilson loop

We will consider now a Wilson loop defined on a circle of radius r_0 localized on some plane, that we can take to be along the x^μ, $\mu = 1, 2$ directions without loss of generality. In this case it is more convenient to work with polar coordinates in the plane. The metric in the Poincaré patch is:

$$ds^2 = \frac{L^2}{z^2} \left(dz^2 + dr^2 + r^2 d\theta^2 + \sum_{\mu=3}^{d-1} (dx^\mu)^2 \right). \tag{3.17}$$

The solution for a string ending on a circle of radius R on the boundary lies on the spherical surface [10, 21]

$$z^2 + r^2 = r_0^2. \tag{3.18}$$

This surface can be parametrized by the worldsheet embedding

$$X^\theta \equiv \Theta = \tau, \qquad X^r \equiv R = \frac{r_0}{\cosh \sigma}, \qquad Z = r_0 \tanh \sigma, \qquad X^M = 0,\ M \neq r, \theta, z. \tag{3.19}$$

Where both τ and θ have 2π periodicity.

This yields global AdS_2 in conformally flat coordinates as the induced metric on the string

$$ds_2^2 = g_{ab} d\sigma^a d\sigma^b = \frac{1}{\sinh^2 \sigma} \left(d\tau^2 + d\sigma^2 \right). \tag{3.20}$$

Given the topology of the string worldsheet, we will select the string metric to be the same

$$h_{ab} = \frac{1}{\sinh^2 \sigma} \delta_{ab}. \tag{3.21}$$

Let us now consider a general reparametrization of the embedding of the form

$$\Theta = q\tau + \theta(\tau, \sigma), \qquad R = \frac{r_0}{\cosh S}, \qquad Z = r_0 \tanh S, \qquad S = q\sigma + s(\tau, \sigma). \tag{3.22}$$

Where the periodicity of τ is now $2\pi p$ and p, q are nonzero integers. Both θ and s are taken to be periodic functions of τ

$$\theta(\sigma, \tau + 2\pi p) = \theta(\sigma, \tau), \qquad s(\sigma, \tau + 2\pi p) = s(\sigma, \tau). \tag{3.23}$$

At the AdS boundary $\sigma = 0$ we impose the conditions

$$\Theta(\sigma = 0, \tau) = \Theta_0(\tau) = q\tau + \theta_0(\tau), \qquad S(\sigma = 0, \tau) = 0, \qquad \theta_0(\tau + 2\pi p) = \theta_0(\tau). \tag{3.24}$$

The term in Θ_0 that is linear in τ indicates that the Wilson line is winding $w = pq$ times over the circle. However, unless $q = 1/p$ (so $w = 1$) the induced metric will have a conical singularity. We will ignore this and proceed with general values of p, q.

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With this embedding, the induced metric and worldsheet metric become

$$g_{ab} = \frac{1}{\sinh^2 S} \begin{pmatrix} \dot{S}^2 + \dot{\Theta}^2 & \dot{S}S' + \dot{\Theta}\Theta' \\ \dot{S}S' + \dot{\Theta}\Theta' & (S')^2 + (\Theta')^2 \end{pmatrix}, \qquad h_{ab} = \frac{q^2}{\sinh^2(q\sigma)}\delta_{ab}. \tag{3.25}$$

This takes the same form as for the straight line (2.13), so the induced metric can be made conformally flat for embedding solutions satisfying the same set of equations as given in (2.15)

$$S' = \dot{\Theta}, \qquad \Theta' = -\dot{S}, \qquad S'' + \ddot{S} = 0, \qquad \Theta'' + \ddot{\Theta} = 0. \tag{3.26}$$

The linear terms proportional to q in the embedding functions (3.22) automatically satisfy these equations. We can give a solution generalizing the straight line results to account for the periodicity of τ. First we define the functions

$$\begin{aligned} G_\Theta(\sigma, \tau - \tau_0) &= \sum_{n=-\infty}^{\infty} \frac{1}{\pi} \frac{\sigma}{\sigma^2 + (\tau - \tau_0 + 2\pi n p)^2} = \frac{1}{2\pi p} \frac{\sinh \frac{\sigma}{p}}{\cosh \frac{\sigma}{p} - \cos \frac{\tau - \tau_0}{p}}, \\ G_S(\sigma, \tau - \tau_0) &= \frac{1}{2\pi p} \frac{\sin \frac{\tau - \tau_0}{p}}{\cosh \frac{\sigma}{p} - \cos \frac{\tau - \tau_0}{p}}. \end{aligned} \tag{3.27}$$

Then, the solutions for the embedding functions are

$$\begin{aligned} \theta(\sigma, \tau) &= \int_{-\pi p}^{\pi p} d\tau_0 \, G_\Theta(\sigma, \tau - \tau_0)\theta_0(\tau_0), \\ s(\sigma, \tau) &= \int_{-\pi p}^{\pi p} d\tau_0 \, G_S(\sigma, \tau - \tau_0)\theta_0(\tau_0). \end{aligned} \tag{3.28}$$

One can recover the straight line expressions (2.16) by taking the $p \to \infty$ limit.

As before, it will be more convenient for us to use an expansion of the solutions for small τ derivatives relative to $1/\sigma$, which is actually of the same form as for the straight line (2.17) (see appendix A)

$$\begin{aligned} \Theta &= \cos\left(\sigma \frac{d}{d\tau}\right) \Theta_0(\tau) = \Theta_0 - \frac{1}{2}\sigma^2 \ddot{\Theta}_0 + \frac{1}{24}\sigma^4 \Theta_0^{(4)} + \cdots, \\ S &= \sin\left(\sigma \frac{d}{d\tau}\right) \Theta_0(\tau) = \sigma \dot{\Theta}_0 - \frac{1}{6}\sigma^3 \dddot{\Theta}_0 + \frac{1}{120}\sigma^5 \Theta_0^{(5)} + \cdots. \end{aligned} \tag{3.29}$$

However, before doing the reparametrization, we end up with a slightly modified result, since the conformal factor in the induced metric was different:

$$\Omega = \frac{1}{\sigma^2} - \frac{2}{3}\left\{\tan\frac{\Theta_0}{2}, \tau\right\} + \sigma^2 \left(\frac{1}{15}\partial_\tau^2 \left\{\tan\frac{\Theta_0}{2}, \tau\right\} + \frac{4}{15}\left\{\tan\frac{\Theta_0}{2}, \tau\right\}^2\right) + \cdots. \tag{3.30}$$

Where the Schwarzian terms are now

$$\left\{\tan\frac{\Theta_0}{2}, \tau\right\} = \{\Theta_0, \tau\} + \frac{1}{2}\dot{\Theta}_0^2. \tag{3.31}$$

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In this case the terms that appear in the expansion are invariant under boundary reparametrizations of the form

$$e^{i\Theta_0(\tau)} \longrightarrow \frac{\alpha e^{i\Theta_0}+\bar{\beta}}{\beta e^{i\Theta_0}+\bar{\alpha}}, \qquad \alpha,\beta\in\mathbb{C},\ |\alpha|^2-|\beta|^2=1. \tag{3.32}$$

This can be understood as the boundary limit of the SU(1, 1) isometry transformations of the global AdS_2 metric. The symmetry is more easily realized in the coordinates

$$\cosh\sigma=\frac{1}{\tanh\rho};\quad \zeta=\tanh\frac{\rho}{2}e^{i\tau},\ \bar{\zeta}=\tanh\frac{\rho}{2}e^{-i\tau}, \tag{3.33}$$

leading to the metric

$$ds_2^2=\frac{4d\zeta d\bar{\zeta}}{(1-|\zeta|^2)^2}. \tag{3.34}$$

The SU(1, 1) isometry transformations in these coordinates are

$$\zeta\longrightarrow\frac{\alpha\zeta+\bar{\beta}}{\beta\zeta+\bar{\alpha}}\underset{|\zeta|\to 1}{\longrightarrow}\frac{\alpha e^{i\tau}+\bar{\beta}}{\beta e^{i\tau}+\bar{\alpha}}. \tag{3.35}$$

Thus leading to the transformations (3.32). However, contrary to the straight line, the Schwarzian derivative is nonzero for the trivial embedding, instead for $\Theta_0=\tau$ or any SU(1, 1) equivalent,

$$\left\{\tan\frac{\Theta_0}{2},\tau\right\}=\frac{1}{2}. \tag{3.36}$$

This is of course necessary in order to recover the expansion of the conformal factor

$$\Omega=\frac{1}{\sinh^2\sigma}=\frac{1}{\sigma^2}-\frac{1}{3}+\frac{\sigma^2}{15}+\cdots. \tag{3.37}$$

Aside from the difference on the isometry group of the induced metric and the corresponding invariant Schwarzian derivatives, the analysis of the straight Wilson line of section 2.2 can be generalized without any other modifications to the circular Wilson loop. Therefore, replacing

$$\{x_0,\tau\}\to\left\{\tan\frac{\Theta_0}{2},\tau\right\}, \tag{3.38}$$

the cutoff action takes the form (2.29) if we introduce the cutoff in the worldsheet coordinates $\sigma=1/(L\Lambda)$. If the cutoff is in the radial coordinate of the background $z=1/(L\Lambda)$, then the cutoff action is (2.37) changing

$$\{t(\bar{\tau}),\bar{\tau}\}\to\left\{\tan\frac{t(\bar{\tau})}{2},\bar{\tau}\right\}. \tag{3.39}$$

Where $\bar{\sigma}=S(\sigma,\tau)$, $\bar{\tau}=\Theta(\sigma,\tau)$ define the barred coordinates and $t(\bar{\tau})$ is the inverse of the boundary reparametrization $\Theta_0[t(\theta)]=\theta$.

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3.3 Polyakov loop

At finite temperature a Polyakov loop is defined as a Wilson line wrapping the time direction after a Wick rotation to Euclidean signature. A nonzero expectation value for the Polyakov loop implies a spontaneous breaking of center symmetry and it is taken as an indication of deconfinement in pure Yang-Mills. The holographic dual is a string wrapped around the Euclidean time direction of the Wick rotated AdS_{d+1} black brane (3.1), with metric

$$ds^2 = \frac{L^2}{z^2}\left(\frac{dz^2}{f(z)} + f(z)dt_E^2 + \delta_{ij}dx^i dx^j\right), \qquad f(z) = 1 - \left(\frac{z}{z_H}\right)^d. \tag{3.40}$$

Where the Euclidean time direction t_E has periodicity $\beta = 1/T$. The string dual to the Polyakov loop will be extended along the (z, t_E) directions, and has the topology of a disk. In order to have a conformally flat induced metric we first rescale all the coordinates $x^M \to z_H x^M$ and then introduce the following change of variables for the radial coordinate

$$du = \frac{dz}{f(z)}. \tag{3.41}$$

The solution to this equation is similar to the one we found for a spatial Wilson loop (3.3)

$$u = \frac{B_{z^d}\left(\frac{1}{d}, 0\right)}{d}. \tag{3.42}$$

In this case the radial coordinate is not bounded, close to the horizon $z \to 1$

$$u \sim -\frac{1}{d}\log(1-z) \to +\infty. \tag{3.43}$$

We now follow similar steps, and rescale the coordinates as follows

$$u = \frac{1}{2\pi z_H T} r, \qquad t_E = \frac{1}{2\pi z_H T}\theta. \tag{3.44}$$

After this, the periodicity of θ is 2π, and the metric takes the form

$$\begin{aligned} ds^2 &= \frac{L^2}{z(r)^2}\left(\frac{2}{d}\right)^2 f\left[z(r)\right]\left(dr^2 + d\theta^2\right) + \frac{L^2}{z(r)^2}\delta_{ij}dx^i dx^j, \\ f(z) &= 1 - z^d, \qquad z(r)^d = B_{2r}^{-1}\left(\frac{1}{d}, 0\right), \end{aligned} \tag{3.45}$$

where $B_z^{-1}(x, y)$ is the inverse of the incomplete beta function. The simplest choice for the string embedding of the Polyakov loop is

$$X^\theta \equiv \Theta = \tau, \qquad X^r \equiv R = \sigma, \qquad X^i = 0, \tag{3.46}$$

where τ is periodic, with periodicity 2π. The induced metric is, after removing an overall L factor,

$$ds_2^2 = g_{ab}dx^a dx^b = \left(\frac{2}{d}\right)^2 \frac{f[z(\sigma)]}{z(\sigma)^2}\left(d\tau^2 + d\sigma^2\right). \tag{3.47}$$

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This is the same type of induced metric we found for the circular Wilson loop (3.20). The asymptotic behavior for $\sigma \to 0$ is that of AdS_2, with the conformal factor $\sim 1/\sigma^2$. For $\sigma \to \infty$ we see from (3.43) that $z^d \simeq 1 - e^{-2\sigma}$, in such a way that the conformal factor is

$$\left(\frac{2}{d}\right)^2 \frac{f[z(\sigma)]}{z(\sigma)^2} \sim 4e^{-2\sigma} \sim \frac{1}{\sinh^2 \sigma}, \tag{3.48}$$

which is the expected behavior for AdS_2. We are thus driven to take a worldsheet metric corresponding to global AdS_2

$$h_{ab} = \frac{1}{\sinh^2 \sigma} \delta_{ab}. \tag{3.49}$$

From this point onwards we can proceed following the same steps as for a circular Wilson loop, introducing a reparametrization that is the analog of (3.22)

$$\Theta = q\tau + \theta(\tau, \sigma), \qquad R = S = q\sigma + s(\tau, \sigma). \tag{3.50}$$

The details for the solutions and the symmetries of the worldsheet metric are the same as for the circular Wilson loop, so we arrive to the same result of a Schwarzian action for the worldsheet diffeomorphisms (3.39).

4 Discussion

Our motivation was to study the low energy effective description of a Wilson loop using the gauge/gravity duality, by taking a string in some IR region of the dual geometry determined by a cutoff in the holographic radial direction. In principle the string embedding and action that codifies the value of the dual Wilson loop is determined by the full geometry and the location of the string at the asymptotic boundary, associated to the UV in the dual field theory.

When the radial cutoff is introduced the region between the cutoff and the asymptotic boundary is replaced by an action at the cutoff. A systematic way to construct the effective action is by performing a derivative expansion of the embedding functions close to the cutoff with respect to the worldline coordinate of the Wilson loop, which remains valid as long as derivatives are small compared to a scale set by the value of the cutoff. For instance, if the Wilson loop describes a trajectory $(t(\tau), x(\tau))$ at the asymptotic boundary then, in the static gauge $t = \tau$ the cutoff action takes the form [5]

$$S_c = \int d\tau \sqrt{-\gamma} \left[\Lambda - \frac{k_0}{2}(x_c - x)^2 + \frac{k_1}{2}(\dot{x}_c - \dot{x})^2 + \frac{k_2}{2}\dot{x}_c^2 + \frac{k_3}{2}\dot{x}^2 + \cdots \right], \tag{4.1}$$

where γ is the induced metric at the cutoff and x_c corresponds to the intersection of the string embedding with the cutoff. The coefficients Λ, $k_{0,1,2,3}$ that appear in the cutoff action depend on the radial position of the cutoff z_c and obey a series of RG flow equations that guarantee that the total action is independent of z_c. These equations only depend on the geometry close to the cutoff, but the solutions depend on the UV geometry through integration constants that can be explicitly computed when the UV geometry is known.

Since the embedding can change dynamically away from the asymptotic boundary x_c is not fixed, i.e. the string does not obey Dirichlet boundary conditions at the cutoff. Nevertheless the variation of the total on-shell action (IR string plus cutoff) should still be invariant when x_c varies. This leads to mixed boundary conditions at the cutoff for the string that guarantees that the embedding profile in the IR region and the value of the total action remains unchanged when the cutoff is introduced. In other words, introducing the cutoff does not change any physical quantity but allows to drop the UV region of the geometry. When the UV geometry is not known, we can parametrize our ignorance in the integration constants of the RG flow equations of the coefficients in the cutoff action.

In previous works worldline reparametrizations of the Wilson loop were fixed by a suitable gauge choice. However, this is an important symmetry that should also be realized in the holographic dual and our purpose in this work was to address this for the low energy action of the string. In the Polyakov formulation reparametrizations could in principle correspond to a combination of (large) worldsheet diffeomorphisms with Weyl transformations of the worldsheet metric, as the first already induce a worldline reparametrization and the classical string action is invariant under both. When the cutoff is introduced and the UV region is replaced by the cutoff action, the total action should still be invariant under these symmetries. However, there is no unique way in which this could happen. For any of these symmetries one possibility is that the cutoff and IR string action are both invariant, or the other possibility is that both are not invariant but their variations under a symmetry transformation cancel out. An analogous situation for the second possibility is when conformal symmetry is spontaneously broken in a $3+1$-dimensional theory and there is an IR fixed point. The contributions to the conformal anomaly come from the IR CFT and from the dilaton, and they should add up to the value of the UV CFT. In the case at hand there can be reparametrization and Weyl anomalies in the cutoff action that compensate with the non-invariance of the IR string.

We have first identified a map between reparametrizations of the Wilson loop and conformal transformations in the worldsheet of the dual string. We have then shown that the string with a cutoff in the worldsheet is not invariant under reparametrizations of the Wilson loop, so that it is necessary to add a cutoff action proportional to the Schwarzian derivative of said reparametrizations, as well as higher derivative terms further suppressed by the cutoff scale. On the other hand, if the cutoff is set in the target space, the string is invariant under Wilson loop reparametrizations, but not under worldsheet diffeomorphisms or Weyl transformations that do not belong to the subset of conformal transformations. Therefore, new terms for the Polyakov string action should be added at the cutoff before fixing the gauge. Note that in both situations an $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SU}(1,1)$ symmetry is an essential ingredient. For a worldsheet cutoff it is a symmetry of the induced metric, so for more general shapes of the Wilson loop or different geometries we do not expect that leading contribution to the effective action is just the Schwarzian. On the other hand, for a target space cutoff the symmetry is an isometry of the worldsheet metric, which in principle we can fix to be AdS_2, global or in Poincaré patch depending on the topology. Thus we expect the result to be more general in this case. However note that there can still be modifications of the effective action if the dilaton is not constant, so corrections to the Schwarzian action are possible in non-conformal holographic duals.

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In the case of a cutoff on the worldsheet, the new terms have the appropriate structure to describe the effective theory of Goldstone bosons for a spontaneously broken reparametrization invariance of the Wilson loop, similarly to nearly-AdS_2 dynamics. They are necessary because a conformal transformation on the worldsheet would change the physical location of the cutoff in the target space, thus modifying the area of the string in the IR region. This is compensated by the change in the cutoff action, so the total area defined as the string action including the cutoff remains the same. Although we have only considered strings in AdS spacetimes, in principle conformal invariance of the worldsheet should be maintained in any string background, so we expect reparametrization invariance of Wilson loops to hold in general. It would be interesting to explore if the Schwarzian in the cutoff action is related with maximal chaos as observed in strings with worldsheet horizons [22, 23], similarly to SYK [19] and JT gravity [24].

Another interesting extension of this work would be to include in the analysis non-trivial profiles of the string in the transverse directions, generalizing the results of [4, 5]. It should be noted that a Schwarzian action for transverse fluctuations can also appear when the string is embedded in AdS_3 [25, 26], but it is related to diffeomorphisms in the target space, rather than to worldsheet transformations.

Finally, let us comment further on the connection and differences between the Wilson loop and SYK. A supersymmetric Wilson loop in higher representations can be described as a brane intersection with dynamical fields on the defect [27]. A similar defect theory may be expected to describe a Wilson loop in the fundamental representation. For a 1/2 BPS loop, or at lower order in perturbation theory, the defect action of a straight or circular Wilson loop is reparametrization invariant [28] and this reduces the calculation of the expectation value to a random matrix integral. The SYK model could be seen similarly as a 0+1 defect theory,[2] where reparametrization invariance emerges at low energies. Contrary to the Wilson loop, in this case it is not a true symmetry, and there is a Schwarzian effective action for reparametrizations. However, as the Schwarzian is an irrelevant deformation in the 0+1 theory, it seems natural that the IR is captured by a random matrix theory as the late time analysis of partition functions suggests [30]. Similarly, random matrix models have been shown to determine partition functions in JT gravity [31].

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[2]They are not exactly the same type of theories as the SYK model has disorder as an additional ingredient. However there are other models without disorder that capture similar physics [29], so this does not seem to be a crucial ingredient for the physics.

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A Series expansions in σ

First we check the values of X and Z at $\sigma = 0$ for the straight Wilson line. Taking (2.16) and doing the change of variables in the integral $\tau_0 = \tau + \sigma v$, the expressions for X and Z become

$$\begin{aligned} X(\sigma,\tau) &= \int_{-\infty}^{\infty} dv \frac{1}{\pi}\frac{1}{1+v^2} x_0(\tau+\sigma v), \\ Z(\sigma,\tau) &= \fint_{-\infty}^{\infty} dv \frac{1}{\pi}\frac{v}{1+v^2} x_0(\tau+\sigma v), \end{aligned} \tag{A.1}$$

It is immediate to check that $X(\sigma = 0, \tau) = x_0(\tau)$ and $Z(\sigma = 0, \tau) = 0$. From the equations (2.15), X is an even function of σ and Z an odd function. Therefore, they have the expansions

$$\begin{aligned} X(\sigma,\tau) &= \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{(2n)!} \partial_\sigma^{2n} X\Big|_{\sigma=0}, \\ Z(\sigma,\tau) &= \sum_{n=0}^{\infty} \frac{\sigma^{2n+1}}{(2n+1)!} \partial_\sigma^{2n+1} Z\Big|_{\sigma=0}. \end{aligned} \tag{A.2}$$

We can use (2.15) to trade σ derivatives by τ derivatives

$$\begin{aligned} X(\sigma,\tau) &= \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{(2n)!} (-1)^n \partial_\tau^{2n} X\Big|_{\sigma=0}, \\ Z(\sigma,\tau) &= \sum_{n=0}^{\infty} \frac{\sigma^{2n+1}}{(2n+1)!} (-1)^{2n} \partial_\tau^{2n+1} X\Big|_{\sigma=0}. \end{aligned} \tag{A.3}$$

But, using that the kernel in the integrand of (2.16) depends only on $\tau - \tau_0$,

$$\partial_\tau^N X = (-1)^N \int_{-\infty}^{\infty} d\tau_0\, \partial_{\tau_0}^N \left(\frac{1}{\pi}\frac{\sigma}{\sigma^2+(\tau-\tau_0)^2}\right) x_0(\tau_0) = \int_{-\infty}^{\infty} d\tau_0 \frac{1}{\pi}\frac{\sigma}{\sigma^2+(\tau-\tau_0)^2} x_0^{(N)}(\tau_0). \tag{A.4}$$

Therefore, $\partial_\tau^N X\Big|_{\sigma=0} = x_0^{(N)}(\tau)$, so we arrive at (2.17).

For the circle we can proceed in a similar way. Starting with (3.28) and doing the change of variables

$$\tau_0 = \tau + 2\arctan\left(\tan\frac{\sigma}{2} v\right), \qquad \sigma = \operatorname{arcsinh}(s). \tag{A.5}$$

We arrive at

$$\begin{aligned} \Theta(s,\tau) =& q\tau + \int_{-\infty}^{\infty} dv \frac{1}{\pi}\frac{1}{1+v^2} \theta_0\left(\tau + 2\arctan\left(\frac{sv}{1+\sqrt{1+s^2}}\right)\right), \\ S(s,\tau) =& q\sigma(s) + \fint_{-\infty}^{\infty} dv \frac{1}{\pi}\frac{v}{1+v^2} \frac{1+\sqrt{1+s^2}}{1+\sqrt{1+s^2}+\frac{1+v^2}{2}s^2} \theta_0\left(\tau + 2\arctan\left(\frac{sv}{1+\sqrt{1+s^2}}\right)\right). \end{aligned} \tag{A.6}$$

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So, indeed $\Theta(\sigma = 0, \tau) = \Theta(s = 0, \tau) = \Theta_0(\tau)$ and $S(\sigma = 0, \tau) = S(s = 0, \tau) = 0$. Since Θ and S for the circle satisfy the same equations as for the straight line, we can apply the same derivation and arrive at the same result for the expansions.

In both cases the form of the expansion can also be explicitly checked by taking the σ derivative of the integrands, performing the same changes of variables in the integrals we have introduced now, and then computing the integrals explicitly.

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Chapter 7

Conclusions

In this thesis we have presented a way to obtain a low energy effective action for the gravity dual of a Wilson line. We have begun by laying out the basics of a formalism that allows us to perform Wilson renormalization on a Nambu-Goto string, where the leading correction is introduced with a double trace deformation. In chapters 4 and 5 we have focused on phenomenological aspects that can be computed using this formalism. We have first studied the $q\bar{q}$ potential for different examples. In the case of an IR fixed point we obtained that the long distance potential behaves as $1/L$ with a leading correction $1/L^4$ which is consistent with a flow in a one-dimensional theory between a double trace term with scaling dimension $\Delta = -2$ in the UV to a scaling dimension $\Delta = 4$ in the IR. For confining gauge theories holographic models predict that the effective action of a flux tube is given by 4 dimensional Nambu-Goto action plus internal massive modes. We have identified the contribution of these modes with the exponentially decaying term $\sim e^{-ML}$. This term cannot be identified with the rigidity term in [24, 25, 26] but corresponds to massive modes that have not been observed yet. Lattice computations show that a long flux tube in 2+1 and 3+1 dimensions agree with the Nambu-Goto action from long to string length distances, except for some parity odd channel [27]. This deviation does not match to the massive mode we have found since ours is parity even. A way to observe this new mode might be comparing the quark-antiquark potential and the energy of the flux tube, as in confining theories they only differ at the classical contributions induced by the sources at the endpoints and finite size contributions, which are expected to vanish in the large-N limit [27]. The apparition of this massive mode is expected for a general confining theory, which makes it an interesting check for holography. This formalism could be contrasted with other holographic expectations to study the effect of massive modes of the flux tube on meson Regge trajectories to provide an additional test.

We also studied more general trajectories in 5, identifying the contributions to the force suffered by a quark with those found on previous work by [28, 15, 16, 17, 18]. It could be interesting to extend this formalism to time dependent geometries that emulate the dual of a heavy ion colision. As the expansion of the action can be done in a systematic way, this work could be extended to less constrained quark trajectories with sudden changes in the trajectory or movement in more spatial directions. Another generalization could be applying this method to Wilson lines in different representations, 't Hooft lines or the expectation values of local operators obtained from the background geometry.

The formalism of holographic Wilsonian renormalization could also be combined with other phenomenological approaches to fit QCD lattice computations or experimental results. For example there is a recent development in using machine learning to constrain the background geometry [29, 30] that could be applied to Wilson loops predictions[11], as the holographic description of UV physics is expected to be problematic due to asymptotic freedom and this approach limits the range of energy scales.

In both these approaches we have chosen the reparametrization of the Wilson loop with gauge fixing. However reparametrization invariance is an important feature of these observables, so we studied how can we realize it in the holographic dual. We have first identified a map between reparametrizations of the Wilson loop and conformal transformations in the worldsheet of the dual string, showing that a string with a cutoff in the worldsheet is not invariant under reparametrizations of the Wilson loops. This makes neccesary to add a cutoff action proportional to the Schwarzian derivative of the reparametrization with some higher derivative terms that are suppressed by the cutoff scale. If the cutoff is set in the target space the string is now invariant under Wilson loop reparametrizations but not under worldsheet diffeomorphisms or Weyl transformations that do not belong to the subset of conformal transformations. This means that we should add new terms to the Polyakov action at the cutoff before fixing the gauge. In the case of the cutoff on the worldsheet, the new terms have the structure of an effective theory of Goldstone bosons for a spontaneously broken reparametrization invariance that preserves the isometries of the nearly AdS_2 space $SL(2,\mathbb{R})$ or $SU(1,1)$. These terms are necessary as a conformal transformation would change the physical location of the cutoff in the target space. This is compensated by a change in the cutoff action so that the total area of the string remains invariant. Further analysis could be directed to check whether the Schwarzian in the cutoff is related to maximal chaos as observed in strings with worldsheet horizons or applying this analysis to more general geometries and string profiles.

Conclusiones

En esta tesis hemos presentado una forma de obtener una acción efectiva a bajas energías para el dual gravitatorio de una línea de Wilson. En primer lugar hemos establecido un formalismo qu nos permite realizar una renormalización de Wilson en una cuerda de Nambu-Goto, donde las correcciones a primer orden aparecen como una deformación de traza doble. En los capítulos 4 y 5 nos hemos centrado en aspectos fenomenológicos que podemos calcular empleando este formalismo. En primer lugar hemos estudiado el potencial $q\bar{q}$ en distintos ejemplos. En el caso de un punto fijo en el infrarrojo hemos obtenido que el potencial a largas distancias se comporta como $1/L$ con una corrección del tipo $1/L^4$, que es consistente con el flujo de una teoría en una dimensión entre un término de doble traza con dimensión de escalado $\Delta = -2$ en el ultravioleta que fluye a una $\Delta = 4$ en el infrarrojo. Para teorías confinantes los modelos holográficos predicen que la acción efectiva de un tubo de flujo está dada por la acción de Nambu-Goto en 4 dimensiones más modos masivos internos. Hemos identificado la contribución de estos modos con un término que decae exponencialmente como $\sim e^{-ML}$. Este término no puede ser identificado con el término de rigidez encontrado en [24, 25, 26] y corresponde con modos masivos que no se han identificado aun. Cálculos en lattice concuerdan con la acción de Nambu-Goto desde largas distancias a longitudes del orden del tamaño de la cuerda, excepto en algunos canales de paridad impar [27]. Esta desviación no encaja con el modo que hemos encontrado puesto que el nuestro tiene una paridad par. Una forma de observar este nuevo modo podría ser comparar el potencial quark-antiquark y la energía del tubo de flujo, puesto que en teorías confinantes solo se diferencian en contribuciones clásicas inducidas por las fuentes al final de la cuerda y contribuciones de tamaño finito, que se espera que se desvanezcan en el límite de gran N [27]. La aparición de este modo masivo es una predicción general para una teoría confinante, lo que lo convierte en una prueba interesante para el principio holográfico. Este formalismo podría ser contrastado con otras predicciones en holografía para estudiar el efecto de modos masivos en un tubo de flujo en las trayectorias de Regge para proporcinal una prueba adicional.

También estudiamos trayectorias más generales en 5, identificando las contribuciones a la fuerza experimentada por un quark con las encontradas en trabajos previos por [28, 15, 16, 17, 18]. Podría ser interesante extender este formalismo a teorías dependientes del tiempo que simulen el dual de una colisión de iones pesados. Puesto que la expansión de la acción puede llevarse a cabo de una forma sistemática, este trabajo se puede extender a trayectorias menos limitadas con cambios bruscos en la trayectoria o movimiento en más direcciones espaciales. Otra generalización podría ser aplicar este método a líneas de Wilson en diferentes representaciones, lineas de 't Hooft o valores de expectación de operadores locales obtenidos en la geometría de fondo.

El formalismo de la renormalización holográfica de Wilson también se podría combinar con otras aproximaciones fenomenológicas para ajustarse a cálculos de lattice en QCD o resultados experimentales. Por ejemplo existe un trabajo reciente empleando machine learning para delimitar la geometría del fondo [29, 30] que podría aplicarse a predicciones de Wilson loops [11], puesto que se espera que la descripción holográfica de la física en el ultravioleta sea problemática debido a la libertad asintótica, y este método limita el rango de energías.

En los casos anteriores hemos escogido una reparametrización del Wilson loop escogiendo un gauge. Sin embargo la invarianza bajo reparametrizaciones es una característica importante de estos observables, por lo que estudiamos como podemos trasladarla al dual holográfico. En primer lugar hemos identificado una equivalencia entre la reparametrización del Wilson loop y transformaciones conformes en la hoja de mundo de la cuerda dual, mostrando que una cuerda con un corte en la hoja de mundo no es invariante bajo las reparametrizaciones del Wilson loop. Esto hace necesario añadir una acción en el corte proporcional a la Schwarziana de la reparametrización con algunos términos a mayor orden suprimidos por la escala de corte. Si el corte se coloca en la geometría de fondo la cuerda es ahora invariante bajo las reparametrizaciones del Wilson Loop pero no bajo difeomorfismos de la hoja de mundo o transformaciones de Weyl que no pertenezcan al subjconjunto de transformaciones conformes. Esto implica que debemos añadir términos nuevos a la acción de Polyakov en el corte antes de fijar el gauge. En el caso de un corte en la hoja de mundo los términos nuevos tienen la estructura de una teoría efectiva de bosones Goldstone por la ruptura espontánea de simetría de la invarianza bajo reparametrizaciones que conserva las isometrías de la geometría del espacio cuasi-AdS_2: $SL(2,\mathbb{R})$ o $SU(1,1)$. Estos términos son necesarios puesto que una transformación conforme cambiaría la localización física del corte en la geometría de fondo. Esto se compensa por un cambio en la acción en el corte de forma que el área total de la cuerda permanece invariante. Futuros análisis podrías estudiar si la Schwarziana en el corte está relacionada con caos maximal como se observa en cuerdas con horizonte en la hoja de mundo o aplicando este análisis en geometrías y perfiles de la cuerda más generales.

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